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MATHEMATICS

ON THE PRODUCT OF CONSECUTIVE INTEGERS. III [1]

BY

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It has been conjectured a long time ago that the product

$$A_k(n)=n(n+1)\dots(n+k-1)$$

of k consecutive integers is never an l-th power if $k>1$, $l>1$ [2]). RIGGE [3]) and a few months later I [1]) proved that $A_k(n)$ is never a square, and later RIGGE and I [4]) proved using the Thue–Siegel theorem that for every $l>2$ there exists a $k_0(l)$ so that for every $k>k_0(l)$ $A_k(n)$ is not an l-th power. In 1940 SIEGEL and I proved that there is a constant c so that for $k>c$, $l>1$ $A_k(n)$ is not an l-th power, in other words that $k_0(l)$ is independent of l. Our proof was very similar to that used in [1]) and was never published. A few years ago I obtained a new proof for this result which does not use the result of THUE–SIEGEL and seems to me to be of sufficient interest to deserve publication. The value of c could be determined explicitly by a somewhat laborious computation and it probably would turn out to be not too large, and perhaps the proof that the product of consecutive integers is never a power could be furnished by a manageable if long computation (the cases $k\leqq c$ would have to be settled by a different method). A method similar to the one used here was used in a previous paper [5]).

Now we prove

Theorem 1. *There exists a constant c so that for* $k>c$, $l>1$ $A_k(n)$ *is never an l-th power.*

As stated in the introduction RIGGE and I proved that $A_k(n)$ is never a square, thus we can assume $l>2$. Further assume that

$$A_k(n)=x^l. \tag{1}$$

First we need some lemmas.

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[1]) I had two previous papers by the same title, Journal London Math. Soc. **14**, 194–198 (1939) and ibid. 245–249. These papers will be referred to as I and II.

[2]) A great deal of the early litterature of this problem can be found in the paper of R. OBLATH, Tohoku Math. Journal **38**, 73–92 (1933).

[3]) O. RIGGE, Über ein diophantisches Problem, 9. Congr. des Math. scand. 155–160 (1939) and P. ERDÖS I.

[4]) P. ERDÖS II, As far as I know Rigges proof, which was similar to mine, has not been published.

[5]) P. ERDÖS, On a diophantine equation, Journal London Math. Soc. **26**, 176–178 (1951).

Lemma 1. $n > k^l$.

First we show $n \geqslant k$. If $n < k$ it follows from the theorem of TCHEBICHEFF that there is a prime p satisfying $n \leqslant \frac{n+k-1}{2} < p \leqslant n+k-1$. Thus the product $A_k(n)$ is divisible by p but not by p^2, or (1) is impossible.

Assume now $n \geqslant k$. A theorem of SYLVESTER and SCHUR [6]) then asserts that there is a prime $p > k$ which divides $A_k(n)$. But clearly only one of the numbers $n, n+1, \ldots, n+k-1$ can be a multiple of p, say $n+i \equiv 0 \pmod{p}$. But then we have from (1) $n+i \equiv 0 \pmod{p^l}$ or $n+k-1 \geqslant n+i \geqslant \geqslant (k+1)^l$. Thus $n > k^l$ as stated.

Assume that (1) holds. Since all primes greater or equal to k can occur in at most one term of (1), we must have

$$n+i = a_i x_i^l, \quad 0 \leqq i \leqq k-1$$

where all the prime factors of a_i are less than k and a_i is not divisible by an l-th power.

Lemma 2. *The products $a_i \cdot a_j$, $0 \leqq i, j \leqq k-1$, are all different.*

Assume $a_i \cdot a_j = a_r \cdot a_s = A$. Then we would have

$$(n+i)(n+j) = A(x_i x_j)^l, \quad (n+r)(n+s) = A(x_r x_s)^l.$$

First we show that $(n+i)(n+j) = (n+r)(n+s)$ implies $i = r$, $j = s$. Assume first $i+j \neq r+s$, say $i+j > r+s$. Then

$$n^2 + (i+j)n + ij = n^2 + (r+s)n + rs, \text{ or } n \leqq rs < k^2$$

which contradicts Lemma 1. Hence $i+j = r+s$, therefore $ij = rs$.

Assume now without loss of generality $(n+r)(n+s) > (n+i)(n+j)$. Then $x_r x_s \geqq x_i x_j + 1$ and we would have by Lemma 1

$$2kn > (n+k-1)^2 - n^2 \geqq (n+r)(n+s) - (n+i)(n+j) \geqq A\,[(x_i x_j + 1)^l - (x_i x_j)^l] >$$
$$> lA\,(x_i x_j)^{l-1} \geqq l\,[A\,(x_i x_j)^l]^{(l-1)/l} \geqq l\,(n^2)^{(l-1)/l} \geqq 3\,n^{4/3}.$$

Thus we would have $n < k^3$, which contradicts Lemma 1. This contradiction proves Lemma 2.

Lemma 3. *There exists a sequence $0 \leqq i_1 < i_2 < \ldots < i$ so that $t \geqq k - \pi(k)$ and*

$$\prod_{r=1}^{t} a_{i_r} \mid k!. \tag{2}$$

For each $p < k$ denote by a_{j_p} one of the a_j's, $0 \leqq j < k$, which have the property that no other a_r, $0 \leqq r < k$, is divisible by p to a higher power than a_{j_p} (i.e. if a_j is divisible by p to the power d_j then $d_{j_p} = \max\limits_{0 \leqq j < k} d_j$). Denote by $a_{i_1}, a_{i_2}, \ldots a_{i_t}$ the sequence of a's from which all the a_{j_p}'s have been omitted. Clearly $t \geqq k - \pi(k-1) \geqq t - \pi(k)$.

[6]) P. ERDÖS, On a theorem of Sylvester and Schur, Journal London Math. Soc. 9, 282–288 (1934).

To show that (2) holds it suffices to prove that if p^d divides the product

$$\prod_{r=1}^{t} a_{i_r}$$

then $d \leqq [k/p]+[k/p^2]+ \ldots$. This is easy to see, since the number of multiples of p^β among the integers $n, n+1, \ldots, n+k-1$ is at most $[k/p^\beta]+1$, or the number of multiples of p^β amongst the a_i's, $0 \leqq i \leqq k-1$, is at most $[k/p^\beta]+1$. But then the number of multiples of p^β among the a_{i_r}, $1 \leqq r \leqq t$, is at most $[k/p^\beta]$, since if there is an $a_j \equiv 0 \pmod{p^\beta}$, then $a_{i_p} \equiv 0 \pmod{p^\beta}$ and a_{i_p} does not occur among the a_{i_r}, $1 \leqq r \leqq t$. This completes the proof of the Lemma.

By slightly more complicated arguments we could prove that

$$\prod_{r=1}^{t} a_r \,|\, (k-1)!\,.$$

Denote now by $N(x)$ the maximum number of integers $1 \leqq b_1 < b_2 < \ldots$ $\ldots < b_u \leqq x$ so that the products $b_i b_j$, $1 \leqq i, j \leqq u$, are all different.

Lemma 4. *For sufficiently large x we have*

$$N(x) < 2x/\log x.$$

In a previous paper [7]) I proved

(3) $$N(x) < \pi(x) + 8x^{3/4} + x^{1/2}.$$

Using the well known inequality $\pi(x) < \frac{3}{2}\frac{x}{\log x}$ we immediately obtain Lemma 4.

For the sake of completeness I will outline a proof of a formula similar to (3) at the end of the paper.

Now we can prove our Theorem. Consider the integers $a_{i_1}, a_{i_2}, \ldots, a_{i_t}$ of Lemma 3, order them according to size. Thus we obtain the sequence $b_1 < b_2 < \ldots < b_t$ where by Lemma 2 the numbers $b_i b_j$ are all different. Let now $i > i_0$ be sufficiently large. Putting $b_i = x$ and using Lemma 4 we obtain

(4) $$i \leqq N(b_i) < \frac{2b_i}{\log b_i} \text{ or } b_i > (i \log i)/2\,.$$

Thus from (4) we have for sufficiently large i_0 and $t > 2\,i_0$

(5) $$\prod_{i=1}^{t} b_i > i_0! \prod_{i=i_0+1}^{t} (i \log i)/2 > t!\,(\log i_0)^{t/2}/2^t > t!\,10^t.$$

Now $t \geqq k - \pi(k) > k - \frac{3k}{2\log k}$. Thus

(6) $$t! > \frac{k!}{k^{k-t}} > k!\,k^{-\frac{3k}{2\log k}} > k!/5^k.$$

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[7]) P. Erdös, On sequences of integers no one of which divides the product of two others and on some related problems. Mitt. Forsch. Inst. Math. u. Mech. Univ. Tomsk 2, 74–82 (1938).

Thus finally from (5) and (6) we have for sufficiently large k

$$\prod_{r=1}^{t} a_{i_r} = \prod_{i=1}^{t} b_i > k!\,\frac{10^t}{5^k} > k! \tag{7}$$

since

$$10^t > 10^{k-\frac{3k}{\log k}} > 5^k.$$

(7) clearly contradicts Lemma 3, and this contradiction proves the theorem for sufficiently large k.

One could easily make the estimations more precise and obtain a better value for c, but the method used in this paper does not seem suitable to get a really good value for c. The problem clearly is to determine the least constant c so that for all $k>c$ one can not have integers $a_1, a_2, \ldots, a_t$ satisfying (2) $t \geqq k-\pi(k)$ and the products $a_i \cdot a_j$ are all distinct.

It is clear from the proof of Theorem 1 that in fact we proved the following slightly stronger result: For $k>c$ there exists a prime $p>k$ so that if $p^\beta \,\|\, A(n)$ then $\beta \not\equiv 0 \pmod{l}$ ($p^\beta \,\|\, A(n)$ means: $p^\beta \mid A(n)$, $p^{\beta+1} \nmid A(n)$).

By a slightly more careful estimation at the end of the proof of Theorem 1 we could obtain the following

Theorem 2. *Let $l>2$, and ε an arbitrary positive number. Then there exists a constant $c=c(\varepsilon)$ so that if $k>c$, $n>k^l$ and we delate from the numbers $n, n+1, \ldots, n+k-1$ in an arbitrary way less than $(1-\varepsilon)k \log\log k/\log k$ of them. Then the product of the remaining numbers is never an l-th power.*

The condition $n>k^l$ can not entirely be omitted. In fact if $n=1$ it is easy to see that one can delate $r \leqq \pi(k)$ integers from $n, n+1, \ldots, n+k-1$ so that the product of the remaining numbers is an l-th power.

I can not prove Theorem 2 for $l=2$, I can only prove it with $ck/\log k$ instead of $(1-\varepsilon)k \log\log k/\log k$.

In the proof of Lemma 3 (1) was not used. Thus if we put

$$A_i^{(n)} = \prod_p p^d,\ p^d \,\|\, n+i,\ p<k,\ 0 \leqq i \leqq k-1,$$

we can prove by arguments used in the proof of Lemma 3 that there exists a sequence $i_1, i_2, \ldots, i_t$, $t>k-\pi(k)$ so that

$$\prod_{r=1} A_{i_r}^{(n)} \mid (k-1)!. \tag{8}$$

From (8) it easily follows from the prime number theorem that for $k>k_0=k_0(\varepsilon)$

$$\min_{0\leqslant i\leqslant k-1} A_i^{(n)} < (1+\varepsilon)\,k. \tag{9}$$

It is possible that (9) can be sharpened considerably. In fact it is probable that

$$\lim_{k\to\infty} \frac{1}{k}\Big(\max_{1\leqslant n<\infty}\ \min_{0\leqslant i\leqslant k-1} A_i^{(n)}\Big) = 0.$$

To complete our proof we now outline the estimation of $N(x)$. Instead of (3) we shall prove

$$N(x) < \pi(x) + 3x^{7/8} + 2x^{1/2}. \tag{10}$$

It is clear that Lemma 4 is an easy consequence of (10).

Let $1 \leqq b_1 < b_2 < \dots < b_s \leqq x$ be such that all the products $b_i b_j$, $1 \leqq i$, $j \leqq s$, are different. Write $b_i = u_i v_i$, where u_i is the greatest divisor of b_i which is not greater than $x^{1/2}$. First of all it is clear that the numbers $u_1 \cdot v_1$, $u_1 \cdot v_2$, $u_2 \cdot v_1$, $u_2 \cdot v_2$ can not all be b's for if $b_1 = u_1 v_1$, $b_2 = u_1 v_2$, $b_3 = u_2 v_1$, $b_4 = u_2 v_2$ we would have $b_1 b_4 = b_2 b_3$.

Now we distinguish several cases. In case I we have $u_i < x^{1/4}$. In this case v_i must be a prime. For if not let p be the least prime factor of v_i. If $p < x^{1/4}$ then $pu_i < x^{1/2}$ which contradicts the maximum property of u_i. Thus $x^{1/4} \leqq p \leqq x^{1/2}$ (since v_i was assumed to be composite we evidently have $p \leqq x^{1/2}$). But then $p > u_i$ which again contradicts the maximum property of u. Thus v_i must be a prime as stated.

Now we distinguish two subcases. In the first subcase are the b's of the form $p\, u_i$, $u_i < x^{1/4}$ for which there is no other b of the form $p\, u_i'$. The number of these b's is clearly less than or equal to $\pi(x)$.

Consider now the b's of the second subcase. They are clearly of the form

$$p_i\, u_j^{(i)}, \ 1 \leqq i \leqq r, \ 1 \leqq j \leqq l_i, \ l_i > 1, \ u_j^{(i)} < x^{1/4}.$$

By what has been previously said each pair of the sets U_i, $1 \leqq i \leqq r$

$$\{U_i\} = \bigcup_j u_j^{(i)}, \quad 1 \leqq j \leqq l_i$$

can have at most one element in common, or the pairs

$$(u_{j_1}^{(i)}, u_{j_2}^{(i)}), \quad 1 \leqq j_1, j_2 \leqq l_i, \quad 1 \leqq i \leqq r$$

are all distinct. But since $u < x^{1/4}$ the number of these pairs is less than $x^{1/2}$. Thus $(l_i > 1)$

$$\sum_{i=1}^{r} \binom{l_i}{2} < x^{1/2} \quad \text{or} \quad \sum_{i=1}^{r} l_i < 2x^{1/2}.$$

Hence the number of b's belonging to the second subcase is less than $2x^{1/2}$.

In the second case $x^{1/4} \leqq u \leqq x^{1/2}$. Again we consider two subcases. In the first subcase are the b's of the form $v u_i$ for which there are at most $x^{1/8}$ other b's of the form $v u_i'$. From $u_i \geqq x^{1/4}$ we have $v_i \leqq x^{3/4}$. Thus the number of b's of the first subcase is clearly less than or equal to $(x^{1/8} + 1) \cdot x^{3/4} \leqq 2x^{7/8}$.

Denote the b's of the second subcase by

$$v_i\, u_j^{(i)}, \quad 1 \leqq i \leqq r, \quad 1 \leqq j \leqq l_i, \quad l_i > x^{1/8} + 1.$$

Again the sets U_i, $1 \leqq i \leqq r$

$$U_i = \bigcup_j u_j^{(i)}, \quad 1 \leqq j \leqq l_i$$

can have at most one element in common. Thus the pairs $(u_{j_1}^{(i)}, u_{j_2}^{(i)})$, $1 \leqq j_1, j_2 \leqq l_i$, $1 \leqq i \leqq r$ are all distinct. The number of pairs (u_j, u_{j_2}) is clearly less than

$$\binom{[x^{1/2}]}{2} < \frac{x}{2}.$$

Thus we have $(l_i > x^{1/8} + 1)$

$$\sum_{i=1}^{r} \binom{l_i}{2} < \frac{x}{2} \quad \text{or} \quad \sum_{i=1}^{r} l_i < x^{7/8}.$$

Thus finally

$$N(x) < \pi(x) + 3\, x^{7/8} + 2\, x^{1/2}$$

which proves (10).