ON THE ROLE OF THE LEBESGUE FUNCTIONS IN THE THEORY OF THE LAGRANGE INTERPOLATION [1]

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To Prof. dr. L. FEJÉR *on the occasion of his* 75^{th} *birthday*

1. Let there be given a triangular matrix

$$A \equiv \begin{matrix} x_{11} & & & \\ x_{12}\, x_{22} & & & \\ \vdots & \ddots & & \\ x_{1n}\, x_{2n} \ldots x_{nn} & & & \\ \vdots & & \ddots & \end{matrix} \tag{1.1}$$

where for $n = 1, 2, \ldots$ we have

$$1 \geqq x_{1n} > x_{2n} > \cdots > x_{nn} \geqq -1. \tag{1.2}$$

Then, as it is well known, for given values $y_{\nu n}$ there is exactly one polynomial $g(x)$ of degree $\leqq n-1$ such that

$$g(x_{\nu n}) = y_{\nu n} \qquad (\nu = 1, 2, \ldots, n).$$

If the values $y_{\nu n}$ are the values $f(x_{\nu n})$ of a function $f(x)$ defined in $[-1, +1]$, then we call the corresponding $g(x)$ polynomial "the n^{th} interpolatory polynomial of $f(x)$ belonging to A" and denote it by $L_n(f, A)$ or — if misunderstanding cannot arise — by $L_n(f)$. The abscissae $x_{\nu n}$ are called the n^{th} fundamental points of the matrix A and are sometimes denoted also by x_ν. It is well known that $L_n(f, A)$ can be written in the form

$$L_n(f, A) = \sum_{\nu=1}^{n} f(x_{\nu n})\, l_{\nu n}(x, A), \tag{1.3}$$

where the polynomials $l_{\nu n}(x, A)$, the so-called fundamental-functions belonging

[1] A part of the results (assertions a) and b) of this paper) was the subject of a lecture made by one of us at a colloquium for the constructive function-theory in Eger (Hungary), 29 Nov. 1953; they were found twenty years ago. The new results showing they are best possibles were a subject of another lecture in Pécs (Hungary), 18 Sept. 1954.

to A, depend only upon A and have the representation

(1.4) $$l_{\nu n}(x, A) = \frac{\omega_n(x, A)}{\omega_n'(x_{\nu n}, A)\ (x - x_{\nu n})}$$

where

(1.5) $$\omega_n(x, A) = \prod_{\nu=1}^{n} (x - x_{\nu n}).$$

As easy to verify, we have

(1.6) $$\sum_{\nu=1}^{n} l_{\nu n}(x, A) \equiv 1$$

and if $h(x)$ denotes an arbitrary polynomial of degree $\leqq n-1$, then

(1.7) $$L_n(h, A) \equiv h(x).$$

From (1.3) it follows that for an $f(x)$ bounded in $[-1, +1]$ we have for $-1 \leqq x \leqq +1$

(1.8) $$|L_n(f, A)| \leqq \left(\sum_{\nu=1}^{n} |l_{\nu n}(x, A)| \right) \sup_{-1 \leqq x \leqq +1} |f(x)|.$$

2. One would be inclined to think that if A is such that for an arbitrarily small $\varepsilon > 0$

(2.1) $$x_{\nu n} - x_{\nu+1, n} \leqq \varepsilon, \quad x_{0n} = -x_{n+1, n} = 1 \qquad (\nu = 0, 1, \ldots, n;\ n > n_0(\varepsilon)),$$

then the sequence $L_n(f, A)$ converges uniformly in $[-1, +1]$ to $f(x)$ whenever[2] $f(x) \in C$. It was a great surprise at the end of the last century when RUNGE and BOREL discovered that the sequence $L_n(f, B)$, belonging to the "most classical" matrix B defined by

$$x_{\nu n} = 1 - \frac{2\nu}{n+1} \qquad (\nu = 1, 2, \ldots, n;\ n = 1, 2, \ldots),$$

can diverge in a whole subinterval of $[-1, +1]$ for such a simple function as $f(x) = \frac{1}{1+x^2}$. This would leave open the possibility, the situation can be saved by choosing another "better" A matrix. But G. FABER[3] discovered in 1910 the shaking fact that *no* matrix A can assure the uniform convergence of the polynomials $L_n(f, A)$ for every $f \in C$. His proof showed that it is essential for this phenomenon a property of the quantity

(2.2) $$M_n(A) \equiv \max_{-1 \leqq x \leqq +1} \lambda_n(x, A) = \max_{-1 \leqq x \leqq +1} \sum_{\nu=1}^{n} |l_{\nu n}(x, A)|,$$

namely that for *all* A-matrices we have

(2.3) $$\overline{\lim_{n \to \infty}}\ M_n(A) = +\infty.$$

[2] C denotes, as usual, the class of functions continuous for $-1 \leqq x \leqq +1$.

[3] G. FABER, Über die interpolatorische Darstellung stetiger Funktionen, *Jahresber. der Deutsch. Math. Ver.*, **23** (1914), pp. 190–210.

We call the quantity $\lambda_n(x, A)$ the n^{th} Lebesgue function, the quantity $M_n(A)$ the n^{th} Lebesgue constant belonging to the matrix A. Moreover, S. BERNSTEIN[4] proved the still more surprising fact that to every matrix A there belongs an x_0 in $-1 \leqq x_0 \leqq +1$ and an $f_0 \in C$ with

$$\overline{\lim_{n\to\infty}} |L_n(f_0, A)|_{x=x_0} = +\infty \tag{2.4}$$

what roots in the relation

$$\overline{\lim_{n\to\infty}} \lambda_n(x_0, A) = +\infty. \tag{2.5}$$

3. These facts show that for the divergence theory of the Lagrange interpolation the functions $\lambda_n(x, A)$ are essential. But it was observed by FEJÉR[5] that these Lebesgue functions $\lambda_n(x, A)$ play also a role in the convergence theory. His simple reasoning reproduced for the sake of completeness in **9** gives the following theorem:

If the quantities $M_n(A)$ of (2.2) satisfy the inequality

$$M_n(A) < c_1 n^{\beta} \tag{3.1}$$

with a fixed $0 < \beta < 1$ and numerical[6] c_1, then the polynomials $L_n(f, A)$ converge for $-1 \leqq x \leqq +1$ uniformly to $f(x)$ if[7] $f \in \mathrm{Lip}\,\gamma$, $\gamma > \beta$.

4. These results are responsible for the rather general opinion that the convergence-divergence theory of the Lagrange interpolation is by and large identical with the study of the Lebesgue constants $M_n(A)$. We have set ourselves the task to investigate to which extent this is true. We have found that going a little beyond the mere continuity this fails to be true; the result can quite vaguely expressed saying that *there is a "rough" and a "fine" convergence-divergence theory for the Lagrange interpolation.* To be more exact, let us consider, if

$$0 < \beta < 1, \tag{4.1}$$

the class $A(\beta)$ of all A-matrices for which with arbitrarily small positive ε we have

$$\overline{\lim_{n\to\infty}} M_n(A) n^{-\beta-\varepsilon} < c_2(\varepsilon), \tag{4.2}$$

$$\overline{\lim_{n\to\infty}} M_n(A) n^{-\beta+\varepsilon} > c_3(\varepsilon), \tag{4.3}$$

[4] S. BERNSTEIN, Sur la limitation des valeurs etc., *Bull. Acad. Sc. De l'URSS*, **8** (1931), pp. 1025—1050.

[5] L. FEJÉR, Lagrangesche Interpolation und die zugehörigen konjugierten Punkte, *Math. Ann.*, **106** (1932), pp. 1—55.

[6] Later on $c_2, c_3, \ldots$ denote generally also numerical constants. If some c_ν depends upon some parameters, the dependence will be explicitely stated.

[7] As usual, the class $\mathrm{Lip}\,\gamma$ denotes the totality of those functions which satisfy uniformly a Lipschitz-condition with the exponent γ in $-1 \leqq x \leqq +1$.

i. e. for our matrices the Lebesgue constants $M_n(A)$ increase roughly speaking like n^β. We call for a fixed β with (4. 1) the Lip γ class of functions

"a good class of functions $f(x)$ for $A(\beta)$",	if for all $A \in A(\beta)$ and $f \in$ Lip γ $L_n(f, A)$ tend uniformly in $[-1, +1]$ to $f(x)$ for $n \to \infty$,

respectively

"a bad class of functions $f(x)$ for $A(\beta)$",	if for all $A \in A(\beta)$ there is an $f_1(x) \in$ Lip γ such that $L_n(f_1, A)$ is unbounded in $[-1, +1]$ for $n \to \infty$.

If a Lip γ class is a good or a bad one, then its convergence-divergence behaviour is by and large determined by the numbers $M_n(A)$; one would be inclined to think that all Lip γ classes with $\gamma < \beta$ are bad and all with $\gamma > \beta$ are good classes and thus the finer structure of the matrix A cannot essentially influence the convergence-divergence behaviour for the respective Lip γ class. A closer investigation showed, howewer, that this is not the case, there are values γ depending only upon β for which the Lip γ classes are neither good nor bad ones. This means that the convergence-divergence behaviour is certainly not determined for the respective Lip γ class alone by the Lebesgue constants $M_n(A)$; hence the convergence-divergence behaviour for the respective Lip γ class depends upon the finer structure of A and thus the determination of the convergence-divergence behaviour belongs to a "finer" theory. Thus even the existence of this "finer" theory is somewhat surprising; moreover we shall see that *for fixed* $0 < \beta < 1$ *all* Lip γ *classes with*

$$\gamma < \frac{\beta}{\beta + 2} \tag{4. 4}$$

are bad classes, all Lip γ *classes with*

$$\gamma > \beta \tag{4. 5}$$

are good ones and the Lip γ *classes with*

$$\frac{\beta}{\beta + 2} < \gamma < \beta \tag{4. 6}$$

form the exact domain of the finer convergence-divergence theory.

Similar questions arise in connection with orthogonal expansions, singular integrals, mechanical quadrature and generally with linear operations. Also other scales than the Lip-classes can be used. Further, the convergence behaviour can instead of uniform convergence refer e. g. to pointwise convergence. Finally, perhaps the matrix-class $A(\beta)$ can be defined more suitably than in (4. 2)—(4. 3).

5. Our result is given under (4. 4), (4. 5) and (4. 6). In order to prove it we brake the assertion into four parts. β is fixed with

$$0 < \beta < 1. \tag{5. 1}$$

a) If $\gamma < \frac{\beta}{\beta+2}$ and $A \in A(\beta)$, then there is an $f_2 \in \mathrm{Lip}\,\gamma$ such that the sequence $L_n(f_2, A)$ is unbounded for $-1 \leqq x \leqq 1$, i. e. the class Lip γ is bad in this case.

b) If $\gamma > \beta$ and $A \in A(\beta)$, then the sequence $L_n(f, A)$ converges uniformly in $[-1, +1]$ to $f(x)$ whenever $f \in \mathrm{Lip}\,\gamma$, i. e. the class Lip γ is good in this case.

c) If $\gamma > \frac{\beta}{\beta+2}$, then there is a matrix $A_0 \in A(\beta)$ such that the sequence $L_n(f, A_0)$ converges uniformly in $[-1, +1]$ to $f(x)$ whenever $f \in \mathrm{Lip}\,\gamma$, i. e. the class Lip γ is certainly not a bad class.

d) If $\gamma < \beta$, then there is a matrix $A_1 \in A(\beta)$ and $f_3 \in \mathrm{Lip}\,\gamma$ such that the sequence $L_n(f_3, A_1)$ is unbounded for $-1 \leqq x \leqq +1$, i. e. the class Lip γ is certainly not a good one.[8]

6. To prove the assertion a) we need the simple

LEMMA I. *If $A \in A(\beta)$, then we have for $\nu = 1, 2, \ldots, (n-1)$*

(6. 1) $$x_{\nu n} - x_{\nu+1, n} > c_4(\varepsilon) n^{-\beta-2-\varepsilon}.$$

For the proof we consider the quantity

$$\frac{1}{x_{\nu n} - x_{\nu-1, n}}.$$

Owing to the definition of the fundamental functions and the mean-value theorem

(6. 2) $$\frac{1}{x_{\nu n} - x_{\nu-1, n}} = \frac{l_{\nu n}(x_{\nu n}, A) - l_{\nu n}(x_{\nu+1, n}, A)}{x_{\nu n} - x_{\nu-1, n}} = l'_{\nu n}(\zeta, A),$$

where

$$x_{\nu-1, n} \leqq \zeta \leqq x_{\nu n}.$$

But using MARKOV's well-known theorem we have

$$|l'_{\nu n}(\zeta)| \leqq (n-1)^2 \max_{-1 \leqq x \leqq +1} |l_{\nu n}(x, A)|,$$

i. e. a fortiori

(6. 3) $$|l'_{\nu n}(\zeta, A)| < n^2 M_n(A).$$

From (4. 2) we have $M_n(A) \leqq c_5(\varepsilon) n^{\beta+\varepsilon}$, i. e. (6. 2) and (6. 3) give

$$\frac{1}{x_{\nu n} - x_{\nu+1, n}} < c_5(\varepsilon) n^{\beta+2+\varepsilon}$$

which proves (6. 1)

[8] By modifications of the construction we could prove that in cases c) and d) also a matrix with $c_1 n^\beta \leqq \max_{-1 \leqq x \leqq +1} \sum_{\nu=1}^{n} |l_{\nu n}(x)| \leqq c_2 n^\beta$ could have been constructed with the other required properties. Also the investigation of the limiting cases is of interest.

7. Now we turn to the proof of the assertion a). Let

$$\sum_{\nu=1}^{n} |l_{\nu n}(x, A)|$$

maximal in $[-1, +1]$ for $x = \zeta_n$. With the convention

$$x_{0n} = -x_{n+1, n} = 1, \quad l_{0n}(x, A) \equiv l_{1n}(x, A), \quad l_{n+1, n}(x, A) \equiv l_{nn}(x, A)$$

we consider the function $\varphi_n(x)$ defined by the broken line with the vertices at the points

$$P_\nu \equiv (x_{\nu n}, \operatorname{sign} l_{\nu n}(\zeta_n, A)) \qquad (\nu = 0, 1, \ldots, n, n+1).$$

Then we have obviously

$$L_n(\varphi_n, A)_{x=\zeta_n} = \sum_{\nu=1}^{n} |l_{\nu n}(\zeta_n, A)| = M_n(A). \tag{7. 1}$$

According to the Lemma I all slopes of $\varphi_n(x)$ are absolutely

$$< c_5(\varepsilon) n^{\beta+2+\varepsilon}, \tag{7. 2}$$

and for $-1 \leqq x \leqq +1$

$$|\varphi_n(x)| \leqq 1. \tag{7. 3}$$

Now we can construct $f_2(x)$ in a way which is a suitably modified form of the resonance principle of LEBESGUE. Since $\gamma < \frac{\beta}{\beta+2}$, we may choose ε so small that

$$0 < \varepsilon < \frac{\beta}{10} \tag{7. 4}$$

and

$$\frac{\beta - 2\varepsilon}{\beta + 2 + 2\varepsilon} > \gamma; \tag{7. 5}$$

we fix ε. According to (4. 3) there is an infinite sequence

$$2^2 \leqq n_1 < n_2 < \cdots$$

such that for $\nu = 1, 2, \ldots$

$$M_{n_\nu}(A) = \lambda_{n_\nu}(\zeta_{n_\nu}, A) > \frac{c_3(\varepsilon)}{2} n_\nu^{\beta-\varepsilon}. \tag{7. 6}$$

Now we select a suitable sub-sequence of the n_ν's which we shall denote by r_ν's. Let

$$r_1 = n_1$$

and we suppose

$$r_1, r_2, \ldots, r_{\nu-1}$$

are already defined. We distingish two cases.

CASE I. Denoting

$$F_{\nu-1}(x) = \sum_{j=1}^{\nu-1} \frac{1}{r_j^{\beta-2\varepsilon}} \varphi_{r_j}(x)$$

the sequence

$$L_\mu(F_{\nu-1}, A) \qquad (\mu = 1, 2, \ldots)$$

is unbounded. In this case for

$$-1 \leqq x < x+h \leqq 1$$

we have

$$|F_{\nu-1}(x+h) - F_{\nu-1}(x)| \leqq \sum_{j=1}^{\nu-1} \frac{1}{r_j^{\beta-2\varepsilon}} |\varphi_{r_j}(x+h) - \varphi_{r_j}(x)|,$$

i. e. using (7. 2)

$$|F_{\nu-1}(x+h) - F_{\nu-1}(x)| \leqq c_5(\varepsilon) h \sum_{j=1}^{\nu-1} r_j^{2+3\varepsilon} < c_6(\varepsilon) h,$$

i. e. $F_{\nu-1}(x)$ belongs even to Lip 1.

CASE II. The sequence $L_\mu(F_{\nu-1}, A)$ is bounded, i. e. for $[-1, +1]$ and $\mu = 1, 2, \ldots$

(7. 7) $$|L_\mu(F_{\nu-1}, A)| \leqq C_{\nu-1}.$$

Then let r_ν be the smallest integer satisfying the conditions

(7. 8) $$r_\nu > r_{\nu-1}^2 \qquad \text{(i. e. also } > 2r_{\nu-1}),$$

(7. 9) $$r_\nu > C_{\nu-1}^{\frac{2}{\varepsilon}}.$$

We may suppose that we have for all ν's the Case II and we assert that in this case we may choose as $f_2(x)$ of assertion a)

(7. 10) $$f_2(x) = \sum_{j=1}^{\infty} \frac{1}{r_j^{\beta-2\varepsilon}} \varphi_{r_j}(x).$$

In order to show that $f_2(x) \in \text{Lip}\,\gamma$ we write for an x and h satisfying

$$-1 \leqq x < x+h \leqq +1, \quad 0 < h < \frac{1}{100}$$

$$f_2(x+h) - f_2(x) = \sum_{j=1}^{p} + \sum_{j=p+1}^{\infty} \frac{1}{r_j^{\beta-2\varepsilon}} (\varphi_{r_j}(x+h) - \varphi_{r_j}(x))$$

where the index p is defined uniquely by

(7. 11) $$r_p \leqq \left(\frac{1}{h}\right)^{\frac{1}{\beta+2+2\varepsilon}} < r_{p+1}.$$

Using for $\nu \leqq p$ (7. 2) and for $\nu \geqq p+1$ (7. 3), we get

$$|f_2(x+h) - f_2(x)| \leqq c_5(\varepsilon) h \sum_{j=1}^{p} r_j^{2+3\varepsilon} + 2 \sum_{j=p+1}^{\infty} r_j^{-\beta+2\varepsilon} <$$

$$< 2c_5(\varepsilon) \left\{ hp r_p^{2+3\varepsilon} + \frac{1}{r_{p+1}^{\beta-2\varepsilon}} \sum_{j=p+1}^{\infty} \left(\frac{r_{p+1}}{r_j}\right)^{\beta-2\varepsilon} \right\},$$

i. e. using (7. 8)

(7. 12) $$|f_2(x+h) - f_2(x)| \leqq c_6(\varepsilon, \beta) \left\{ hp r_p^{2+3\varepsilon} + \frac{1}{r_{p+1}^{\beta-2\varepsilon}} \right\}.$$

(7. 8) gives $r_\nu > 2^{\nu-1} r_1 \geqq 2^\nu$, i. e. $p < 2\log r_p$; hence from this and (7. 11)

$$h p r_p^{2+3\varepsilon} < c_7(\varepsilon) h r_p^{2+4\varepsilon} < c_7(\varepsilon) h^{\frac{\beta-2\varepsilon}{\beta+2+2\varepsilon}}, \quad \frac{1}{r_{p-1}^{\beta-2\varepsilon}} < h^{\frac{\beta-2\varepsilon}{\beta+2+2\varepsilon}},$$

i. e. (7. 12) and (7. 5) give at once

$$|f_2(x+h)-f_2(x)| \leqq c_8(\varepsilon, \beta) h^{\frac{\beta-2\varepsilon}{\beta+2+2\varepsilon}} < c_8(\varepsilon, \beta) h^\gamma.$$

Thus our $f_2(x)$ belongs to the class Lip γ as stated.

8. We have also to show that the polynomials $L_n(f_2, A)$ are unbounded in $[-1, +1]$. Let $s \geqq 2$ and write

$$f_2(x) = F_{s-1}(x) + \frac{1}{r_s^{\beta-2\varepsilon}} \varphi_{r_s}(x) + \Phi_s(x) \tag{8. 1}$$

with

$$\Phi_s(x) = \sum_{j=s+1}^{\infty} \frac{1}{r_j^{\beta-2\varepsilon}} \varphi_{r_j}(x).$$

(7. 7) gives for $-1 \leqq x \leqq +1$

$$|L_{r_s}(F_{s-1}, A)| \leqq C_{s-1}. \tag{8. 2}$$

From (7. 1) and (4. 3) we have

$$r_s^{-\beta+2\varepsilon} |L_{r_s}(\varphi_{r_s}, A)|_{x=\zeta_{r_s}} > c_9(\varepsilon) r_s^\varepsilon. \tag{8. 3}$$

Finally, (1. 8) gives for $-1 \leqq x \leqq +1$

$$|L_{r_s}(\Phi_s, A)| \leqq M_{r_s}(A) \max_{-1 \leqq x \leqq +1} |\Phi_s(x)|;$$

hence from (4. 2) and (7. 3), by (7. 8),

$$|L_{r_s}(\Phi_s, A)| \leqq c_{10}(\varepsilon) r_s^{\beta+\varepsilon} \sum_{j=s+1}^{\infty} \frac{1}{r_j^{\beta-2\varepsilon}} < c_{11}(\varepsilon, \beta) r_s^{-\beta+2\varepsilon}.$$

From this, (8. 2) and (8. 3) we obtained

$$|L_{r_s}(f_2, A)|_{x=\zeta_{r_s}} > c_9(\varepsilon) r_s^\varepsilon - C_{s-1} - c_{11}(\varepsilon, \beta) r_{s+1}^{-\beta+2\varepsilon}$$

what proves the unboundedness, using also (7. 9).

9. Next we turn to the proof of the assertion b) in **5.** This is based on an idea of S. BERNSTEIN adapted to interpolation by L. FEJÉR;[5] as told, we only reproduce it for the sake of completeness. Let $P_{n-1}(x)$ be the best-approximating polynomial of $(n-1)^{\text{th}}$ degree of $f(x)$ in $[-1, +1]$ in CHEBYSEV's sense. If $f \in \text{Lip}\,\gamma$, then according to S. BERNSTEIN we have in $[-1, +1]$

$$|f(x) - P_{n-1}(x)| \leqq c_{12} n^{-\gamma}. \tag{9. 1}$$

Owing to (1. 7) we have

$$L_n(P_{n-1}, A) \equiv P_{n-1}(x),$$

i. e.

$$L_n(f, A) - f(x) = L_n(f - P_{n-1}, A) + (P_{n-1}(x) - f(x))$$

and thus

$$|L_n(f, A)-f| \leqq c_{12} n^{-\gamma} + |L_n(f-P_{n-1}, A)|. \tag{9.2}$$

But using (1. 3) and (9. 1) we obtain

$$|L_n(f-P_{n-1}, A)| \leqq c_{12} n^{-\gamma} \sum_{\nu=1}^{n} |l_{\nu n}(x, A)|. \tag{9.3}$$

Choosing in (4. 2) ε so small that $\beta+\varepsilon < \gamma-\varepsilon$ and fixing it, we obtain

$$\sum_{\nu=1}^{n} |l_{\nu n}(x, A)| \leqq c_{13}(\varepsilon) n^{\beta+\varepsilon} < c_{13}(\varepsilon) n^{\gamma-\varepsilon}$$

and from (9. 3)

$$|L_n(f-P_{n-1}, A)| \leqq c_{14}(\varepsilon) n^{-\varepsilon}.$$

This and (9. 2) prove already the assertion b).

10. Next we turn to the proof of the assertion c) in **5.** We shall show that the matrix whose n^{th} row consists of

$$x_{1n} = \cos\left(\frac{3\pi}{2n} - \frac{1}{n^{1+\beta}}\right), \quad x_{\nu n} = \cos\frac{2\nu-1}{2n}\pi \qquad (\nu = 2, 3, \ldots, n) \tag{10.1}$$

belongs to $A(\beta)$ and fulfils the requirements for A_0 of the assertion c). In what follows, we shall speak about one line of the matrix so that instead of $x_{\nu n}$ and $l_{\nu n}(x, A_0)$ we may write x_ν and $l_\nu(x, A_0)$. We have obviously

$$x_1 - x_2 = 2\sin\frac{1}{2n^{1+\beta}} \sin\left(\frac{3\pi}{2n} - \frac{1}{2n^{1+\beta}}\right) = \frac{3\pi}{2} n^{-\beta-2} + O(n^{-2-2\beta}), \tag{10.2}$$

i. e. for $n > n_0$

$$x_1 - x_2 < 2\pi n^{-\beta-2}. \tag{10.3}$$

Hence

$$\frac{n^{\beta+2}}{2\pi} < \frac{1}{x_1-x_2} = \frac{l_1(x_1, A_0) - l_1(x_2, A_0)}{x_1 - x_2} = \left| \frac{l_1(x_1, A_0) - l_1(x_2, A_0)}{x_1 - x_2} \right| \leqq$$

$$\leqq \max_{-1\leqq x\leqq +1} \left| \frac{d}{dx} l_1(x, A_0) \right| < n^2 \max_{-1\leqq x\leqq 1} |l_1(x, A_0)| \leqq n^2 M_n(A_0),$$

using again MARKOV's theorem. Hence

$$M_n(A_0) \geqq \frac{1}{2\pi} n^{\beta}. \tag{10.4}$$

If we show that

$$M_n(A_0) \leqq c_{15} n^{\beta}, \tag{10.5}$$

then A_0 belongs indeed to $A(\beta)$.

11. For the proof of (10. 5) we need some lemmas. We shall denote by T the matrix the n^{th} row of which consists of the numbers

$$x_\nu^* = \cos \frac{2\nu-1}{2n}\pi \qquad (\nu = 1, 2, \ldots, n) \tag{11. 1}$$

and by $l_\nu(x, T)$ the fundamental functions belonging to T. Then we need the

LEMMA II. *We have for* $-1 \leqq x \leqq +1$

$$\sum_{\nu=1}^{n} |l_\nu(x, T)| \leqq c_{16} \log n.$$

This lemma is well known.[9] We need further the

LEMMA III. *If* $n \geqq 4$, *we have for* $-1 \leqq x \leqq +1$

$$\sum_{\nu=3}^{n} |l_\nu(x, A_0)| \leqq c_{17} \log n.$$

PROOF. For $\nu \geqq 2$ we have

$$\frac{l_\nu(x, A_0)}{l_\nu(x, T)} = \frac{x-x_1}{x_\nu-x_1} \cdot \frac{x_\nu-x_1^*}{x-x_1^*} = \left(1-\frac{x_1^*-x_1}{x_1^*-x}\right)\left(1+\frac{x_1^*-x_1}{x_1-x_\nu}\right). \tag{11. 2}$$

Since for $\nu \geqq 3$, $n \geqq 3$

$$0 < 1 + \frac{x_1^*-x_1}{x_1-x_\nu} < 1 + \frac{\cos\frac{\pi}{2n} - \cos\frac{3\pi}{2n}}{\cos\frac{3\pi}{2n} - \cos\frac{5\pi}{2n}} = 1 + \frac{1}{2\cos\frac{\pi}{n}} \leqq 1 + \frac{1}{\sqrt{2}},$$

we have

$$\left|\frac{l_\nu(x, A_0)}{l_\nu(x, T)}\right| \leqq \left(1+\frac{1}{\sqrt{2}}\right)\left|1-\frac{x_1^*-x_1}{x_1^*-x}\right|. \tag{11. 3}$$

If $-1 \leqq x \leqq x_1$, then owing to

$$0 < 1 - \frac{x_1^*-x_1}{x_1^*-x} < 1$$

we have from (11. 3) for $-1 \leqq x \leqq x_1$

$$|l_\nu(x, A_0)| \leqq 2|l_\nu(x, T)|,$$

i. e. summing for $\nu = 3, 4, \ldots, n$ and using Lemma II we obtain

$$\sum_{\nu=3}^{n} |l_\nu(x, A_0)| \leqq 2c_{17} \log n, \tag{11. 4}$$

i. e. Lemma III is proved for $-1 \leqq x \leqq x_1$. To prove it also for $x_1 \leqq x \leqq 1$ we write (11. 3) in the form

$$|l_\nu(x, A_0)| < 2\left|\frac{x-x_1}{x-x_1^*}\right| |l_\nu(x, T)| = 2|x-x_1|\left|\frac{l_\nu(x, T)}{x-x_1^*}\right|.$$

[9] See S. N. BERNSTEIN, Quelques remarques sur l'interpolation, *Math. Ann.*, **79** (1918), pp. 1—12.

Let us observe that for $\nu \geqq 3$ and $x_1 \leqq x \leqq x_1^*$ we have

(11.5) $$\operatorname{sign} l_\nu(x, T) = (-1)^\nu$$

and for $x_1^* \leqq x \leqq 1$

(11.6) $$\operatorname{sign} l_\nu(x, T) = (-1)^{\nu+1}.$$

Hence for $x_1 \leqq x \leqq x_1^*$, $\nu \geqq 3$

$$|l_\nu(x, A_0)| < 2|x-x_1| \frac{(-1)^\nu l_\nu(x, T)}{|x-x_1^*|},$$

i. e.

(11.7) $$\sum_{\nu=3}^{n} |l_\nu(x, A_0)| < 2|x-x_1| \left| \frac{\sum_{\nu=3}^{n} (-1)^\nu l_\nu(x, T)}{x-x_1^*} \right|.$$

A similar reasoning shows, (11.7) holds also for $x_1^* \leqq x \leqq 1$, i. e. (11.7) holds for $x_1 \leqq x \leqq 1$. Applying the mean-value theorem to the expression

$$\frac{\sum_{\nu=3}^{n} (-1)^\nu l_\nu(x, T)}{x-x_1^*}$$

and also

$$|x-x_1| < 1-\cos\frac{3\pi}{2n} < \frac{10^2}{n^2},$$

we obtain from (11.7) for $x_1 \leqq x \leqq 1$

$$\sum_{\nu=3}^{n} |l_\nu(x, A_0)| < 2\frac{10^2}{n^2} \max_{-1 \leqq x \leqq +1} \left| \frac{d}{dx} \sum_{\nu=3}^{n} (-1)^\nu l_\nu(x, T) \right|.$$

Using again MARKOV's theorem the right side is

$$< \frac{200}{n^2}(n-1)^2 \max_{-1 \leqq x \leqq +1} \left| \sum_{\nu=3}^{n} (-1)^\nu l_\nu(x, T) \right| < 200 \max_{-1 \leqq x \leqq 1} \sum_{\nu=1}^{n} |l_\nu(x, T)| \leqq$$
$$\leqq 200\, c_{16} \log n$$

by Lemma II. By this and (11.4) Lemma III is proved.

12. It follows from Lemma III that (10.5) will be proved if we succeed in showing that for $-1 \leqq x \leqq +1$ we have

(12.1) $$|l_1(x, A_0)| \leqq c_{18}\, n^\beta, \quad |l_2(x, A_0)| \leqq c_{18}\, n^\beta.$$

This will follow as a byproduct from the following Lemma which we shall need in **13.**

LEMMA IV. *For $-1 \leqq x \leqq +1$ we have*

$$|l_1(x, A_0) + l_2(x, A_0)| \leqq c_{19}.$$

PROOF. From (1.4) we have

(12.2) $$\frac{l_1(x, A_0)}{l_1(x, T)} = \prod_{j=2}^{n} \frac{x_1^*-x_j}{x_1-x_j} = \frac{T_n'(x_1^*)}{T_n'(x_2)} \left\{ (x_1-x_1^*) \frac{T_n'(x_2)}{T_n(x_1)} \right\}.$$

The factor in the bracket is

$$\left\{\cos\frac{\pi}{2n}-\cos\left(\frac{3\pi}{2n}-\frac{1}{n^{1+\beta}}\right)\right\}\left\{\frac{n}{\sin\frac{3\pi}{2n}}\cdot\frac{1}{\cos\left(\frac{3\pi}{2}-n^{-\beta}\right)}\right\}=-2\frac{\pi}{2n}\left\{1+O\left(\frac{1}{n^{\beta}}\right)\right\}\cdot$$

(12. 3)

$$\cdot\frac{\pi}{n}\left\{1+O\left(\frac{1}{n^{\beta}}\right)\right\}\left\{\frac{2}{3\pi}n^{2}\left\{1+O\left(\frac{1}{n^{2}}\right)\right\}\right\}n^{\beta}\left\{1+O\left(\frac{1}{n^{2\beta}}\right)\right\}=-\frac{2\pi}{3}n^{\beta}+O(1).$$[10]

Taking in account that

$$\frac{x_2-x_1^*}{x_2-x_1}=\frac{\cos\frac{3\pi}{2n}-\cos\frac{\pi}{2n}}{\cos\frac{3\pi}{2n}-\cos\left(\frac{3\pi}{2n}-n^{-1-\beta}\right)}=\frac{\pi^2 n^{-2}\left\{1+O\left(\frac{1}{n^2}\right)\right\}}{\frac{3\pi}{2}n^{-2-\beta}\{1+O(n^{-\beta})\}}=$$

$$=\frac{2\pi}{3}n^{\beta}\{1+O(n^{-\beta})\}=\frac{2\pi}{3}n^{\beta}+O(1),$$

this and (12. 3) means that

$$(x_1-x_1^*)\frac{T_n'(x_2)}{T_n(x_1)}=-\frac{x_2-x_1^*}{x_2-x_1}+O(1)$$

and from (12. 2), using the explicit form of $l_1(x, T)$,

$$l_1(x, A_0)=\frac{T_n(x)}{x-x_1^*}\cdot\frac{1}{T_n'(x_2)}\left\{-\frac{x_2-x_1^*}{x_2-x_1}+O(1)\right\}=$$

(12. 4)

$$=-\frac{x_2-x_1^*}{x_2-x_1}\cdot\frac{T_n(x)}{(x-x_1^*)T_n'(x_2)}+O(1)|l_1(x, T)|\left|\frac{T_n'(x_1^*)}{T_n'(x_2)}\right|.$$

As well known,[5] we have for $-1\leqq x\leqq +1$

(12. 5) $$|l_\nu(x, T)|\leqq\sqrt{2}\qquad(\nu=1, \ldots, n),$$

i. e. the second term on the right of (12. 4) is

$$O(1)\left|\frac{T_n'(x_1^*)}{T_n'(x_2)}\right|=O(1)\frac{\sin\frac{3\pi}{2n}}{\sin\frac{\pi}{2n}}=O(1);$$

thus from (12. 4)

(12. 6) $$l_1(x, A_0)=-\frac{x_2-x_1^*}{x_2-x_1}\cdot\frac{T_n(x)}{x-x_1^*}\cdot\frac{1}{T_n'(x_2)}+O(1).$$

Further we have

$$\frac{l_2(x, A_0)}{l_2(x, T)}=\frac{x-x_1}{x-x_1^*}\cdot\frac{x_2-x_1^*}{x_2-x_1},$$

[10] The O-sign refers to $n\to\infty$ but always uniformly for $-1\leqq x\leqq +1$.

i. e. using the explicit form of $l_2(x, T)$

$$l_2(x, A_0)=\frac{x_2-x_1^*}{x_2-x_1}\cdot\frac{x-x_1}{x-x_1^*}\cdot\frac{T_n(x)}{x-x_2}\cdot\frac{1}{T_n'(x_2)}.$$

From this and (12. 6)

$$|l_1(x, A_0)+l_2(x, A_0)|=O(1)+\left|\frac{x_2-x_1^*}{x_2-x_1}\right|\left|\frac{T_n(x)}{T_n'(x_2)}\right|\left|-\frac{1}{x-x_1^*}+\frac{x-x_1}{x-x_1^*}\cdot\frac{1}{x-x_2}\right|=$$

$$(12.7)\qquad =O(1)+\left|\frac{x_2-x_1^*}{x_2-x_1}\right|\left|\frac{T_n(x)}{T_n'(x_2)\,(x-x_1^*)}\right|\left|\frac{x_1-x_2}{x-x_2}\right|=$$

$$=O(1)+|x_2-x_1^*|\frac{1}{|T_n'(x_2)|}\left|\frac{T_n(x)}{(x-x_1^*)(x-x_2)}\right|=O(1)+O\left(\frac{1}{n^4}\right)\left|\frac{T_n(x)}{(x-x_1^*)(x-x_2)}\right|.$$

Since we have for $-1\leqq x\leqq\cos\frac{2\pi}{n}$

$$\left|\frac{T_n(x)}{(x-x_1^*)(x-x_2)}\right|\leqq\frac{1}{\left(\cos\frac{\pi}{2n}-\cos\frac{2\pi}{n}\right)\left(\cos\frac{3\pi}{2n}-\cos\frac{2\pi}{n}\right)}=O(n^4)$$

for $\cos\frac{2\pi}{n}\leqq x\leqq\cos\frac{\pi}{n}$,

$$\left|\frac{T_n(x)}{(x-x_1^*)(x-x_2)}\right|\leqq\frac{1}{\cos\frac{\pi}{2n}-\cos\frac{\pi}{n}}\cdot\max_{-1\leqq x\leqq+1}\left|\frac{T_n(x)}{x-x_2}\right|=O(n^4)$$

and similarly for $\cos\frac{\pi}{n}\leqq x\leqq+1$, Lemma IV follows from (12. 7) indeed.

In this proof is at the same time a proof for (12. 1) contained. We may write namely (12. 6) in the form

$$l_1(x, A_0)=-\frac{x_2-x_1^*}{x_2-x_1}\,\frac{T_n'(x_1^*)}{T_n'(x_2)}\,l_1(x, T)+O(1)=\frac{x_2-x_1^*}{x_2-x_1}\cdot\frac{\sin\frac{3\pi}{2n}}{\sin\frac{\pi}{2n}}\,l_1(x, T)+O(1).$$

Taking in account (12. 5), further

$$\frac{\sin\frac{3\pi}{2n}}{\sin\frac{\pi}{2n}}=O(1)$$

and

$$\frac{x_2-x_1^*}{x_2-x_1}=O(n^\beta),$$

the relation $l_1(x, A_0)=O(n^\beta)$ follows indeed. For $l_2(x, A_0)$ the assertion follows using Lemma IV. Thus our matrix belongs to $A(\beta)$ indeed.

13. In order to finish the proof of the assertion c) in **5** we have to show that the polynomials $L_n(f, A_0)$ converge uniformly in $[-1, +1]$ to $f(x)$ whenever $f\in \text{Lip}\,\gamma$, $\gamma > \frac{\beta}{\beta+2}$. Let η be so small that

$$\gamma > \frac{\beta}{\beta+2} + \eta \tag{13.1}$$

and fixed. Let further

$$\left[\frac{n}{\log n}\right] = m \tag{13.2}$$

and $P_m(x)$ should stand for the best-approximating polynomial of $f(x)$ in $[-1, +1]$ in CHEBYSEV's sense. Then we have again

$$L_n(P_m, A_0) \equiv P_m(x)$$

and proceeding exactly as in **9** we obtain, using the abbreviation

$$f(x_\nu) - P_m(x_\nu) = y_\nu, \tag{13.3}$$

that

$$L_n(f, A_0) - f = (P_m - f) + \sum_1^n y_\nu l_\nu(x, A_0). \tag{13.4}$$

Using again S. BERNSTEIN's theorem mentioned in **9** we have

$$|f - P_m| \leqq c_{20} m^{-\gamma} < c_{21} n^{-\gamma} \log^\gamma n, \tag{13.5}$$

i. e. from (13. 4) we get

$$|L_n(f, A_0) - f| \leqq c_{21} n^{-\gamma} \log^\gamma n + |y_1 l_1(x, A_0) + y_2 l_2(x, A_0)| + \sum_{\nu=3}^{n} |y_\nu|\, |l_\nu(x, A_0)|. \tag{13.6}$$

Taking in account (13. 5) and Lemma III the last sum is $O(m^{-\gamma} \log n) = O(n^{-\gamma} \log^{\gamma+1} n)$; hence from (13. 6) we have

$$\begin{aligned} &|L_n(f, A_0) - f| \leqq \\ &\leqq c_{21} n^{-\gamma} \log^{\gamma+1} n + |y_1|\, |l_1(x, A_0) + l_2(x, A_0)| + |y_1 - y_2|\, |l_2(x, A_0)|. \end{aligned} \tag{13.7}$$

Taking in account Lemma IV and (12. 1), (13. 7) gives

$$|L_n(f, A_0) - f| \leqq c_{22} \{ n^{-\gamma} \log^{\gamma+1} n + n^\beta |y_1 - y_2| \}. \tag{13.8}$$

For the estimation of $|y_1 - y_2|$ we write

$$|y_1 - y_2| \leqq |f(x_1) - f(x_2)| + |P_m(x_1) - P_m(x_2)|. \tag{13.9}$$

Since $f \in \text{Lip}\,\gamma$ and

$$|x_1 - x_2| = \cos\left(\frac{3\pi}{2n} - \frac{1}{n^{1+\beta}}\right) - \cos\frac{3\pi}{2n} < \frac{100}{n^{2+\beta}}, \tag{13.10}$$

we have, using (13. 1),

$$|f(x_1) - f(x_2)| \leqq c_{23} \left(\frac{100}{n^{2+\beta}}\right)^\gamma \leqq c_{24} n^{-(2+\beta)\left(\frac{\beta}{\beta+2}+\eta\right)} = c_{24} n^{-\beta} n^{-(\beta+2)\eta}. \tag{13.11}$$

Further, using again MARKOV's theorem and (13. 10),

$$|P_m(x_1)-P_m(x_2)| \leqq \frac{100}{n^{2+\beta}} \max_{-1\leqq x\leqq +1} |P'_m(x)| \leqq \frac{100\, m^2}{n^{2+\beta}} \max_{-1\leqq x\leqq +1} |P_m(x)| <$$
$$< \frac{100}{n^{\beta}\log^2 n}\,(1+\max_{-1\leqq x\leqq +1}|f(x)|).$$

From this, (13. 8) and (13. 11) it follows that the polynomials $L_n(f, A_0)$ converge in $[-1, +1]$ uniformly to $f(x)$, i. e. the proof of the assertion c) is finished.

14. Finally we shall prove the assertion d) in **5.** Since the proof follows the line of that of a) we shall not go into all details. It is obviously sufficient to define A_1 only for even $n=2k$. With our β we define

$$\vartheta = \frac{\beta^2}{2}\left(\frac{\log k}{k}\right)^2 \tag{14. 1}$$

and the $2k^{th}$ line of our matrix A_1 should be given by the zeros $\zeta_1, \ldots, \zeta_{2k}$ of

$$\omega_{2k}(x, A_1) \equiv \omega_{2k}(x) \equiv T_k(1+\vartheta-(2+\vartheta)x^2) = 0 \quad (T_k(\cos\vartheta) = \cos k\vartheta), \tag{14. 2}$$

i. e. with $x_\nu = \cos\dfrac{2\nu-1}{2k}\pi$

$$\begin{aligned} \zeta_\nu &= \sqrt{\frac{1+\vartheta-x_\nu}{2+\vartheta}} \qquad (\nu=1,2,\ldots,k),\\ \zeta_{-\nu} &= -\sqrt{\frac{1+\vartheta-x_\nu}{2+\vartheta}} \qquad (\nu=1,2,\ldots,k). \end{aligned} \tag{14. 3}$$

We have obviously

$$\left(\frac{\sqrt{\vartheta}}{2}<\right)\ \sqrt{\frac{\vartheta}{2+\vartheta}} < \zeta_1 < \zeta_2 < \cdots < \zeta_k < 1. \tag{14. 4}$$

Since for $1 \leqq |\nu| \leqq k$

$$|\omega'_{2k}(\zeta_\nu)| = \left|\frac{d\,T_k(u)}{du}\right|_{u=1+\vartheta-(2+\vartheta)\zeta_\nu^2} \cdot 2(2+\vartheta)\,|\zeta_\nu| = \frac{k}{\sqrt{1-x_{|\nu|}^2}}\,2(2+\vartheta)|\zeta_\nu|,$$

we get — denoting the fundamental functions by $l_{\pm\nu}(x, A_1)$ $(\nu=1,2,\ldots,k)$ —

$$\lambda_{2k}(x, A_1) = \sum_{1\leqq|\nu|\leqq k} |l_\nu(x, A_1)| = \frac{1}{2(2+\vartheta)k} \sum_{1\leqq|\nu|\leqq k} \frac{\sqrt{1-x_{|\nu|}^2}}{|\zeta_\nu|}\,\frac{|\omega_{2k}(x)|}{|x-\zeta_\nu|}. \tag{14. 5}$$

15. First we are going to show (by rough estimations)

$$\sum_{|\nu|=1}^{k} |l_\nu(x, A_1)| < c_{25}(\beta)\,k^{\beta}\log^2 k. \tag{15. 1}$$

We may obviously suppose $0 \leqq x \leqq 1$. Since for all of our ν's we have

$$\frac{\sqrt{1-x_\nu}}{|\zeta_\nu|} = \sqrt{2+\vartheta}\,\sqrt{\frac{1-x_\nu}{1+\vartheta-x_\nu}} < 2$$

and

$$\sqrt{1+x_\nu} < 10\,\frac{k-\nu+1}{k},$$

we have

(15. 2)
$$M_{2k}(A_1) < \frac{4}{k^2} \sum_{1 \leqq |\nu| \leqq k} \frac{k-|\nu|+1}{|x-\zeta_\nu|} |\omega_{2k}(x)|.$$

What can be said on $(\zeta_{\nu+1}-\zeta_\nu)$? For

$$k \geqq |\nu| \geqq \frac{k}{2}$$

we have from (14. 3)

(15. 3)
$$|\zeta_{\nu+1}-\zeta_\nu| = \frac{1}{\sqrt{2+\vartheta}} \cdot \frac{x_\nu - x_{\nu-1}}{\sqrt{1+\vartheta-x_\nu}+\sqrt{1+\vartheta-x_{\nu+1}}} > \frac{k-|\nu|+1}{20k^2}.$$

For

$$\frac{k}{2} > |\nu| \geqq 1$$

we have from (15. 3)

(15. 4)
$$|\zeta_{\nu+1}-\zeta_\nu| > c_{26}(\beta) \frac{\frac{\nu}{k^2}}{\sqrt{\frac{\nu^2}{k^2}+\frac{\log^2 k}{k^2}}} > \frac{c_{27}(\beta)}{k \log k}.$$

Let first be

$$0 \leqq x \leqq \zeta_1.$$

Then we have from (15. 2)

(15. 5)
$$\lambda_{2k}(x, A_1) < \frac{8}{k^2} |\omega_{2k}(x)| \sum_{\nu=1}^{k} \frac{k-\nu+1}{\zeta_\nu - x}.$$

Owing to (15. 4) we have

$$\sum_{\nu=1}^{k} \frac{k-\nu+1}{\zeta_\nu - x} \leqq \frac{k}{\zeta_1 - x} + k \sum_{2 \leqq \nu < \frac{k}{2}} \frac{1}{\zeta_\nu - x} + k \sum_{\frac{k}{2} \leqq \nu \leqq k} \frac{1}{\zeta_\nu - x} <$$

$$< \frac{k}{\zeta_1 - x} + c_{28}k^2 + k \sum_{2 \leqq \nu < \frac{k}{2}} \frac{1}{(\nu-1)c_{27}(\beta)\frac{1}{k \log k}} < \frac{k}{\zeta_1 - x} + c_{29}(\beta) k^2 \log^2 k,$$

i. e. from (15. 5) for such values x

(15. 6)
$$\lambda_{2k}(x, A_1) < c_{30}(\beta) \left\{ \frac{1}{k} \left| \frac{\omega_{2k}(x)}{x-\zeta_1} \right| + |\omega_{2k}(x)| \log^2 k \right\}.$$

But for $0 \leqq x \leqq \xi_1$ owing to (14. 1) we have for $k > c_{31}$

(15. 7)
$$|\omega_{2k}(x)| \leqq T_k(1+\vartheta) < \frac{1}{2}\{1+(1+\vartheta+\sqrt{(1+\vartheta)^2-1})^k\} <$$
$$< (1+\vartheta+\sqrt{2\vartheta+\vartheta^2})^k < e^{k\left(\vartheta+\sqrt{2\vartheta}\sqrt{1+\frac{\vartheta}{2}}\right)} < 2e^{k\sqrt{2\vartheta}} = 2k^\beta.$$

Further, taking in account that for $0 \leqq x \leqq \zeta_1$

$$\left|\frac{d\omega_{2k}(x)}{dx}\right| = 2(2+\vartheta)|x|\left|\frac{dT_k(u)}{du}\right|_{u=1+\vartheta-(2+\vartheta)x^2} =$$

$$= k(2+\vartheta)|x|\left\{(u+\sqrt{u^2-1})^{k-1}\left(1+\frac{u}{\sqrt{u^2-1}}\right)+\right.$$

$$\left.+(u-\sqrt{u^2-1})^{k-1}\left(1-\frac{u}{\sqrt{u^2-1}}\right)\right\}_{u=1+\vartheta-(2+\vartheta)x^2} =$$

$$= k(2+\vartheta)|x|\left\{\frac{(u+\sqrt{u^2-1})^k-(u-\sqrt{u^2-1})^k}{\sqrt{u^2-1}}\right\}_{u=1+\vartheta-(2+\vartheta)x^2} \leqq$$

$$\leqq k(2+\vartheta)\zeta_1\left\{\quad\right\}_{u=1+\vartheta} < \frac{k(2+\vartheta)}{\sqrt{2\vartheta}}\sqrt{\frac{2\sin^2\frac{\pi}{4k}+\vartheta}{2}}\cdot 2k^\beta < 10k^{\beta+1},$$

we have for $0 \leqq x \leqq \zeta_1$

$$(15.8) \qquad \frac{1}{k}\left|\frac{\omega_{2k}(x)}{x-\zeta_1}\right| \leqq \frac{1}{k}\max_{0\leqq x\leqq\zeta_1}|\omega_{2k}'(x)| < 10k^\beta.$$

(15.6), (15.7) and (15.8) give at once (15.1) for $0 \leqq x \leqq \xi_1$. For $\zeta_1 \leqq x \leqq 1$ the proof of (15.1) runs on similar lines and also a much better estimation of $\lambda_{2k}(x, A_1)$ could have been given; we omit the details. Thus (15.1) is proved.

16. Next we show that

$$(16.1) \qquad \lambda_{2k}(0, A_1) > c_{32}(\beta)\, k^\beta.$$

Since for $k > c_{33}(\beta)$

$$|\omega_{2k}(0)| = T_k(1+\vartheta) > \frac{1}{2}\,(1+\vartheta+\sqrt{2\vartheta+\vartheta^2})^k > \frac{1}{4}k^\beta,$$

we obtain from (14.5)

$$\lambda_{2k}(0, A_1) > \frac{k^{\beta-1}}{4(2+\vartheta)}\sum_{\nu=1}^{k}\frac{\sqrt{1-x_\nu^2}}{\zeta_\nu^2} > \frac{k^{\beta-1}}{4}\sum_{\frac{k}{10}\leqq\nu\leqq\frac{9}{10}k}\frac{\sqrt{1-x_\nu^2}}{1+\vartheta-x_\nu} > c_{33}(\beta)k^\beta.$$

This and (15.1) show that our matrix A_1 belongs indeed to $A(\beta)$.

17. What we actually proved, is a little more and this is what we shall need. We showed

$$\sum_{1\leqq|\nu|\leqq\frac{1}{2}k}|l_\nu(0, A_1)| > c_{34}(\beta)k^\beta,$$

i. e.

$$(17.1) \qquad \sum_{1\leqq|\nu|\leqq\frac{1}{2}k}|l_\nu(\Theta_{2k}, A_1)| \equiv \max_{-\frac{1}{4}\leqq x\leqq\frac{1}{4}}\sum_{1\leqq|\nu|\leqq\frac{1}{2}k}|l_\nu(x, A_1)| \geqq c_{35}(\beta)k^\beta.$$

Having this, the construction of $f_3(x)$ runs as follows. Introducing the notation

(17.2) $$y_{\nu,2k}=\begin{cases}\operatorname{sign} l_{\nu,2k}(\Theta_{2k},A_1) & \text{for } 1\leqq|\nu|\leqq\frac{k}{2},\\ 0 & \text{for } \frac{k}{2}<|\nu|\leqq k,\end{cases}$$

we consider the function $\psi_{2k}(x)$ defined by the broken line with the vertices at the points

$$P_\nu^*\equiv(\zeta_{\nu,2k},\ y_{\nu,2k})\qquad(1\leqq|\nu|\leqq k);$$

we use also here the convention of **7** to complete the graph of $\psi_{2k}(x)$ in the whole interval $[-1,+1]$ by affixing two horizontals at the ends. We have obviously

(17.3) $$L_{2k}(\psi_{2k},A_1)_{x=\Theta_{2k}}=\sum_{1\leqq|\nu|\leqq\frac{k}{2}}|l_{\nu,2k}\,\Theta_{2k},A_1)|>c_{35}(\beta)k^{\beta}.$$

According to (15.4) the slopes of $\psi_{2k}(x)$ are absolutely

$$\leqq\frac{1}{c_{27}(\beta)}k\log k,$$

for $-\frac{51}{100}\leqq x\leqq\frac{51}{100}$ and $k\geqq c_{36}(\beta)$ we have

(17.4) $$|\psi_{2k}(x)|\leqq 1,$$

for $\frac{51}{100}\leqq|x|\leqq 1$ (x real)

$$\psi_{2k}(x)=0$$

and for $-1\leqq x<x+h\leqq+1$

(17.5) $$|\psi_{2k}(x+h)-\psi_{2k}(x)|\leqq c_{37}(\beta)hk\log k.$$

Now our function $f_3(x)$ will be of the form

$$f_3(x)=\sum_{\nu=1}^{\infty}\frac{1}{r_\nu^{\beta-\varepsilon}}\psi_{2r_\nu}(x)$$

with sufficiently quickly increasing r_ν-indices, as in **7.** The proof of unboundedness of $L_n(f_3,A_1)$ needs only slight changes compared to that of **8,** so we can omit the details. As to the Lipschitz exponent of $f_3(x)$ we have (p to be disposed later)

$$|f_3(x+h)-f_3(x)|\leqq\sum_{\nu\leqq p}\frac{1}{r_\nu^{\beta-\varepsilon}}|\psi_{2r_\nu}(x+h)-\psi_{2r_\nu}(x)|+$$

$$+\sum_{\nu\geqq p+1}\frac{1}{r_\nu^{\beta-\varepsilon}}(|\psi_{2r_\nu}(x+h)|+|\psi_{2r_\nu}(x)|)$$

or owing to (17. 5) and (17.4) "essentially" (owing to the quick increase of the r_ν's)

$$|f_3(x+h)-f_3(x)| \leqq c_{38}(\beta) \left\{ h r_p^{1-\beta+2\varepsilon} + \frac{1}{r_{p+1}^{\beta-\varepsilon}} \right\}.$$

Choosing p so that

$$r_p \leqq \frac{1}{h} < r_{p+1},$$

everything follows since ε is arbitrary small. Hence assertion d) is also proved.

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РОЛЬ ФУНКЦИЙ ЛЕБЕГА В ТЕОРИИ ИНТЕРПОЛЯЦИИ ЛАГРАНЖА

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Главным результатом настоящей работы является доказательство того, что в теории сходимости и расходимости интерполяции Лагранжа следует различать между „грубой" и „тонкой" теориями. „Грубой" является часть теории, зависящая только от быстроты возрастания „лебеговых констант"

$$M_n(A) = \max_{-1 \leqq x \leqq +1} \sum_{\nu=1}^{n} |l_\nu(x)|,$$

где A означает основную матрицу интерполяции, „тонкие" же другие результаты. Главная задача состояла в отделении этих двух теорий. Из различных возможных точек зрения мы здесь остановимся на той, при которой матрицы A для произвольно малого положительного числа ε, и для фиксированного $0 < \beta < 1$ подчинены условиям

(1) $$\overline{\lim_{n\to\infty}} \frac{M_n(A)}{n^{\beta+\varepsilon}} < c_1(\varepsilon) \, (< \infty),$$

(2) $$\underline{\lim_{n\to\infty}} \frac{M_n(A)}{n^{\beta-\varepsilon}} > c_2(\varepsilon) \, (> 0),$$

а следует определять те классы функций Lip α в $[-1, +1]$, для которых при любой матрице удовлетворяющей условиям (1) и (2) интерполяция пригодна, и те классы, для которых интерполяция при любых таких матрицах непригодна. Оказывалось, что если исходить из классы матриц удовлетворяющих (1) и (2), то „тонкая" теория есть теория сходимости-расходимости тех класс Lip α, для которых

$$\frac{\beta}{\beta+2} < \alpha < \beta.$$

Более точно, имеет место следующее:

а) Если для A справедливы (1) и (2) и имеет место $\alpha < \frac{\beta}{\beta+2}$, то существует функция $f_1 \in$ Lip α, для которой взятые по A интерполяционные полиномы Лагранжа $L_n(f_1, A)$ неограничены на $[-1, +1]$. (Интерполяция непригодна.)

b) Если A удовлетворяет условиям (1) и (2) и имеет место $\alpha > \beta$, то для любой $f \in \mathrm{Lip}\,\alpha$ интерполяционные полиномы Лагранжа $L_n(f, A)$ соответствующие A стремятся к f равномерно на $[-1, +1]$. (Интерполяция пригодна.)

c) Если $\alpha > \frac{\beta}{\beta+2}$, то уже существует матрица A_0, удовлетворяющая (1) и (2), для которой интерполяционные полиномы $L_n(f, A_0)$ равномерно сходятся к f, если только $f \in \mathrm{Lip}\,\alpha$.

d) Если $\alpha < \beta$, то уже существуют матрица A_1 удовлетворяющая (1) и (2), и функция $f_2 \in \mathrm{Lip}\,\alpha$ такие, что интерполяционные полиномы Лагранжа $L_n(f_2, A_1)$ неограничены в интервале $[-1, +1]$.

Аналогичные вопросы возникают также для разложений в ортогональный ряд, для механических квадратур и для других линейных операций.