MATHEMATIKA

A JOURNAL OF PURE AND APPLIED MATHEMATICS

VOL. 3. PART 1. JUNE, 1956. No. 5.

MONOTONICITY OF PARTITION FUNCTIONS

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1. *Introduction.* Let A be an arbitrary set of positive integers (finite or infinite) other than the empty set or the set consisting of the single element unity*. Let $p(n) = p_A(n)$ denote the number of partitions of the integer n into parts taken from the set A, repetitions being allowed. In other words, $p(n)$ is the number of ways n can be expressed in the form $n_1 a_1 + n_2 a_2 + \dots$, where $a_1, a_2, \dots$ are the distinct elements of A and $n_1, n_2, \dots$ are arbitrary non-negative integers. In this paper we shall prove that $p(n)$ is a strictly increasing function of n for sufficiently large n if and only if A has the following property (which we shall subsequently call property P_1): A contains more than one element, and if we remove any single element from A, the remaining elements have greatest common divisor unity.

We shall obtain this result as a special case of the following more general one. Let k be any integer and suppose we define $p^{(k)}(n) = p_A^{(k)}(n)$ by the formal power-series relation

$$f_k(X) = \sum_{n=0}^{\infty} p^{(k)}(n) X^n = (1-X)^k \sum_{n=0}^{\infty} p(n) X^n$$

$$= (1-X)^k \prod_{a \in A} (1-X^a)^{-1}. \tag{1}$$

Thus $p^{(k)}(n)$ is the k-th difference of $p(n)$ if $k > 0$, $p(n)$ itself if $k = 0$, and the $(-k)$-th order summatory function of $p(n)$ if $k < 0$. For $k \geqslant 0$, we shall prove in the sequel that *$p^{(k)}(n)$ is positive for all sufficiently large positive integers n if and only if A has the following property, which we shall call property P_k: There are more than k elements in A, and if we remove an arbitrary subset of k elements from A, the remaining elements have greatest common divisor unity.* When $k = 0$, this reduces to the well-known result ([3], [6]) that $p^{(0)}(n) = p(n)$ is positive for all sufficiently large n if and

* The inclusion of these two trivial cases would complicate the statements and proofs of some of our theorems and so, for the sake of simplicity, we agree to exclude them throughout the paper.

only if the elements of A have greatest common divisor unity. When $k=1$, it is the result stated in the preceding paragraph.

Although we are primarily interested in positive values of k, we shall find it convenient to agree that when k is a negative integer any set A has property P_k. Then the italicized assertion is trivially true for negative k.

Note that if A has property P_k, it must actually contain at least $k+2$ elements. We remark also that A has property P_k if and only if the following assertion holds for *every* prime number p: there are at least $k+1$ elements of A which are not multiples of p. In particular, if A has the property that for every prime number p there are infinitely many elements of A not divisible by p, then A has property P_k for arbitrary k.

The proof that property P_k is a necessary condition for $p^{(k)}(n)$ to be positive for large n is a straightforward argument with power series (§2). The sufficiency is proved in the following stages. First (§3), we prove it for the case in which A is finite by using the partial fraction decomposition of the generating function $f_k(X)$. Second (§4), we prove that in any case $p(n) = o\left(p^{(-1)}(n)\right)$ as n increases. Third (§5), we prove sufficiency for the case where A is infinite by using these two results. Actually, under the assumption of property P_k we shall prove much more than the mere positivity of $p^{(k)}(n)$ for large n and shall include the case of negative k in our theorems for completeness (§6). However, all the arguments used are essentially elementary. We conclude the paper with a discussion (§7) of the relative orders of magnitude (as functions of n) of $p^{(k)}(n)$ and $p^{(k+1)}(n)$.

Our attention was drawn to the problems discussed in this paper by remarks of various authors ([4], [5], [6]) on the usefulness of knowing that $p(n)$ is an increasing function of n for large n. In particular, the application of Ingham's Tauberian theorem for partitions becomes considerably simpler in that situation. By our results this holds if and only if A has property P_1. However, it follows from the corollary after our Theorem 6 (in the case $k=0$) that actually property P_0, namely, that the elements of A have greatest common divisor unity, would be sufficient for the purpose of applying Ingham's theorem. This remark has been made previously by Auluck and Haselgrove in [1].

All our theorems refer to the behaviour of $p^{(k)}(n)$ for sufficiently large n. The behaviour of $p^{(k)}(n)$ for small n can be rather erratic, since it depends on the arithmetic properties of the smaller members of A rather than on the arithmetic properties of A as a whole. In particular, if $k>1$ it is impossible for $p^{(k)}(n)$ to be positive for all non-negative n, since $p^{(k)}(1) \leqslant 1-k$.

For partitions into distinct parts the questions analogous to those discussed in this paper are much more difficult. The reason for this is that, in the case of distinct parts, these questions become trivial for finite sets and it is not possible to use the finite case to attack the general case, as we have done in this paper. However, a rather broad sufficient condition for monotonicity is given by Roth and Szekeres in [**10**].

Throughout this paper small Latin letters denote (rational) integers.

2. *Necessity of property* P_k.

THEOREM 1. *If* $k \geqslant 2$ *and if* $p^{(k)}(n)$ *is non-negative for all sufficiently large* n, *then* A *has property* P_k. *If* $p^{(1)}(n)$ *is non-negative for all sufficiently large* n, *then either* A *has property* P_1 *or* A *contains the element* 1.

Proof. Suppose $k > 0$ and A is such that $p^{(k)}(n)$ is non-negative for all sufficiently large n. Since the empty set and the set consisting of the single element 1 have been excluded from consideration, it follows from (1) that $f_k(X)$ is not a polynomial. Thus $p^{(k)}(n)$ is positive for infinitely many n and so $f_k(X) \to +\infty$ as X approaches 1 from below. Thus A must contain more than k elements, since otherwise $f_k(X)$ would be a rational function which, when expressed in reduced form, has a denominator not divisible by $1-X$. Suppose that $B = \{a_1, a_2, \ldots, a_k\}$ is an arbitrary subset of A having exactly k elements and let d be the greatest common divisor of the elements of $A-B$. Then the left-hand side of the identity

$$\prod_{a \in A-B} (1-X^a)^{-1} = f_k(X) \prod_{m=1}^{k} (1+X+X^2+\ldots+X^{a_m-1}) \tag{2}$$

is expressible as a power-series in X^d. On the right-hand side of (2), the power-series for $f_k(X)$ has non-negative coefficients from some point on and an infinite number of positive coefficients, while the coefficient of X in the expansion of

$$\prod_{m=1}^{k} (1+X+X^2+\ldots+X^{a_m-1})$$

is positive unless $k = 1 = a_1$. Thus, unless $k = 1$ and A contains the element 1, the right-hand side of (2) has infinitely many pairs of consecutive coefficients both of which are positive, so that d must be 1. Hence, unless $k = 1$ and A contains 1, A must have property P_k. Accordingly Theorem 1 is proved.

THEOREM 2. *For arbitrary* k, *if* $p^{(k)}(n)$ *is positive for all sufficiently large* n, *then* A *has property* P_k.

Proof. The theorem is vacuous if $k < 0$ and follows immediately from Theorem 1 either if $k > 1$ or if $k = 1$ and A does not contain the element 1. If $k = 1$ and A contains 1,

$$f_1(X) = \prod_{a \in A,\, a \neq 1} (1-X^a)^{-1}$$

and thus $f_1(X)$ is expressible as a power series in X^d, where d is the greatest common divisor of the elements of A other than 1; hence, if $p^{(1)}(n) > 0$

for all sufficiently large n, $d=1$ and so A has property P_1. If $k=0$,

$$f_0(X)=\prod_{a\in A}(1-X^a)^{-1}$$

and thus $f_0(X)$ is expressible as a power-series in X^d, where d is the greatest common divisor of all the elements of A; hence, if $p^{(0)}(n)$ is positive for all sufficiently large n, $d=1$ and A has property P_0.

3. *Sufficiency of condition P_k when A is finite.*

LEMMA 1. *Suppose A has exactly r elements $a_1, \dots, a_r$ and suppose $k<r$. Then if $n>0$*

$$p^{(k)}(n)=g(n)+O(n^{q-1}),$$

where $g(n)$ is a polynomial in n of degree $r-k-1$ with highest coefficient $[(r-k-1)!\,a_1a_2\dots a_r]^{-1}$, and where q is the largest number of elements of A which have a common divisor greater than 1.

Proof. The proof is based on the methods of Cayley, Glaisher, and Sylvester (*cf.* [2] and [7]). From (1) we see that $1/f_k(X)$ is a polynomial whose factorization into linear factors is

$$\frac{1}{f_k(X)}=(1-X)^{r-k}\prod_{m=1}^{r}\prod_{l=1}^{a_m-1}(1-e^{2\pi il/a_m}X). \tag{3}$$

Thus the terms in $1-X$ in the decomposition of $f_k(X)$ into partial fractions have the form

$$\frac{\alpha_1}{1-X}+\frac{\alpha_2}{(1-X)^2}+\dots+\frac{\alpha_{r-k}}{(1-X)^{r-k}}, \tag{4}$$

where $\alpha_{r-k}=(a_1a_2\dots a_r)^{-1}$. If $g(n)$ is the coefficient of X^n in the power series expansion of (4), we have

$$g(n)=\sum_{h=1}^{r-k}\alpha_h\binom{n+h-1}{h-1},$$

which is a polynomial in n of degree $r-k-1$ and highest coefficient $[(r-k-1)!\,a_1a_2\dots a_r]^{-1}$. If $d>1$ and ζ is a primitive d-th root of unity, the multiplicity of the factor $1-\zeta X$ in the factorization (3) is equal to the number q_d of multiples of d among $a_1, a_2, \dots, a_r$. Since by definition q is the largest value which q_d can have for any positive integer d greater than 1, the terms in $1-\zeta X$ in the decomposition of $f_k(X)$ into partial fractions have the form

$$\frac{\beta_1}{1-\zeta X}+\frac{\beta_2}{(1-\zeta X)^2}+\dots+\frac{\beta_q}{(1-\zeta X)^q}, \tag{5}$$

where of course some of the β's may be zero. The coefficient of $(\zeta X)^n$ in the power series expansion of (5) is a polynomial in n of degree at most $q-1$. Summing over all possible ζ, we get the result of the lemma.

THEOREM 3. *Suppose k is arbitrary, A has exactly r elements $a_1, a_2, \ldots, a_r$, and A has property P_k. Then as n increases $p^{(k)}(n) \to +\infty$ in such a way that*

$$\text{(i)}\ p^{(k)}(n) = \frac{n^{r-k-1}}{(r-k-1)!\, a_1 a_2 \ldots a_r} + O(n^{r-k-2}),$$

$$\text{(ii)}\ \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = 1 - \frac{p^{(k)}(n-1)}{p^{(k)}(n)} = O\left(\frac{1}{n}\right).$$

Proof. By the assumptions of the theorem and the definition of property P_k, we have $k < r$. Thus Lemma 1 is applicable. Further, the number q in Lemma 1 does not exceed $r-k-1$. Hence (i) follows immediately from Lemma 1 and (ii) follows from (i).

If A is finite, conclusion (i) of Theorem 3 shows that property P_k is a sufficient condition for $p^{(k)}(n)$ to be positive for all large n. The special case $k = 0$ of Theorem 3 is well known (*cf.* [**8**]).

4. *Proof that* $p(n) = o\left(p^{(-1)}(n)\right)$. Conclusion (ii) of Theorem 3 shows in particular that when A is finite $p(n) = o\left(p^{(-1)}(n)\right)$ as n increases. We now show that this relation is also true when A is infinite.

THEOREM 4. *Suppose A is infinite. Then as n increases*

(i) $p^{(-1)}(n)\, n^{-c} \to +\infty$ *for any fixed* c,

$$\text{(ii)}\ \frac{p(n)}{p^{(-1)}(n)} = 1 - \frac{p^{(-1)}(n-1)}{p^{(-1)}(n)} \to 0.$$

Proof. In proving assertion (i) we may assume $c > 0$. Let B be a finite subset of A having at least $c+1$ elements. Conclusion (i) of Theorem 3 shows that $p_B^{(-1)}(n)\, n^{-c} \to +\infty$ as n increases. But

$$p_A^{(-1)}(n) \geqslant p_B^{(-1)}(n)$$

and so (i) follows.

It remains to prove assertion (ii). Suppose that $n > 0$ and that $n = n_1 a_1 + n_2 a_2 + \ldots + n_q a_q$ is a partition of n into parts $a_1, a_2, \ldots, a_q$ taken from A, where $n_1, n_2, \ldots, n_q$ are positive integers. From this we construct a partition of each of the q integers $n-a_1, n-a_2, \ldots, n-a_q$ in the following way:

$$n-a_1 = (n_1-1)\, a_1 + n_2 a_2 + \ldots + n_q a_q,$$

$$n-a_2 = n_1 a_1 + (n_2-1)\, a_2 + n_3 a_3 + \ldots + n_q a_q,$$

$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots \qquad \cdots$$

$$n-a_q = n_1 a_1 + \ldots + n_{q-1} a_{q-1} + (n_q-1)\, a_q.$$

In this construction no two distinct partitions of n give rise to the same partition of some integer less than n. Let $P_q(n)$ denote the number of partitions of n into parts taken from A in which the number of distinct elements of A which actually occur is exactly q. Then the above construction shows that

$$\sum_{q=1}^{n} q\, P_q(n) \leqslant \sum_{m=0}^{n-1} p(m). \tag{6}$$

Now $P_q(n)$ is the coefficient of X^n in the expansion of

$$\Sigma \frac{X^{a_1}}{1-X^{a_1}} \cdots \frac{X^{a_q}}{1-X^{a_q}}, \tag{7}$$

where the summation is taken over all subsets $\{a_1, \ldots, a_q\}$ of A containing *exactly q elements. Hence $P_q(n)$ does not exceed the coefficient of X^n in*

$$\Sigma(X^{a_1}+X^{2a_1}+\ldots)\ldots(X^{a_q}+X^{2a_q}+\ldots),$$

where the summation is taken over all subsets $\{a_1, \ldots, a_q\}$ of the set $\{1, 2, \ldots, n\}$ which contain exactly q elements. Since there are $\binom{n}{q}$ such subsets and since the coefficient of X^n in

$$(X^{a_1}+X^{2a_1}+\ldots)\ldots(X^{a_q}+X^{2a_q}+\ldots)$$

does not exceed that in

$$(X+X^2+\ldots)^q = \left(\frac{X}{1-X}\right)^q = \sum_{m=q}^{\infty} \binom{m-1}{q-1} X^m,$$

we have

$$P_q(n) \leqslant \binom{n}{q}\binom{n-1}{q-1} \leqslant n^{2q-1}. \tag{8}$$

Now if t is any fixed positive integer, we have by (6) and (8)

$$\begin{aligned} p^{(-1)}(n) &= \sum_{m=0}^{n} p(m) \\ &\geqslant p(n) + \sum_{q=1}^{n} q\, P_q(n) = (t+1)p(n) + \sum_{q=1}^{n} (q-t)\, P_q(n) \\ &\geqslant (t+1)p(n) - (t-1)\sum_{q=1}^{t-1} P_q(n) \geqslant (t+1)p(n) - (t-1)^2 n^{2t-3}. \end{aligned}$$

Hence

$$\frac{p(n)}{p^{(-1)}(n)} \leqslant \frac{1}{t+1} + \frac{(t-1)^2}{t+1} \frac{n^{2t-3}}{p^{(-1)}(n)}. \tag{9}$$

By conclusion (i) the second term on the right-hand side of (9) has limit zero as n increases. Therefore

$$\overline{\lim_{n\to\infty}} \frac{p(n)}{p^{(-1)}(n)} \leqslant \frac{1}{t+1}.$$

Since t can be chosen arbitrarily large, conclusion (ii) follows.

5. *Sufficiency of condition P_k when A is infinite.*

LEMMA 2. *For any k, an infinite set of positive integers has property P_k if and only if some finite subset has property P_k.*

Proof. The assertion is trivial if $k < 0$, so suppose $k \geqslant 0$. If some finite subset of an infinite set A has property P_k, then clearly A has also, since enlarging a set with property P_k cannot destroy that property. Suppose now that A is an infinite set with property P_k. Let

$$A_0 = \{a_{01}, a_{02}, \ldots, a_{0,k+1}\}$$

be some subset of A containing exactly $k+1$ elements. Suppose the prime divisors of the product $a_{01}\, a_{02} \ldots a_{0,k+1}$ are $p_1, p_2, \ldots, p_b$. By the definition of property P_k there exists, for $i = 1, 2, \ldots, b$, a subset

$$A_i = \{a_{i1}, a_{i2}, \ldots, a_{i,k+1}\}$$

of A containing exactly $k+1$ elements none of which is divisible by p_i. If p is a prime other than $p_1, \ldots, p_b$, clearly no element of A_0 is divisible by p. Then the union B of the (not necessarily disjoint) subsets $A_0, A_1, \ldots, A_b$ of A is a finite subset of A with property P_k. For given any prime number, we can find at least $k+1$ elements of B which it does not divide.

THEOREM 5. *Suppose k is arbitrary, A is infinite, and A has property P_k. Then as n increases*

(i) $p^{(k)}(n)\, n^{-c} \to +\infty$ *for any fixed c,*

(ii) $\dfrac{p^{(k+1)}(n)}{p^{(k)}(n)} = 1 - \dfrac{p^{(k)}(n-1)}{p^{(k)}(n)} \to 0.$

Proof. The assertions of the theorem have been proved for $k = -1$ (Theorem 4) and they follow immediately for $k < -1$ by summation. So suppose $k \geqslant 0$ and A is an infinite set with property P_k. By Lemma 2 there is a finite subset A_1 of A which has property P_k. If $k = 0$ we may assume that A_1 contains at least two elements. Let us put $A_2 = A - A_1$ and write

$$p_A^{(k)}(n) = p^{(k)}(n), \quad p_{A_1}^{(k)}(n) = p_1^{(k)}(n), \quad p_{A_2}(n) = p_2(n), \quad p_{A_2}^{(-1)}(n) = p_2^{(-1)}(n).$$

We shall use the identity

$$p^{(k)}(n) = \sum_{m=0}^{n} p_1^{(k)}(n-m)\, p_2(m), \tag{10}$$

which is an immediate consequence of (1).

Let t be an arbitrary positive integer. By Theorem 3

$$\lim_{n\to\infty} \frac{p_1^{(k)}(n)}{1+|p_1^{(k+1)}(n)|} = +\infty$$

and so there exist positive integers g and h such that

$$\frac{p_1^{(k)}(n)}{1+|p_1^{(k+1)}(n)|} \geqslant \begin{cases} -g & \text{for all } n, \\ t+1 & \text{for } n \geqslant h. \end{cases} \tag{11}$$

Let
$$s = \max_{0 \leqslant n \leqslant h-1} \{1+|p_1^{(k+1)}(n)|\}. \tag{12}$$

By Theorem 4 there is a positive integer h_1 such that

$$\frac{p_2(n)}{p_2^{(-1)}(n)} \leqslant \frac{1}{(g+t+1)hs} \tag{13}$$

for $n \geqslant h_1$. Then if $n \geqslant h+h_1-1$ we obtain from (10)-(13)

$$\begin{aligned}
p^{(k)}(n) &\geqslant (t+1) \sum_{m=0}^{n-h} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) \\
&\qquad\qquad - g \sum_{m=n-h+1}^{n} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) \\
&\geqslant (t+1) \sum_{m=0}^{n} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) - (g+t+1)s \sum_{m=n-h+1}^{n} p_2(m) \\
&\geqslant (t+1) \sum_{m=0}^{n} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) \\
&\qquad\qquad - (g+t+1)s \left\{ \sum_{m=n-h+1}^{n} p_2(m)/p_2^{(-1)}(m) \right\} p_2^{(-1)}(n) \\
&\geqslant \left\{ t+1-(g+t+1)s \sum_{m=n-h+1}^{n} p_2(m)/p_2^{(-1)}(m) \right\} \\
&\qquad\qquad \times \sum_{m=0}^{n} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) \\
&\geqslant t \sum_{m=0}^{n} \{1+|p_1^{(k+1)}(n-m)|\} p_2(m) \\
&\geqslant t \sum_{m=0}^{n} p_2(m) + t \left| \sum_{m=0}^{n} p_1^{(k+1)}(n-m) p_2(m) \right| \\
&= t p_2^{(-1)}(n) + t |p^{(k+1)}(n)|.
\end{aligned}$$

Now t can be chosen arbitrarily large. Therefore $\lim_{n\to\infty} p^{(k)}(n)/p_2^{(-1)}(n) = +\infty$ and thus, since A_2 is infinite, assertion (i) of the present theorem follows from assertion (i) of Theorem 4. Also $\lim_{n\to\infty} p^{(k+1)}(n)/p^{(k)}(n) = 0$ and assertion (ii) is proved.

Theorem 5 shows in particular that, when A is infinite, property P_k is a sufficient condition for $p^{(k)}(n)$ to be positive for all large n. Thus property P_k is a sufficient condition for the eventual positivity of $p^{(k)}(n)$ in any case.

COROLLARY. *For arbitrary k, if A has property P_k, then as n increases*

$$p^{(k)}(n) \to +\infty, \quad \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = 1 - \frac{p^{(k)}(n-1)}{p^{k}(n)} \to 0.$$

6. *Further implications of property P_k.* Although P_k is a necessary condition for $p^{(k)}(n)$ to be positive for all large n, we have already seen (Theorems 3 and 5) that if we assume property P_k we can actually assert much more about $p^{(k)}(n)$ than the mere fact that it is positive for large n. In this section we go further in this direction.

THEOREM 6. *Suppose k is arbitrary and A has property P_k. Then there is a positive integer b such that*

$$p^{(k)}(m) > p^{(k)}(n) \quad \textit{if} \quad m-b \geqslant n \geqslant 0.$$

Proof. If $k < -1$, the result stated is trivially true with $b = 1$, since $p^{(k+1)}(n) = p^{(k)}(n) - p^{(k)}(n-1)$ is positive for all non-negative n. If $k = -1$ and A is non-empty, we can take b as any element of A. For then if $m-b \geqslant n \geqslant 0$, there is an integer j such that $n < jb \leqslant m$, so that

$$p^{(-1)}(m) = \sum_{h=0}^{m} p(h) \geqslant p(jb) + \sum_{h=0}^{n} p(h) = p(jb) + p^{(-1)}(n) > p^{(-1)}(n).$$

When $k \geqslant 0$ we first settle the case in which A is finite. Suppose $A = \{a_1, a_2, \ldots, a_r\}$ and A has property P_k. Then by Theorem 3 there is a positive integer c such that

$$\left| p^{(k)}(n) - \frac{n^{r-k-1}}{(r-k-1)!\, a_1 a_2 \ldots a_r} \right| \leqslant cn^{r-k-2} + 1$$

for all non-negative n. Therefore if $m > n \geqslant 0$ we have

$$\begin{aligned} p^{(k)}(m) - p^{(k)}(n) &\geqslant \frac{m^{r-k-1} - n^{r-k-1}}{(r-k-1)!\, a_1 a_2 \ldots a_r} - cm^{r-k-2} - cn^{r-k-2} - 2 \\ &\geqslant \frac{(m-n)\, m^{r-k-2}}{(r-k-1)!\, a_1 a_2 \ldots a_r} - (2c+2)\, m^{r-k-2}. \end{aligned}$$

The last expression is positive if $m-n \geqslant b$, where

$$b = (2c+2)(r-k-1)!\, a_1 a_2 \ldots a_r + 1.$$

Thus the assertion of the theorem is proved if A is finite.

Finally, suppose $k \geqslant 0$, A is infinite, and A has property P_k. As in the proof of Theorem 5 let A_1 be a finite subset of A which has property P_k and contains at least two elements, put $A_2 = A - A_1$, and write

$p_A^{(k)}(n) = p^{(k)}(n)$, $p_{A_1}^{\prime(k)}(n) = p_1^{\prime(k)}(n)$, $p_{A_2}(n) = p_2(n)$, $p_{A_2}^{(-1)}(n) = p_2^{(-1)}(n)$. By Theorem 3 there exist non-negative integers g and h such that $p_1^{(k)}(n) \geqslant -g$ for all n and $p_1^{\prime(k)}(n) \geqslant 1$ for $n \geqslant h$. By Theorem 4 there is a positive integer h_1 such that $p_2(n)/p_2^{(-1)}(n) \leqslant 1/(2gh+2h)$ for $n \geqslant h_1$. By the previous paragraph there is a positive integer b such that $b \geqslant h+h_1-1$ and

$$p_1^{(k)}(m) > p_1^{(k)}(n) \text{ for } m-b \geqslant n \geqslant 0.$$

Hence if $m-b \geqslant n \geqslant 0$ we have

$$\begin{aligned} p^{(k)}(m)-p^{(k)}(n) &= \sum_{l=0}^{n} \{p_1^{(k)}(m-l)-p_1^{(k)}(n-l)\}p_2(l)+\sum_{l=n+1}^{m} p_1^{(k)}(m-l)\,p_2(l) \\ &\geqslant \sum_{l=0}^{n} p_2(l)+\sum_{l=n+1}^{m-h} p_2(l)-g\sum_{l=m-h+1}^{m} p_2(l) \\ &= p_2^{(-1)}(m)\left\{1-(g+1)\sum_{l=m-h+1}^{m}\frac{p_2(l)}{p_2^{(-1)}(m)}\right\} \\ &\geqslant p_2^{(-1)}(m)\left\{1-(g+1)\sum_{l=m-h+1}^{m}\frac{p_2(l)}{p_2^{(-1)}(l)}\right\} \geqslant \tfrac{1}{2}p_2^{(-1)}(m) > 0. \end{aligned}$$

Thus Theorem 6 is proved.

COROLLARY. *Suppose k is arbitrary and A has property P_k. If h is any fixed positive integer, then*

$$\frac{p^{(k-1)}(n+h)-p^{(k-1)}(n)}{h} = \{1+o(1)\}\,p^{(k)}(n)$$

as n increases. If h is a fixed integer not less than the integer b of Theorem 6, then

$$\frac{p^{(k-1)}(n+h)-p^{(k-1)}(n)}{h}$$

is a strictly increasing function of n for non-negative n.

Proof. The first assertion follows from conclusion (ii) of Theorem 3 and conclusion (ii) of Theorem 5:

$$p^{(k-1)}(n+h)-p^{(k-1)}(n) = \sum_{m=n+1}^{n+h} p^{(k)}(m) = h\{1+o(1)\}p^{(k)}(n).$$

The second assertion follows from Theorem 6:

$$\begin{aligned} &\{p^{(k-1)}(n+h)-p^{(k-1)}(n)\}-\{p^{(k-1)}(n-1+h)-p^{(k-1)}(n-1)\} \\ &\qquad = \sum_{m=n+1}^{n+h} p^{(k)}(m)-\sum_{m=n}^{n-1+h} p^{(k)}(m) = p^{(k)}(n+h)-p^{(k)}(n) > 0, \end{aligned}$$

provided $h \geqslant$ the integer b of Theorem 6 and $n > 0$.

THEOREM 7. *Suppose k is arbitrary, A is infinite, A has property P_k, and $0 < \alpha < 1$. Then*

$$\lim_{n\to\infty} \frac{1}{p^{(k)}(n)} \max_{0\leqslant m\leqslant \alpha n} p^{(k)}(m) = 0.$$

Proof. Let A_1 be a finite subset of A having property P_k and containing at least $k+2$ elements. Then A_1 has $q+k+1$ elements, where q is some positive integer. By conclusion (i) of Theorem 3 there is a positive integer s such that

$$\lim_{n\to\infty} \frac{sp_1^{(k)}(n)}{n^q} = 1.$$

Hence there are positive integers t and u such that

$$\tfrac{1}{2}n^q \leqslant sp_1^{(k)}(n) \leqslant 2n^q \quad \text{for} \quad n \geqslant t$$

and

$$p_1^{(k)}(n) \leqslant u \quad \text{for} \quad n < t.$$

Hence if $n \geqslant \max\{(us)^{1/q}/(\alpha-\alpha^2),\ t/(1-\alpha)\}$ and $t \leqslant m \leqslant \alpha n$, we have

$$\begin{aligned} p^{(k)}(m) &= \sum_{l=0}^{m} p_1^{(k)}(m-l)\,p_2(l) \\ &\leqslant \sum_{l=0}^{m-t} 2s^{-1}(m-l)^q p_2(l) + \sum_{l=m-t+1}^{m} up_2(l) \\ &\leqslant \sum_{l=0}^{m-t} 2s^{-1}(\alpha n-\alpha l)^q p_2(l) + \sum_{l=m-t+1}^{m} s^{-1}\alpha^q(n-\alpha n)^q p_2(l) \\ &\leqslant 2\alpha^q \sum_{l=0}^{m} s^{-1}(n-l)^q p_2(l) \leqslant 4\alpha^q \sum_{l=0}^{m} p_1^{(k)}(n-l)\,p_2(l). \qquad (14) \end{aligned}$$

Suppose now that $k \leqslant 0$. Then $p_1^{(k)}(n-l) \geqslant 0$ for any l and so if $0 \leqslant m \leqslant \alpha n$ we have

$$\sum_{l=0}^{m} p_1^{(k)}(n-l)\,p_2(l) \leqslant \sum_{l=0}^{n} p_1^{(k)}(n-l)\,p_2(l) = p^{(k)}(n). \qquad (15)$$

Combining (14) and (15), we find that

$$p^{(k)}(m) \leqslant 4\alpha^q p^{(k)}(n), \qquad (16)$$

provided $t \leqslant m \leqslant \alpha n$ and n is sufficiently large. By Theorem 5 the inequality (16) also holds if $0 \leqslant m \leqslant t-1$ and n is sufficiently large. Hence

$$\overline{\lim_{n\to\infty}} \frac{1}{p^{(k)}(n)} \max_{0\leqslant m\leqslant \alpha n} p^{(k)}(m) \leqslant 4\alpha^q.$$

Since we can make q as large as we please by arbitrarily enlarging A_1, the assertion of the theorem is proved when $k \leqslant 0$.

To prove the assertion of the theorem when $k > 0$ it is again sufficient to show that (15) holds if $0 \leqslant m \leqslant \alpha n$ and n is sufficiently large. By Theorem 3 there are non-negative integers t_1 and g_1 such that $p_1^{(k)}(n) \geqslant 1$ for $n \geqslant t_1$ and $p_1^{(k)}(n) \geqslant -g_1$ for all n. By Theorem 4 there is a positive integer t_2 such that $p_2(n)/p_2^{(-1)}(n) \leqslant 1/(3t_1 g_1 + 3t_1)$ for $n \geqslant t_2$. By the result of the preceding paragraph there is a positive integer $t_3 \geqslant t_1 + t_2 - 1$ such that $p_2^{(-1)}(m)/p_2^{(-1)}(n) \leqslant \frac{1}{3}$ provided $0 \leqslant m \leqslant \alpha n$ and $n \geqslant t_3$. Accordingly, if $n \geqslant \max\{t_3, (t_1+1)/(1-\alpha)\}$ and $0 \leqslant m \leqslant \alpha n$ we have

$$
\begin{aligned}
p^{(k)}(n) - \sum_{l=0}^{m} p_1^{(k)}(n-l)\, p_2(l) &= \sum_{l=m+1}^{n} p_1^{(k)}(n-l)\, p_2(l) \\
&\geqslant \sum_{l=m+1}^{n-t_1} p_2(l) - g_1 \sum_{l=n-t_1+1}^{n} p_2(l) \\
&= p_2^{(-1)}(n) - p_2^{(-1)}(m) - (g_1+1) \sum_{l=n-t_1+1}^{n} p_2(l) \\
&\geqslant p_2^{(-1)}(n) \left\{ 1 - \frac{p_2^{(-1)}(m)}{p_2^{(-1)}(n)} - (g_1+1) \sum_{l=n-t_1+1}^{n} \frac{p_2(l)}{p_2^{(-1)}(l)} \right\} \\
&\geqslant p_2^{(-1)}(n) \left\{ 1 - \tfrac{1}{3} - (g_1+1)\, t_1 \frac{1}{3t_1 g_1 + 3t_1} \right\} = \tfrac{1}{3} p_2^{(-1)}(n) > 0.
\end{aligned}
$$

This completes the proof.

7. *The order of magnitude of $p^{(k+1)}(n)/p^{(k)}(n)$.* Suppose A has property P_k and h is the smallest positive integer such that $p^{(k)}(n) > 0$ for $n \geqslant h$. (Such an integer h exists if and only if A has property P_k.) Then the quantity

$$\rho^{(k)}(n) = \frac{p^{(k+1)}(n)}{p^{(k)}(n)} = 1 - \frac{p^{(k)}(n-1)}{p^{(k)}(n)}$$

is defined for $n \geqslant h$. If A is finite we know that $\rho^{(k)}(n) = O(1/n)$ [conclusion (ii) of Theorem 3]. If A is infinite we have proved that $\rho^{(k)}(n)$ tends to zero as n increases [conclusion (ii) of Theorem 5] without getting a more explicit estimate of its order of magnitude for large n. In fact we have been unable to obtain any result in this direction. However, we feel that there is reason to believe that the following assertion may be true.

CONJECTURE. *If k is arbitrary and A has property P_k, then*

$$\rho^{(k)}(n) = O(1/n^{1/2}).$$

It is easy to see that the assertion of the conjecture, if true, is best possible. For if A is the set of all positive integers, then as n increases $\rho^{(k)}(n) = \{1+o(1)\}\pi/(6n)^{1/2}$ by Rademacher's exact formula for $p(n)$ in this case [9]. On the other hand, the following theorem shows that $\rho^{(k)}(n)$ cannot be $O(1/n)$ for any infinite set A, that is, conclusion (ii) of Theorem 3 is definitely false for infinite sets.

THEOREM 8. *Suppose k is arbitrary and A is infinite. If A has property P_k, then $n\rho^{(k)}(n)$ is unbounded above for large n. If A has property P_{k+1}, then as n increases $n\rho^{(k)}(n) \to +\infty$.*

Proof. Suppose A is infinite and has property P_k. As above let h be the smallest integer such that $p^{(k)}(n) > 0$ for $n \geqslant h$. Suppose $n\rho^{(k)}(n)$ were bounded above by the integer g for $n \geqslant h$. Let $c = \max(g, h)$. Then if $n \geqslant c$

$$\frac{p^{(k)}(n)}{p^{(k)}(c)} = \prod_{m=c+1}^{n} \frac{p^{(k)}(m)}{p^{(k)}(m-1)} \leqslant \prod_{m=c+1}^{n} \frac{m}{m-g} \leqslant \prod_{m=c+1}^{n} \frac{m}{m-c} = \binom{n}{c} \leqslant n^c.$$

But this contradicts conclusion (i) of Theorem 5. Thus the first assertion of the present theorem is proved.

Suppose now that A is infinite and has property P_{k+1}. By Theorem 6 there is a positive integer b such that $p^{(k+1)}(n) > p^{(k+1)}(m)$ if $n-b \geqslant m \geqslant 0$. Since the ratios

$$\frac{p^{(k+1)}(n-1)}{p^{(k+1)}(n)}, \quad \frac{p^{(k+1)}(n-2)}{p^{(k+1)}(n)}, \quad \ldots, \quad \frac{p^{(k+1)}(n-b+1)}{p^{(k+1)}(n)}$$

all have the limit 1 as n increases, there is a positive integer $h_1 \geqslant b$ such that all these ratios are less than 2 if $n \geqslant h_1$. Thus if $n \geqslant h_1$ and $m \leqslant n$ we have $p^{(k+1)}(m) < 2p^{(k+1)}(n)$. Now let ϵ be an arbitrary positive number. By Theorem 7

$$\lim_{n\to\infty} \frac{1}{p^{(k+1)}(n)} \max_{0 \leqslant m \leqslant (1-\epsilon)n} p^{(k+1)}(m) = 0$$

and hence there is a positive integer h_2 such that

$$\max_{0 \leqslant m \leqslant (1-\epsilon)n} p^{(k+1)}(m) \leqslant \epsilon p^{(k+1)}(n)$$

for any $n \geqslant h_2$. Hence if $n \geqslant \max(h_1, h_2, 1/\epsilon)$ we have

$$\begin{aligned} p^{(k)}(n) &= \sum_{m=0}^{[(1-\epsilon)n]} p^{(k+1)}(m) + \sum_{m=[(1-\epsilon)n]+1}^{n} p^{(k+1)}(m) \\ &\leqslant \sum_{m=0}^{[(1-\epsilon)n]} \epsilon p^{(k+1)}(n) + \sum_{m=[(1-\epsilon)n]+1}^{n} 2p^{(k+1)}(n) \\ &\leqslant n\epsilon p^{(k+1)}(n) + (\epsilon n+1)\, 2p^{(k+1)}(n) \leqslant 5\epsilon n p^{(k+1)}(n). \end{aligned}$$

Since ϵ is arbitrary, this proves the latter assertion of Theorem 8.

References.

1. F. C. Auluck and C. B. Haselgrove, " On Ingham's Tauberian theorem for partitions ", *Proc. Cambridge Phil. Soc.*, 48 (1952), 566–570.
2. P. Bachmann, *Niedere Zahlentheorie:* Zweiter Teil, *Additive Zahlentheorie*, Ch. 3 (Leipzig, Teubner, 1910).
3. A. Brauer, " On a problem of partitions ", *American J. of Math.*, 64 (1942), 299–312.
4. N. A. Brigham, " A general asymptotic formula for partition functions ", *Proc. American Math. Soc.*, 1 (1950), 182–191, especially p. 183.
5. A. E. Ingham, " A Tauberian theorem for partitions ", *Ann. of Math.* (2), 42 (1941), 1075–1090, especially pp. 1084–1086.
6. K. Knopp, " Asymptotische Formeln der additiven Zahlentheorie ", *Schriften der Königsberger gelehrten Gesellschaft* (Naturwissenschaftliche Klasse), 2 (1925), 45–74, especially pp. 60–63.
7. E. Netto, *Lehrbuch der Combinatorik*, Ch. 6 (Leipzig, Teubner, 1901 and 1927).
8. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Vol. 1, Part I, Problem 27 (Berlin, Springer, 1925).
9. H. Rademacher, " On the expansion of the partition function in a series ", *Ann. of Math.* (2), 44 (1943), 416–422.
10. K. F. Roth and G. Szekeres, " Some asymptotic formulae in the theory of partitions ", *Quart. J. of Math.* (2), 5 (1954), 241–259.

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(*Received 5th August*, 1955.)

Printed by C. F. Hodgson & Son, Ltd., Pakenham Street, London, W.C.1.