LEMMA 5. *If* $\phi_\alpha(t) \to s$ *then, for any* $\beta > \alpha$, $\phi_\beta(t) \to s$.

It is sufficient to assume that $s = 0$. By (3), since $F(t)$ is increasing, $\frac{1}{\beta}\phi_\beta(t) \leqslant \frac{1}{\alpha}\phi_\alpha(t)$ and the required result follows immediately.

LEMMA 6. *Let* $\gamma(t)$ *be an increasing function. If* $g(t) = \int_0^t \gamma(u)\,du \sim st^\alpha$ *as* $t \to 0$, *then* $\gamma(t) \sim s\alpha t^{\alpha-1}$.

The proof is similar to that of a result due to Hardy and Littlewood*, to which the lemma reduces when $\gamma(t)$ is a derivative.

Let $0 < \theta < 1$ be a fixed number; then

$$(1-\theta)t\gamma(\theta t) < g(t) - g(\theta t) < (1-\theta)t\gamma(t).$$

Since $g(t) - g(\theta t) \sim s(1-\theta^\alpha)t^\alpha$, we have

$$\underline{\lim}\, t^{1-\alpha}\gamma(t) \geqslant s\frac{1-\theta^\alpha}{1-\theta},$$

$$\overline{\lim}\,(\theta t)^{1-\alpha}\gamma(\theta t) \leqslant \frac{s(1-\theta^\alpha)}{(1-\theta)\theta^{\alpha-1}}, \quad i.e. \ \overline{\lim}\, t^{1-\alpha}\gamma(t) \leqslant \frac{s(1-\theta^\alpha)}{(1-\theta)\theta^{\alpha-1}},$$

and the required result follows on allowing θ to tend to 1.

To complete the proof of (ii) we employ the Littlewood technique of repeated differentiation. Let

$$F(t) = F_1(t), \ F_k(t) = \int_0^t F_{k-1}(u)\,du, \ k = 2, 3, \ldots.$$

By Lemma 5, if $\phi_\alpha(t) \to s$ then $\phi_k(t) \to s$ for an integer k, and this is equivalent to $F_k(t) \sim st^k/k!$ By repeated application of Lemma 6 we obtain $F(t) \sim st$, and the desired result $u(r) \to s$ follows by Lemma 4.

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ON A PROBLEM OF ADDITIVE NUMBER THEORY†

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Let $a_1, a_2, \ldots$ be an infinite sequence of integers, such that

$$0 \leqslant a_1 \leqslant a_2 \leqslant \ldots.$$

* See, *e.g.* G. H. Hardy, *loc. cit.* 170.

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Denote by $f(n)$ the number of solutions of $a_i+a_j=n$, by $r(n)$ the number of solutions of $a_i+a_j \leqslant n$; thus $r(n)=f(0)+f(1)+\ldots+f(n)$. In a previous paper [2] Erdös-Turán conjectured that

$$r(n)-cn=O(1)$$

cannot hold. In the present paper we prove

Theorem 1. *If $c>0$, then*

$$r(n)=cn+o(n^{1/4}\log^{-1/2}n). \tag{1}$$

cannot hold.

Remarks. (i) The assumption $c \neq 0$ is clearly necessary, for if $c=0$ were permitted, (1) could clearly hold, if $a_k \to \infty$ sufficiently fast.

(ii) The leading term cn on the right-hand side of (1) could be replaced by more general functions such as, *e.g.*, $cn+dn^{\alpha}$ $(0<\alpha<1)$.

If $a_k=k^2$ the estimation of $r(n)$ is the classical problem about lattice points in a circle. Here it follows from the results of Hardy and Landau that $r(n)\neq cn+dn^{1/2}+o(n^{1/4}\log^{1/4}n)$. It is rather surprising that our result for a general a_k is almost as good while its proof is much simpler.

We can also prove

$$\sup_{1\leqslant l\leqslant n}|r(l)-cl| > \frac{K_\beta n^{1/4}}{\log^\beta n} \quad (\beta>\tfrac{1}{2}).$$

Theorem 1 remains true for sequences of non-negative numbers $\{a_k\}$, not necessarily integers. The proof can be reduced to that of Theorem 1 by defining $a_k{}^*$ as the nearest integer to a_k. Then the inequalities

$$r(n-2)\leqslant r^*(n)\leqslant r(n+2)$$

show that Theorem 1 is true for $r(n)$, if it is true for $r^*(n)$.

We use the letter K as a generic notation for a positive number possibly depending on the sequence $\{a_k\}$, but on nothing else. The numerical value of K will differ at different occurrences. It seems likely that one can construct an infinite sequence $a_1<a_2<\ldots$ for which

$$r(n)=cn+O(n^{1/4}).$$

This we have not succeeded in doing.

Another conjecture of Erdös-Turán was that if $f(n)>0$ for all large n, then $\overline{\lim}\, f(n)=\infty$, and an even stronger conjecture would be that if $a_k<ck^2$ for all k, then $\overline{\lim}\, f(n)=\infty$. Here our method does not seem to

be of any use, since one can construct a sequence $\{a_k\}$ such that $a_k < ck^2$ and

$$\overline{\lim}\, \frac{1}{n}\left(\sum_{k=1}^{n} f^2(k)\right) < \infty.$$

The number $f(b)$ can be interpreted in three different ways: (I) We can count $i \neq j$ twice and $i = j$ once; (II) $i \neq j$ can be counted once and $i = j$ also once; (III) $i \neq j$ can be counted once and $i = j$ excluded altogether. If we put $g(z) = \sum_{k=1}^{\infty} z^{a_k}$, then the generating functions $\Sigma f(n) z^n$ in the three cases are

$$\text{(I) } g^2(z); \quad \text{(II) } \tfrac{1}{2}\big(g^2(z) + g(z^2)\big); \quad \text{(III) } \tfrac{1}{2}\big(g^2(z) - g(z^2)\big).$$

Our theorem is true for every one of the three interpretations.

In cases (I) and (II) Dirac and Newman [**1**] proved that $f(n)$ cannot be a constant for $n > n_0$. We can prove in all three cases

THEOREM 2. *If $c > 0$, or $c = 0$ and $a_k < Ak^2$, then*

$$\overline{\lim_{n\to\infty}}\, \frac{1}{n} \sum_{k=0}^{n} \big(f(k) - c\big)^2 > 0.$$

We shall need the following:

LEMMA. *Suppose $\phi(z) = \sum_{n=0}^{\infty} b_n z^n$ is convergent for $|z| < 1$ and suppose that all b_n are non-negative, real numbers. Then for $0 < \alpha \leqslant \pi$, $z = re^{i\theta}$ $(0 < r < 1)$,*

$$\frac{1}{2\alpha}\int_{-\alpha}^{\alpha} |\phi(z)|^2 d\theta \geqslant \frac{1}{6\pi}\int_{-\pi}^{\pi} |\phi(z)|^2 d\theta.$$

Proof. The function

$$h(\theta) = 1 - |\theta/\alpha| \;\; (|\theta| < |\alpha|), \quad h(\theta) = 0 \;\; (\alpha \leqslant |\theta| \leqslant \pi)$$

has the Fourier series

$$h(\theta) = \sum_{k=-\infty}^{\infty} c_k e^{ik\theta}, \quad c_k = (1 - \cos \alpha k)/\pi\alpha k^2 \geqslant 0.$$

By Parseval's formula

$$\int_{-\pi}^{\pi} |h(\theta)\phi(z)|^2 d\theta = 2\pi \sum_{\nu=-\infty}^{\infty} \Big(\sum_{k+l=\nu} c_k b_l r^l\Big)^2$$

$$\geqslant 2\pi \sum_{\nu=-\infty}^{\infty} \sum_{k+l=\nu} c_k^2 b_l^2 r^{2l},$$

since all the terms in the first inner sum are non-negative. The sum on the right-hand side is now

$$2\pi(\Sigma c_k^2)(\Sigma b_l^2 r^{2l}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} h^2(\theta)\,d\theta \int_{-\pi}^{\pi} |\phi(z)|^2 d\theta$$

$$= \frac{\alpha}{3\pi}\int_{-\pi}^{\pi} |\phi(z)|^2 d\theta.$$

But when $|\theta| < |\alpha|$, $|h(\theta)| \leqslant 1$ and when $|\alpha| \leqslant |\theta| \leqslant \pi$, $h(\theta) = 0$. Hence

$$\int_{-\alpha}^{\alpha} |\phi|^2 d\theta \geqslant \int_{-\pi}^{\pi} |h(\theta)\,\phi(z)|^2 d\theta \geqslant \frac{\alpha}{3\pi}\int_{-\pi}^{\pi} |\phi(z)|^2 d\theta.$$

Proof of Theorem 1. Let

$$g(z) = \sum_{k=1}^{\infty} z^{a_k} = \Sigma c_\nu z^{a_\nu},$$

where the summation in the last sum is over all *different* a_ν and c_ν is an integer $\geqslant 1$. We give the proof for the case where the generating function is $g^2(z)$. Then $(1-z)^{-1} g^2(z) = \sum_{n=0}^{\infty} r(n) z^n$.

We must therefore prove that we cannot have

$$(1-z)^{-1} g^2(z) = c \sum_{n=0}^{\infty} nz^n + h(z)$$

$$= cz(1-z)^{-2} + h(z); \tag{2}$$

$$h(z) = \sum_{n=0}^{\infty} v_n z^n, \quad v_n = o(n^{1/4} \log^{-1/2} n).$$

Let $\frac{1}{2} < r < 1$, $z = re^{i\theta}$, $1-r < \alpha < \pi$. By (2)

$$\int_{-\alpha}^{\alpha} |g^2(z)|\,d\theta = \int_{-\alpha}^{\alpha} |cz(1-z)^{-1} + (1-z)\,h(z)|\,d\theta$$

$$\leqslant c\int_{-\pi}^{\pi} |1-z|^{-1} d\theta + \int_{-\alpha}^{\alpha} |1-z|\,|h(z)|\,d\theta. \tag{3}$$

Now $(1-z)^{-1/2} = \Sigma \gamma_n z^n$, $\gamma_n = O(n^{-1/2})$, so that

$$\int_{-\pi}^{\pi} |1-z|^{-1} d\theta = \int_{-\pi}^{\pi} |(1-z)^{-1/2}|^2 d\theta = 2\pi \Sigma |\gamma_n|^2 r^{2n} < K \Sigma r^{2n}/n$$

$$< K \log \frac{1}{1-r}. \tag{4}$$

If $|\theta| \leqslant \alpha$, $|1-z| < K\alpha$. Therefore

$$\int_{-\alpha}^{\alpha} |1-z||h(z)|\,d\theta < K\alpha \int_{-\alpha}^{\alpha} |h(z)|\,d(\theta)$$

$$< K\alpha\left(\int_{-\alpha}^{\alpha} d\theta\right)^{1/2}\left(\int_{-\pi}^{\pi} |h(z)|^2 d\theta\right)^{1/2}$$

$$\leqslant K\alpha^{3/2}\left(\sum_{n=0}^{\infty} |v_n|^2 r^{2n}\right)^{1/2},$$

and

$$\sum_{n=0}^{\infty} |v_n|^2 r^{2n} \leqslant K \sum_{n \leqslant (1-r)^{-1/2}} n^{1/2} r^{2n} + \eta(r) \log^{-1}\frac{1}{1-r} \sum_{n > (1-r)^{-1/2}} n^{1/2} r^{2n},$$

where $\eta(r) < \epsilon$ for $1 > r > r_0(\epsilon)$, since $|v_n| < \epsilon n^{1/4} \log^{-1/2} n$ for large n. The first sum has $[(1-r)^{-1/2}]$ terms each one less than $(1-r)^{-1/4}$. For the second sum comparison with the binomial expansion shows that

$$\sum_{n=1}^{\infty} n^{1/2} r^{2n} < K(1-r^2)^{-3/2} < K(1-r)^{-3/2}.$$

Hence

$$\sum_{n=1}^{\infty} |v_n|^2 r^{2n} < K(1-r)^{-3/4} + K\eta(r)(1-r)^{-3/2} \log^{-1}\frac{1}{1-r}$$

$$< K\delta^2(1-r)^{-3/2} \log^{-1}\frac{1}{1-r} \quad \bigl(r > r_1(\delta),\ \delta > 0\bigr).$$

Therefore, for all r sufficiently close to one,

$$\int_{-\alpha}^{\alpha} |1-z||h(z)|\,d\theta \leqslant K\alpha^{3/2}\delta(1-r)^{-3/4} \log^{-1/2}\frac{1}{1-r}. \tag{5}$$

Collecting (3), (4), (5) gives

$$\int_{-\alpha}^{\alpha} |g^2(z)|\,d\theta < K_1 \log\frac{1}{1-r} + K_2\,\alpha^{3/2}\delta(1-r)^{-3/4} \log^{-1/2}\frac{1}{1-r}. \tag{A}$$

By the lemma with $\phi(z) = g(z) = \Sigma c_\nu r^{a_\nu}$,

$$\int_{-\alpha}^{\alpha} |g^2(z)|\,d\theta > K\alpha \int_{-\pi}^{\pi} |g^2(z)|\,d\theta = K\alpha\, \Sigma c_\nu^2 r^{2a_\nu}$$

$$\geqslant K\alpha\, \Sigma c_\nu r^{2a_\nu} = K\alpha g(r^2).$$

By (2)

$$g^2(r^2) = cr^2(1-r^2)^{-1} + (1-r^2)\,h(r^2)$$

$$= cr^2(1-r^2)^{-1} + (1-r^2)\,O(\Sigma n^{-1/4} r^{2n})$$

$$> K(1-r)^{-1} - O\bigl((1-r)\,.\,(1-r)^{-5/4}\bigr)$$

$$> K(1-r)^{-1}.$$

Hence $$\int_{-\alpha}^{\alpha} |g^2(z)|\, d\theta > K_3\, \alpha(1-r)^{-1/2}. \tag{B}$$

Now choose δ so that $K_3\,\delta^{-2/3} > K_1+K_2$ and put $\alpha = \delta^{-2/3}(1-r)^{1/2} \log \frac{1}{1-r}$. Then (A) and (B) yield the contradiction $K_3\,\delta^{-2/3} < K_1+K_2$.

Proof of Theorem 2. Let

$$t_n = \sum_{k=0}^{n} \big(f(k)-c\big)^2.$$

If $t_n/n \to 0$ as $n \to \infty$, then c must be an integer and therefore

$$t_n \geqslant \sum_{k=0}^{n} |f(k)-c| \geqslant \left| \sum_{k=0}^{n} f(k)-(n+1)c \right| = |r(n)-(n+1)c|.$$

Hence Theorem 2 can only fail to be true, if

$$r(n) = cn+o(n). \tag{6}$$

But $r(a_k) \leqslant k^2$ ($\leqslant$ number of all sums a_i+a_j with $i, j \leqslant k$) and $r(2a_k) \geqslant \frac{1}{2}k(k-1)$ ($\geqslant$ number of all sums a_i+a_j with $i, j \leqslant k$). It follows that (6) can hold only if either

$$c = 0 \quad \text{and} \quad k^2/a_k \to 0$$

or

$$c > 0 \quad \text{and} \quad Bk^2 < a_k < Ak^2.$$

The first possibility is excluded by the hypothesis of the theorem. It remains to discuss the second case. Then, with the previously used notations, for $\frac{1}{2} < r < 1$,

$$|g(z)| < \Sigma\, r^{Bk^2} < K(1-r)^{-1/2}$$

and

$$\int_{-\pi}^{\pi} |g(z)|^2 d\theta = 2\pi \Sigma\, c_\nu^2 r^{2a_\nu} > 2\pi \Sigma\, c_\nu r^{2a_\nu} > K \Sigma\, r^{2Ak^2} > K(1-r)^{-1/2}.$$

We can take care of the cases I, II and III simultaneously by introducing the symbol ϵ which shall stand for one of the numbers 0, 1, -1.

Then

$$\Big(\Sigma\big(f(k)-c\big)^2 r^{2k}\Big)^{1/2} = \left(\frac{1}{8\pi}\int_{-\pi}^{\pi} |g^2(z)+\epsilon g(z^2)-K(1-z)^{-1}|^2 d\theta\right)^{1/2}$$
$$\geqslant K \int_{-\pi}^{\pi} |g^2(z)+\epsilon g(z^2)-K(1-z)^{-1}|\, d\theta$$
$$\geqslant K \int_{-\pi}^{\pi} |g^2(z)|\, d\theta - K\int_{-\pi}^{\pi} |g(z^2)|\, d\theta - K\int_{-\pi}^{\pi} |1-z|^{-1} d\theta$$
$$\geqslant K(1-r)^{-1/2} - O\left((1-r)^{-1/4} + \log\frac{1}{1-r}\right) > K(1-r)^{-1/2},$$

where Schwarz's inequality was used to estimate the second integral. Therefore

$$\Sigma t_n r^{2n} = (1-r^2)^{-1} \Sigma \big(f(k)-c\big)^2 r^{2k} > K(1-r)^{-2} = K \Sigma n r^n.$$

This implies $$\overline{\lim}\, t_n/n > 0.$$

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ON LINEAR INHOMOGENEOUS DIOPHANTINE APPROXIMATION

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1. The problems discussed in this article are concerned with the evaluation of

$$\liminf_{|x|\to\infty} |x(\phi x+y+\alpha)| \tag{1.1}$$

and

$$\liminf_{x\to+\infty} x|\phi x+y+\alpha|, \tag{1.2}$$

where we suppose that ϕ is irrational and that $\phi x+y+\alpha \neq 0$ for any integral x, y.

In §2 I shall show how the algorithm used in [**1**] and [**2**] for inhomogeneous binary quadratic forms (product of two inhomogeneous linear forms) may be adapted to yield an analytic formulation of (1.1) in terms of semi-regular continued fractions. The method is applied in §3 to give a simple proof of the following theorem, which includes as special cases theorems of Cassels [**3**] and Morimoto [**5**]:

THEOREM 1. *For any k with $0 \leqslant k \leqslant \frac{1}{4}$, there exist $\mathfrak{c}$ values of ϕ, to each of which corresponds $\mathfrak{c}$ values of α, such that*

$$\liminf_{|x|\to\infty} |x(\phi x+y+\alpha)| = k \tag{1.3}$$

(where $\mathfrak{c}$ denotes the cardinal number of the continuum).

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