P. ERDÖS
Technion, Haifa, Israel

On the distribution function of additive arithmetical functions and on some related problems [1]

Conferenza tenuta il 31 gennaio 1956

SUNTO. — Si espongono ricerche e risultati sulla classica questione della esistenza, e delle eventuali proprietà, della funzione distribuzione di una funzione additiva o moltiplicativa.

Si accenna anche ad alcune questioni connesse, quali quella della densità dei numeri abbondanti primitivi.

L'esposizione si chiude con una elencazione di problemi tuttora aperti che si possono prospettare in questo campo di ricerche.

The real valued number-theoretic function $f(n)$ is said to be additive if $f(a \cdot b) = f(a) + f(b)$ for $(a, b) = 1$. It is said to be multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ for $(a, b) = 1$, since the logarithm of a multiplicative function is additive, it will suffice to consider additive functions. The additive function is said to have a distribution function if for every c, $-\infty < c < \infty$ the density $\psi(c)$ of the integers satisfying $f(n) < c$ exists and $\psi(-\infty) = 0$, $\psi(+\infty) = 1$. WINTNER and I [2] proved that the necessary and sufficient condition for the existence of the distribution function is that all the three series

$$\sum_{|f(p)|>1} \frac{1}{p}, \qquad \sum_{|f(p)|\leq 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\leq 1} \frac{f(p)^2}{p}$$

should converge.

The necessary and sufficient condition for the continuity of the distribution function is that [3]

$$\sum_{f(p)\neq 0} \frac{1}{p} = \infty \cdot$$

No necessary and sufficient condition for the absolute continuity of the distribution function is known. It is known that the distribution function of $\varphi(n)/n$ and $\sigma(n)/n$ is purely singular [$\varphi(n)$ denotes EULER's φ function and $\sigma(n) = \sum_{d|n} d$], but one can give additive (and of course multiplicative) functions whose distribution function is an entire function [4].

One of the principal tools of our result with WINTNER is the following result of KAC and myself [5]: Let $f(n)$ be an additive function for which

$$\sum_p \frac{f(p)^2}{p} = \infty \quad \text{and} \quad |f(p)| < c\,.$$

Put

$$A_n = \sum_{p<n} \frac{f(p)}{p}\,, \qquad B_n = \sum_{p<n} \frac{f(p)^2}{p}\,.$$

Then the density of integers n for which $f(n) < A_n + \alpha\, B_n^{1/2}$ equals

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-x^2/2}\, dx\,. \tag{1}$$

The condition $|f(p)| < c$ was clearly unnecessarily restrictive. SHAPIRO [6] observed that if

$$B_n \to \infty \quad \text{and for every} \quad \varepsilon > 0 \quad \sum_{\substack{p<n \\ f(p) > \varepsilon B_n^{1/2}}} \frac{1}{p} = o\,(1)\,, \tag{2}$$

then (1) holds, see also a recent paper by KUBELIUS [7]. SHAPIRO [6] conjectured that the necessary and sufficient condition that (1) hodsl is that (2) should hold.

I proved that if

$$\sum_{f(p) \neq 0} \frac{1}{p} = \infty \quad \text{and} \quad |f(p)| < c\,,$$

then the density of integers for which $f(n+1) > f(n)$ equals $1/2$. Clearly $|f(p)| < c$ is unnecessarily restrictive, but no necessary and sufficient condition is known that the density of integers satisfying $f(n+1) > f(n)$ should be $1/2$.

The first result on the distribution function of additive and multiplicative functions was due to SCHOENBERG [9] who proved that $\varphi(n)/n$ has a distribution function. Later DAVENPORT [10] BEHREND and CHOWLA proved that $\sigma(n)/n$ has a distribution function. I gave a subsequent proof of a special case of this fact which is based on the following idea. A number n is said to be primitive abundant if $\sigma(n) \geq 2n$ but for every divisor d of n, $\sigma(d) < 2d$. Let $a_1 < a_2 < \ldots$ be the sequence of primitive abundant numbers, then the abundant numbers are the integers n which are divisible by at least one a. Next I proved that $\Sigma\ 1/a_i < \infty$, and a simple lemma shows that if $\Sigma\ 1/a_i < \infty$ then the density of integers divisible by at least one a exists. Denote by $N(n)$ the number of the primitive abundant number not exceeding n. I [12] proved that

$$\frac{n}{e^{8(\log n\ \log\log n)^{1/2}}} < N(n) < \frac{n}{e^{\frac{1}{25}(\log n\ \log\log n)^{1/2}}}$$

Unfortunately the above method works only in a few special cases, e. g. one can not even prove by this method that for every α the density of integers $\sigma(n)/n \geq \alpha$ exists. To see this denote by $a_i^{(\alpha)}$, $1 \leq i < \infty$ the integers for which $\sigma(a_i^{(\alpha)}) \geq \alpha\, a_i^{(\alpha)}$ but for every $d | a_i^{(\alpha)}$, $\sigma(d) < \alpha d$ (i. e. the $a_i^{(\alpha)}$ are the primitive α—abundant numbers).

Then it [13] can be shown that for almost all α, $\Sigma \dfrac{1}{a_i^{(\alpha)}} < \infty$, but that the α-s for which $\Sigma \dfrac{1}{a_i^{(\alpha)}} = \infty$ form a set of power c in every interval.

BESICOWITCH [14] was the first to observe that there exists a sequence $a_1 < a_2 < \ldots$ so that the density of the integers $b_1, b_2, \ldots$ which are divisible by at least one a does not exist. In this connection I [15] proved the following theorem: The necessary and sufficient condition that the density of the b-s should esixts is that

$$\lim_{\varepsilon = 0}\ \limsup_{n=\infty}\ \frac{1}{n} \sum_{n^{1-\varepsilon} < a_i \leq n} \Phi(n;\, a_i;\, a_1, a_2, \ldots a_{i-1}) = 0$$

where $\Phi(n;\, a_i;\, a_1, a_2, \ldots a_{i-1})$ denotes the number of integers m not exceeding n for which $m \equiv 0 \pmod{a_i}$ but $m \not\equiv 0 \pmod{a_j}$, $1 \leq j < i$. Thus in particular if $a_k > c\, k \log k$ the density in question always exsists.

DAVENPORT and I [16] proved that the logarithmic density of the b-s always exists i. e. that

$$\lim_{n=\infty} \frac{1}{\log n} \sum_{b_i < n} \frac{1}{b_i}$$

exists. In this connection the following question can be asked: Let $a_1 < a_2 < \ldots$ be a sequence of integers and let $r_1^{(i)}, r_2^{(i)}, \ldots r_{l_i}^{(i)}$ be an arbitrary set of residues (mod a_i). Is it true that the sequence of integers t_j which do not satisfy any of the congruences

$$t_j \equiv r_k^{(i)} \pmod{a_i}, \quad 1 \leq k \leq l_i, \quad 1 \leq i < \infty; \quad t_j \geq a_i$$

have a logarithmic density? [17]

In concluding I would like to state a few unsolved problems. I do not know any necessary and sufficient condition which would imply that $f(n+1)/f(n) \to 1$. If $f(n) = g(n) \log n$ where $g(n)$ tends to infinity sufficiently slowly then it is easy to see that $f(n+1)/f(n) \to 1$ Another problem would be to give a necessary and sufficient condition that $f(n+1)/f(n) \to 1$ should hold if we neglect a sequence of density 0.

No necessary and sufficient conditions are known when a distribution function $\psi(c)$ can be the distribution function of an additive arithmetic function.

I conjectured long ago that the density of the integers n which have two divisors d_1 and d_2 satisfying $d_1 < d_2 < 2\, d_1$ is 1. I proved [15] that the density of these integers exists, but could not prove that the density in question is 1.

It is known [8] that if $f(n+1) \geq f(n)$ for all n then $f(n) = c \log n$. Assume that $f(n+1) \geq f(n)$ except for the n-s which form a sequence of density 0. Does it follow that $f(n) = c \log n$?

I [8] proved that if $f(n+1) - f(n) \to 1$ then $f(n) = c \log n$. In fact my proof gives that if $f(n) \neq c \log n$, then $f(n+1) - f(n)$ must have both a positive and a negative limit point. Assume that $f(n+1) - f(n) \to 0$ for all n if we neglect a sequence of density 0. Does it then follow that $f(n) = c \log n$?

Assume that

$$\sum_{k=1}^{n} |f(k+1) - f(k)| = o(n) .$$

Does it follow that $f(n) = c \log n$? [8]

Assume that $|f(n+1) - f(n)| < c_1$. Does it follow that $f(n) = c \log n + g(n)$, where $|g(n)| < c_2$? [8]

SUMMARY. — In this paper researches and results about the existence and properties of the distribution function of an additive function are illustrated. Some associated questions are also considered; for example the density of primitive abundant numbers.

The paper ends with an enumeration of unsolved problems which can be encountered in these researches.

REFERENCES

[1] Much of the material discussed in this paper is also found in the excellent revue article of M. KAC, *Bull. Amer. Math. Soc.* 55 (1949), 641-665, see also my paper which will soon appear in the Proc. International Math. Congress of Amsterdam.

[2] *Amer. Journal of Math.* 61 (1939), 713-721.

[3] *Journal London Math. Soc.* 13 (1938), 119-127. The result in question is a specia case of a result of Paul LÉVY.

[4] *Amer. Journal of Math.* 61 (1939), 722-725.

[5] *Amer. Journal of Math.* 62 (1940), 738-742.

[6] Written communication.

[7] *Dokladi Akademii Nauk SSSR*, 100 (4) (1955), 623-626.

[8] *Annals of Math.* 47 (1946), 1-20.

[9] *Math. Zeitschrift*, 28 (1928), 171-200. A later paper of SCHOENBERG, *Trans Amer. Math. Soc.* 39 (1936), 315-330 contains very much more general results.

[10] *Berliner Sitzungsberichte (der Akademie)* (1934) 830-834.

[11] *Journal London Math. Soc.* 9 (1934), 278-282.

[12] *Journal London Math. Soc.* 10 (1935), 49-58.

[13] The proof of this result is not published.

[14] *Math. Annalen*, 110 (1934), 336-341.

[15] *Bull. Amer. Math. Soc.* 54 (1948), 685-692.

[16] *Acta Arithmetica*, 2 (1936), 147-151, see also *Indian Journal of Math.* 15 (1951), 19-24.

[17] A simple argument shows that if the condition $t_j \geq a_i$ is replaced by $t_j \geq 0$ the logarithmic density in question does not always exist.