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# REMARKS ON A THEOREM OF RAMSAY

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A graph $G$ is called complete if any two of its vertices are connected by an edge, a set of vertices of $G$ are said to be independent if no two of them are connected by an edge. A well known theorem of Ramsay[1] states that an infinite graph either contains an infinite complete subgraph or an infinite set of independent vertices. He also proved the finite form of this theorem, namely that there exists a function $h(n)$ so that if $G$ has $h(n)$ or more vertices, then either $G$ contains a complete graph of $n$ vertices, or a set of $n$ independent points. Another way of formulating the same theorem is to say that if the edges of a complete graph having $h(n)$ vertices are coloured red or blue, then there exists either a red, or a blue complete graph of $n$ vertices.

Define $f(k,l)$ as the least integer so that every graph of $f(k,l)$ vertices either contains a complete graph of $k$ vertices or contains a set of $l$ independent points, $f(k,l)$ has only been determined for small values of $k$ and $l$[2]. Szekeres[3] proved that

$$f(k,l) \leqslant \binom{k+l-2}{k-1} \tag{1}$$

and there has been no substantial improvement of (1) up to the present time.

I[4] proved that

$$2^{k/2} \leqslant f(k,k) \left(\leqslant \binom{2k-2}{k-1}\right) \tag{2}$$

It seems likely that $\lim_{k=\infty} f(k,k)^{1/k}$ exists, but this has not yet been proved.

(1) implies that $f(3,l) \leqslant \binom{l+1}{2}$, but up to the present it has not even been proved that

$$\lim_{l=\infty} f(3,l)/l = \infty. \tag{3}$$

In the present note I shall prove (3) and in fact considerably more; namely that there exists a constant $c_1 > 0$, so that

$$f(3,l) > l^{1+c_1} \tag{4}$$

Thus at least for $l = 3$, (1) is not so very far from being best possible. I can not decide whether $f(3,l) > c_2 l^2$ is true.

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The lower bound in (2) was obtained in some sense by an existence proof, by using combinatorial-probabilistic arguments and not by actually constructing a graph which satisfies $2^{k/2} < f(k,k)$, on the other hand (4) will be proved by an explicit construction.

Define $g(k,l)$ as the least integer so that every graph having $g(k,l)$ or more vertices contains either a closed polygon of $k$ or fewer vertices (we do not here require that any two vertices of this polygon should be connected by an edge i.e. it does not have to be a complete graph), or a set of $l$ independent points. Clearly $g(3,l) = f(3,l)$. In a subsequent paper I am going to prove that

$$c_3 l^{1+c_4/k} < g(k, l) \text{ and } g(2k+1, l) < c_5 l^{1+1/(k+1)}, g(2k+2, l) < c_5 l^{1+1/(k+1)}$$

Further that for $k > 3$ there exists a graph of $n$ vertices which contains no complete graph of $k$ vertices and no set of $c_6 n^{2/(k+1)} \log n$ independent points.

This implies by a simple computation that for a certain $c_7 > 0$ ($c_7$ independent of $k$ and $l$)

$$f(k,l) > l \binom{k+l-2}{k-1}^{c_7}$$

which shows that for no $k$ and $l$ can (1) be improved very much.

First we outline the proof of (3). Let $k$ be a large integer and consider the lattice points in $k$ dimensional space. The vertices of our graph $G$ will be the lattice points which are contained in a sphere of radius $r$ and whose centre is the origin, $r$ large. It is well known that the number of lattice points in this sphere equals $(1+o(1))r^k V_k$ where $V_k$ denotes the volume of the unit $k$ dimensional unit sphere. Two vertices of our graph are connected if their distance is greater than $r\sqrt{3}$. Clearly our graph contains no triangle since a triangle inscribed in a circle of radius $r$ must have at least one of its sides $\leqslant r\sqrt{3}$. Let $S$ be a set of independent points of $G$. Then the distance between any two points of $S$ is $\leqslant r\sqrt{3}$, or the diameter of the convex hull of $S$ is $\leqslant r\sqrt{3}$ and therefore by a well known theorem[5] its volume is less than or equal to the volume of the $k$ dimensional sphere of radius $r\sqrt{3}/2$ i.e. $(r\sqrt{3}/2)^k V_k$. Thus the number of lattice points of $S$ is less than $(1+o(1))$ $(r\sqrt{3}/2)^k V_k$ which is less than $\varepsilon$ times the number of vertices of $G$ if $k$ is sufficiently large. From this (3) follows immediately.

Now we prove (4). First we prove the following simple

*Lemma:* Let there be given a set of $n$ lattice points in $k$ dimensional space. Then the number of unit cubes which contain at least one of these lattice points as vertices is greater than $n$. Similarly if we have $n$ unit cubes, the number of all the vertices incident on it is greater then $n$.

A cube has $2^k$ vertices and the number of unit cubes containing a given lattice point as vertex is $2^k$. Consider all the unit cubes containing at least one of our vertices. Their number is $2^k n$, clearly some of these cubes occur with a multiplicity $< 2^k$ (e.g. the unit cube farthest away from the origin), thus the number of these unit cubes is greater than $n$, which proves the first half of our lemma. The second half is proved in the same way.

Consider now the sphere $S_r$ of radius $r = 100k^{1/2}$, and centre the origin, (instead of 100 we could have chosen any sufficiently large number). The vertices of $G$ are the lattice points in $S_r$. Two points of $G$ are connected if their distance is $> r\sqrt{3}$. As stated before $G$ contains no triangle. First we estimate from below the number of vertices of $G$. Consider all the unit cubes which are entirely contained in $S_r$. Since the diagonal of the unit cube has length $k^{1/2}$, these cubes cover the sphere of radius $99k^{1/2}$ and therefore the volume of these unit cubes — that is their number — is greater than $\left(\Gamma(u+1) < u^{u+1}/e^u \text{ for } u > u_o\right)$

$$99^k k^{k/2} V_k = 99^k k^{k/2} \frac{\pi^{k/2}}{\Gamma\left(\frac{k+2}{2}\right)} > \frac{1}{k} 99^k \pi^{k/2} (2e)^{k/2}.$$

Therefore by our lemma the number of lattice points in $S_r$ i.e. the number of vertices of $G$ is greater than

$$\frac{1}{k} 99^k (2e\pi)^{k/2}. \tag{5}$$

Consider next the set of independent points of $G$. The distance between any two of them is less than or equal to $100(3k)^{1/2}$. Consider next the set of all unit cubes which have a vertex among these points. The diameter of this set formed by these unit cubes is at most $k^{1/2}(100.3^{1/2}+2)$, and therefore by a well known theorem[5] its volume is not greater then the volume of the sphere of radius $k^{1/2}(50.3^{1/2}+1) < 90k^{1/2}$, or the number of these cubes is less than the volume of the sphere of radius $90k^{1/2}$ i.e.

$$90^k k^{k/2} \frac{\pi^{k/2}}{\Gamma\left(\frac{k+2}{2}\right)} < 90^k (2e\pi)^{k/2} \quad \left(\Gamma(u+1) > u^u/e^u \text{ for } u > u_0\right).$$

But by our lemma the number of independent points is less than the number of cubes. thus finally the maximum number of independent points of $G$ is less than

$$90^k (2e\pi)^{k/2}.$$

Now choose $k$ to be the greatest integer for which

$$90^k (2e\pi)^{k/2} \geqslant l. \tag{6}$$

Then the maximum number of independent points of $G$ is $\leqslant l$, and from (5) and (6) we obtain by a simple calculation that the number of vertices of $G$ is greater than $l^{1+c_1}$, which completes the proof of (4).

A graph is called $k$-chromatic if its vertices can be numbered by $k$ integers so that no two vertices which have the same number are connected, and this can not be done with $k - 1$ integers. As far as I know Tutte was the first to construct for every $k$ a finite $k$-chromatic graph which contains no triangle. Denote by $\varphi(k)$ the smallest integer $n$, for which there exists a $k$-chromatic graph of $n$ vertices which does not contain a triangle. It easily follows from (1) that $\varphi(k) > c_8 k^2$, Tutte's upper bound for $\varphi(k)$ was very large. Several authors independently rediscovered Tutte's result, but all of them (as far as I know) obtained very large upper bounds ($\varphi(k) < c_9{}^k$). It is easy to see that (4) implies $\varphi(k) < k^{c_{10}}$.

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