A THEOREM ON PARTIAL WELL-ORDERING OF SETS OF VECTORS

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Let S be an abstract set and n an ordinal number. Denote† by $W_S(n)$ the set of all vectors over S of "length" n, *i.e.* the set of all mappings $\nu \to x_\nu$ of $[0, n)$ into S. As notation for such a vector we use juxtaposition of its components: $X = x_0 x_1 \ldots \hat{x}_n$. Put

$$W_S(<n) = \Sigma(m<n)\, W_S(m).$$

Suppose that $x \leqslant y$ is a quasi-order ([**1**], p. 4) on S. This order induces ([**2**]) a quasi-order on $W_S(<n)$ by means of the following rule. If $X,\ Y \in W_S(<n)$ then the relation $X \leqslant Y$ holds if, and only if,

$$X = x_0 \ldots \hat{x}_r;\ Y = y_0 \ldots \hat{y}_s;\ x_\rho, y_\sigma \in S,\ x_\rho \leqslant y_{t(\rho)} \text{ for } \rho < r$$

for some suitable sequence of ordinals $t(\rho)$ such that

$$t(0) < t(1) < \ldots < \hat{t}(r) < s.$$

Now suppose that, inparticular, S is partially well-ordered (PWO). This means ([**3**]) that whenever $z_0, \ldots, \hat{z}_\omega \in S$ there are indices α, β such that $\alpha < \beta < \omega$; $z_\alpha \leqslant z_\beta$. G. Higman proved ([**2**]) that, whenever S is PWO then $W_S(<\omega)$ is PWO. On the other hand ([**3**]) there are PWO sets S such that $W_S(\omega)$ is not PWO.

Denote by $V_S(n)$ the set of all vectors of $W_S(n)$ which have only a finite number of distinct components, and put $V_S(<n) = \Sigma(m<n)\, V_S(m)$. In [**3**] the conjecture was put forward that, for every n, $V_S(<n)$ is PWO whenever S is PWO, and this was proved for $n = \omega^3$. In the present note we prove the conjecture for every $n < \omega^\omega$. Our method is much simpler than the method used in [**3**] for the special case $n = \omega^3$. The same result, $V_S(<n)$ is PWO for every $n < \omega^\omega$, has been obtained by J. Kruskal (not published) who very kindly showed his manuscript to us and in this way stimulated the present investigation. His proof is considerably more complicated than ours.

The following result is known ([**3**], Theorem 4).

(i) *If $V_S(n)$ is PWO then $V_S(<n\omega)$ is PWO.*

We want to prove the following proposition.

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† $[0, n)$ denotes the semi-open interval of ordinals $\{\nu:\ 0 \leqslant \nu < n\}$. The effect of the "obliteration operator" $\hat{}$ consists in removing from a well-ordered series the term above which it is placed. Set-union is denoted by $A+B$ and $\Sigma(\nu \leqslant n)\, A_\nu$, and instead of ω_0 we write ω.

(ii) *If S is PWO and $n<\omega^\omega$ then $V_S(<n)$ is PWO.*

Denote by N the class of all* ordinals n such that whenever S is PWO then $V_S(n)$ is PWO. Then, by (i), the proposition (ii) is equivalent to

(iii) $[0, \omega^\omega)\subseteq N$.

Clearly, (iii) is a corollary of the following theorem.

THEOREM. *If $n\,\varepsilon\, N$ then $\omega n\,\varepsilon\, N$.*

We now proceed to prove this theorem. Let $n\,\varepsilon\, N$, and let S be PWO.

1. Let $X=x_0\ldots\hat{x}_\omega\,\varepsilon\, V_S(\omega)$; $x_0, \ldots, \hat{x}_\omega\,\varepsilon\, S$. Then we can write

$$X=f(X)g(X),$$

where $$f(X)=x_0\ldots x_{k-1};\ g(x)=x_k\ldots\hat{x}_\omega;\ k<\omega.$$

We choose k so large that

$$\{x_k, \ldots, \hat{x}_\omega\}=\{x_\nu, \ldots, \hat{x}_\omega\} \text{ for } k\leqslant\nu<\omega.$$

This means that every element of S which occurs among the components of $g(X)$ occurs infinitely often among these components.

Let $\{x_k, \ldots, \hat{x}_\omega\}=\{z_0, \ldots, z_{r-1}\}$; $z_\mu\neq z_\nu$ for $\mu<\nu<r$. Put

$$h(X)=(z_0z_1\ldots z_{r-1})^\omega=z_0z_1\ldots z_{r-1}z_0z_1\ldots z_{r-1}z_0\ldots,$$

so that $h(X)\,\varepsilon\, V_S(\omega)$. Then

$$g(X)\leqslant h(X)\leqslant g(X) \tag{1}$$

since the required mappings can obviously be constructed step by step.

2. Let $Y\,\varepsilon\, V_S(\omega n)$. Then $Y=Y_0Y_1\ldots\hat{Y}_n$, where each $Y_\nu\,\varepsilon\, V_S(\omega)$. Put

$$\phi(Y)=f(Y_0)h(Y_0)f(Y_1)h(Y_1)\ldots\hat{f}(Y_n)\hat{h}(Y_n).$$

Then $\phi(Y)\,\varepsilon\, V_T(\gamma)$ for some γ, where $T=S+V_S(\omega)$. The elements of S are considered as incomparable with the elements of $V_S(\omega)$. As components of $\phi(Y)$ we take the components, which are in S, of the finite vectors $f(Y_\nu)$, as well as the whole vectors $h(Y_\nu)$, which are in $V_S(\omega)$. We must note here that in view of the definition of $h(Y_\nu)$ there are only a finite number of distinct vectors among the $h(Y_\nu)$. Let $f(Y_\nu)$ have $p(\nu)-1$ components. Then $1\leqslant p(\nu)<\omega$ and, if $\omega m\leqslant n<w(m+1)$, we have

$$\gamma=p(0)+\ldots+\hat{p}(n)=\Sigma(\mu<m)\Big(p(\omega\mu)+p(\omega\mu+1)+\ldots+\hat{p}(\omega\mu+\omega)\Big)$$
$$+p(\omega m)+\ldots+\hat{p}(n)<\omega m+\omega\leqslant n+\omega.$$

* No logical difficulties arise from this notation.

Thus $\phi(Y) \varepsilon V_T(<n+\omega)$. Now T, as union of two PWO sets, is itself PWO. It follows from (i) that $V_T(<n+\omega)$ is PWO. For if $n<\omega$ then $n+\omega=\omega$, and if $n \geqslant \omega$ then $n+\omega \leqslant n+n < n\omega$.

3. Let $Z_0, \ldots, \hat{Z}_\omega \varepsilon V_S(\omega n)$. Then $\phi(Z_\nu) \varepsilon V_T(<n+\omega)$, and by what has been said at the end of §2 there are indices α, β such that $\alpha<\beta<\omega$,

$$\phi(Z_\alpha) \leqslant \phi(Z_\beta). \tag{2}$$

More accurately, (2) holds in the following sense. We have

$$\begin{gathered}\phi(Z_\alpha)=a_0 a_1 \ldots \hat{a}_p;\ \phi(Z_\beta)=b_0 b_1 \ldots \hat{b}_q;\ a_\mu, b_\nu \varepsilon T, \\ a_\mu \leqslant b_{t(\mu)} \text{ for } \mu<p, \\ t(0)<\ldots<\hat{t}(p)<q.\end{gathered} \tag{3}$$

But then (2) holds also in the ordinary sense of the quasi-order for vectors over S. For in (3) there are two cases to consider.

Case 1. $a_\mu, b_{t(\mu)} \varepsilon S$. For such μ we leave (3) unaltered.

Case 2. $a_\mu, b_{t(\mu)} \varepsilon V_S(\omega)$. For such μ (3) is equivalent to the existence of a certain system of infinitely many inequalities between the components (in S) of the vectors a_μ and $b_{t(\mu)}$, and we replace (3) by such a system of inequalities. The total system of inequalities thus obtained implies that (2) holds also in the sense of the order among vectors over S.

We now have, by (1),

$$Z_\alpha \leqslant \phi(Z_\alpha) \leqslant \phi(Z_\beta) \leqslant Z_\beta.$$

We conclude that the set $V_S(\omega n)$ is PWO. Since S was arbitrary, this implies $\omega n \varepsilon N$, and the theorem is proved.

References.

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