SEQUENCES OF LINEAR FRACTIONAL TRANSFORMATIONS

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A point set E in the extended z-plane will be called an SD (set of divergence) provided there exists a sequence of transformations

$$T_n(z) = (a_n z + b_n)/(c_n z + d_n)$$

that diverges at each z in E and converges at each z in the complement of E. In the present paper, we give a topological characterization of the SD's that lie on a straight line.

We also characterize the denumerable SD's. But for this purpose, topological ideas are not sufficient (see [1, p. 133]), and we introduce a geometric analogue to the concept of a limit point.

1. SETS OF DIVERGENCE ON A STRAIGHT LINE

THEOREM 1. *If a set* E *lies on a straight line, it is an* SD *if and only if it is of type* $G_{\delta\sigma}$.

The necessity of the condition follows immediately from the fact that the transformations T_n are continuous, in the extended plane.

In proving the sufficiency, we may assume, without loss of generality, that the set E lies on the extended real axis. If E coincides with the extended real axis, it is of type G_δ; this case is covered by Theorem 3 of [1]. In the other case, we may assume that the point $z = \infty$ does not belong to E, so that E can be represented in the form

$$E = \bigcup_{j=1}^{\infty} E_j, \qquad E_j = \bigcap_{k=1}^{\infty} E_{jk},$$

where for each j the family $\{E_{jk}\}_{k=1}^{\infty}$ constitutes a decreasing sequence of open sets on the segment $(-j/2, j/2)$ of the real axis. (Even if E is empty, we may assume that none of the sets E_{jk} is empty.) For each index pair (j, k), we denote by $\{E_{jkp}\}$ the finite or denumerable family of components (a_{jkp}, b_{jkp}) of E_{jk}. With each interval E_{jkp}, we associate a domain B_{jkp} bounded by E_{jkp} and by arcs of the two parabolas

$$y = (jkp)^{-1}(x - a_{jkp})^2, \qquad y = (jkp)^{-1}(x - b_{jkp})^2. \tag{1}$$

We construct a denumerable set of circular disks D_{jkpq} (see Figure 1) with centers $z_{jkpq} = x_{jkpq} + i y_{jkpq}$, subject to the following three requirements:

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(i) each disk D_{jkpq} lies in B_{jkp}, and its boundary is tangent to E_{jkp};

(ii) the sequence $\{z_{jkpq}\}_{q=1}^{\infty}$ has a_{jkp} and b_{jkp} as its only limit points;

(iii) each point of E_{jkp} lies on the orthogonal projection of one of the disks D_{jkpq}.

We observe that conditions (i) to (iii) are consistent with the further requirement that

$$y_{jkpq} < (b_{jkp} - a_{jkp})(j^2 kpq)^{-1} < (jkpq)^{-1}, \tag{2}$$

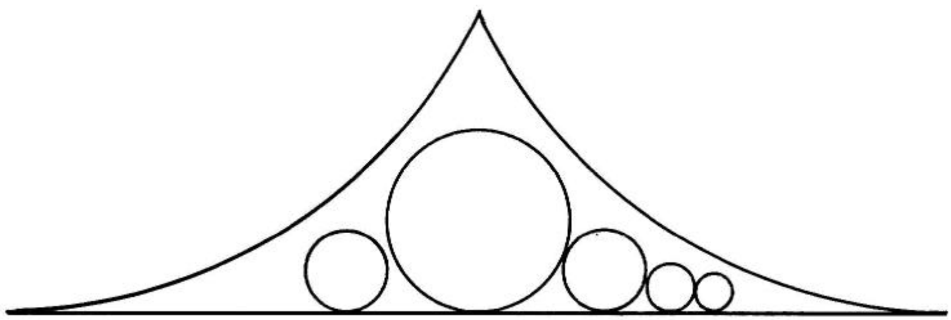

Fig. 1

and we shall assume that this condition is also satisfied. Now let the family of transformations

$$T_{jkpq}(z) = \frac{y_{jkpq}}{j(z - z_{jkpq})}$$

be arranged into a simple sequence $\{T_n\}$. We shall prove that the sequence $\{T_n(z)\}$ diverges everywhere in E and converges to 0 everywhere in the complement of E.

Note first that the value of $|T_{jkpq}(z)|$ is $1/j$ on the boundary of D_{jkpq}, and that it is inversely proportional to $|z - z_{jkpq}|$. By (2), no point z lies in infinitely many of the disks D_{jkpq}, and it follows immediately that

$$\liminf_{n\to\infty} |T_n(z)| = 0,$$

for each z in the plane. Also, if $z \in E_j$, then $|T_n(z)| > 1/2j$, for infinitely many n. This establishes the divergence of $\{T_n(z)\}$ on E.

Suppose next that $z = x + iy \notin E$, and let $\varepsilon > 0$. If $y \neq 0$, then $y_{jkpq} < |y|/2$, except for finitely many index sets (j, k, p, q). For all except these finitely many index sets, condition (2) gives the inequalities

$$|T_{jkpq}(z)| < \frac{y_{jkpq}}{j|y - y_{jkpq}|} < \frac{2y_{jkpq}}{j|y|} < 2(|y|j^2 kpq)^{-1}.$$

Since the last member is less than ε, with at most finitely many exceptions, $T_n(x + iy) \to 0$ if $y \neq 0$.

If $y = 0$, then z lies outside of each of the disks D_{jkpq}, and therefore the inequality $|T_{jkpq}(x)| \leq j^{-1}$ holds for each index set (j, k, p, q). Hence the inequality $|T_{jkpq}(x)| < \varepsilon$ holds for each index set (j, k, p, q) with $j > 1/\varepsilon$. For each of the exceptional values $j = 1, 2, \cdots, [1/\varepsilon]$, there exist at most finitely many index pairs (k, p) for which $x \in E_{jkp}$. Condition (2) implies that if $z \in E_{jkp}$, then

$$|T_{jkpq}(x)| < \frac{(j^2kpq)^{-1}}{|x - x_{jkpq}|},$$

and the right member clearly approaches 0 as $q \to \infty$. Therefore it remains only to deal with the index sets (j, k, p, q) for which $j \leq 1/\varepsilon$ and $x \notin E_{jkp}$. Here we note that

$$|T_{jkpq}(x)| \leq \max\{|T_{jkpq}(a_{jkp})|, |T_{jkpq}(b_{jkp})|\}.$$

By symmetry, it is sufficient to show that the first of the expressions in the braces is less than ε for all except finitely many of the index sets (j, k, p, q) with $j \leq 1/\varepsilon$. By the construction of the parabolas (1),

$$(3) \qquad |T_{jkpq}(a_{jkp})| < \frac{y_{jkpq}}{j(x_{jkpq} - a_{jkp})} < (j^2kp)^{-1}(x_{jkpq} - a_{jkp}),$$

and by condition (2),

$$(4) \qquad |T_{jkpq}(a_{jkp})| < (b_{jkp} - a_{jkp})(j^2kpq)^{-1}(x_{jkpq} - a_{jkp})^{-1}.$$

For those index sets (j, k, p, q) for which x_{jkpq} lies in the left half of E_{jkp}, the last member of (3) is less than ε, with at most finitely many exceptions. For those index sets for which $x_{jkpq} \geq (a_{jkp} + b_{jkp})/2$, the second member of (4) is not greater than $2(j^2kpq)^{-1}$. This concludes the proof of Theorem 1.

DENUMERABLE SETS OF DIVERGENCE

Corresponding to any point set E in the plane, we define the set $gd(E)$ by the rule that $z \in gd(E)$ provided, for each $\varepsilon > 0$, there exists a $\delta > 0$ with the following property: if $|t - z| < \delta$, then some w in E satisfies the inequality $|w - t| < \varepsilon|t - z|$. Roughly speaking, $z \in gd(E)$ provided the complement of E does not contain arbitrarily small disks that subtend a fixed angle $\theta(z)$ at the point z. We point out, for example, that if E is the set $|z| \leq 1$, then $gd(E)$ is the set $|z| < 1$; and that if E is the classical two-dimensional Cantor set, then $gd(E)$ is empty.

The set E^1 is defined by the rule $E^1 = E \cap gd(E)$. For each ordinal α, we write

$$E^\alpha = E^{\alpha-1} \cap gd(E^{\alpha-1}) \qquad (\alpha \text{ of the first kind}),$$

$$E^\alpha = \bigcap_{\beta<\alpha} E^\beta \qquad (\alpha \text{ of the second kind}).$$

THEOREM 2. *A denumerable set* E *is an* SD *if and only if there exists an ordinal* α *such that the set* E^α *is empty.*

To prove the necessity of the condition, suppose that E is a denumerable set for which E^α is not empty, for any α. Then clearly there exists an ordinal β ($\beta < \Omega$, where Ω denotes the first nondenumerable ordinal) such that $E^{\beta+1} = E^{\beta}$. We proceed to show that if a sequence $\{T_n(z)\}$ diverges everywhere in E^{β}, then the SD of $\{T_n\}$ is not denumerable.

Without loss of generality, we may assume that $T_n(z) = a_n/(z - t_n)$, and that $T_n(z) \to 0$ for each z for which the sequence converges (see [1, Section 2]). Let w_0 be any point in E^{β}. Then there exists a constant $h_0 > 0$ and a sequence $\{n_k\}$ such that $|T_{n_k}(w_0)| > h_0$ for all k. We may suppose that $t_{n_k} \to w_0$, since otherwise the SD of $\{T_n\}$ contains an open disk and is therefore not denumerable.

If $t_{n_1} \neq w_0$, let D_0 denote the disk $|z - t_{n_1}| < h_0 |w_0 - t_{n_1}|$. Then the inequality $|T_{n_1}(z)| > 1$ holds throughout D_0. Also, since $E^{\beta+1} = E^{\beta}$, the disk D_0 contains two points w_{00} and w_{11} of E^{β} (provided the point t_{n_1} lies near enough to w_0, a condition which is certainly satisfied if n_1 is chosen large enough).

If $t_{n_1} = w$, there also exists two points w_{00} and w_{01} of E^{β} in whose neighborhoods $|T_{n_1}(z)| > 1$.

In either case, there exist two disjoint disks D_{00} and D_{01} in which $|T_{n_1}(z)| > 1$ and in which some $T_{n_{00}}(z)$ and $T_{n_{01}}(z)$ $(n_{00} > n_0,\ n_{01} > n_0)$, respectively, have modulus greater than 1. By a familiar argument, the continuation of the construction leads to a nondenumerable point set throughout which $\limsup |T_n(z)| \geq 1$. This proves the necessity of the condition.

To prove the sufficiency of the condition, we suppose that E is a denumerable set $\{z_m\}$ $(m = 1, 2, \cdots)$, and that E^{α} is empty for some ordinal α. Then, for each index m, there exists a unique ordinal $\beta = \beta(m)$ such that $z_m \in E^{\beta} - E^{\beta+1}$. Also, for each m, there exists a constant ε_m $(0 < \varepsilon_m < 1/m)$ such that each deleted neighborhood $0 < |z - z_m| < \varepsilon$ $(\varepsilon < \varepsilon_m)$ contains a disk N^* subtending an angle $4\varepsilon_m$ at z_m and containing no points of $E^{\beta(m)}$. Since E is denumerable, we can replace N^* by a concentric subdisk N whose boundary does not meet the set E, whose closure does not meet the set $E^{\beta(m)}$ (and therefore does not meet any of the sets E^{γ} with $\gamma \geq \beta(m)$), and whose center w and radius r satisfy the condition $r > \varepsilon_m |z_m - w|$. Corresponding to each index m, we shall need a sequence $\{N_{mj}\}$ $(j = 1, 2, \cdots)$ of disks having these properties, and subject to the condition that the centers w_{mj} converge to z_m as $j \to \infty$. Our collection of disks N_{mj} $(m, j = 1, 2, \cdots)$ must satisfy the further restriction that if two disks N_{mj} and N_{nk} intersect, then one contains the other, and that if $N_{mj} \subset N_{nk}$, then $\beta(m) < \beta(n)$.

To construct the collection $\{N_{mj}\}$, we order the index pairs (m, j) into a sequence, and corresponding to the first index pair we choose the disk N_{mj} in any manner consistent with the specifications listed in the preceding paragraph. Suppose that a finite number of choices have been made, and that (m, j) is the first of the index pairs for which the disk N_{mj} has not been selected. Since z_m does not lie on the boundary of any of the disks that have been chosen, we can choose N_{mj} in such a way that $|z_m - w_{mj}| < 1/mj$, and in such a way that each of the previously constructed disks that meet N_{mj} contains it entirely. Moreover, since z_m lies in none of the disks N_{nk} with $\beta(n) \leq \beta(m)$, we can stipulate that N_{mj} lies in none of these disks.

Finally, we define the transformations

$$T_{mj}(z) = \varepsilon_m^2 (z_m - w_{mj})/(z - w_{mj})$$

and arrange them into a sequence $\{T_n\}$. Since $|T_{mj}(z)| < \varepsilon_m$ outside of N_{mj}, and $T_{mj}(z_m) = \varepsilon_m^2$, the sequence $\{T_n(z)\}$ diverges at each point of E. Suppose, on the other hand, that z is not one of the points z_m; then $\{T_n(z)\}$ certainly converges if z lies in only finitely many of the disks N_{mj}. But for each z, the disks containing z form a nested sequence, and the corresponding ordinals $\beta(m)$ form a decreasing sequence. Since a decreasing sequence of ordinals is finite, $T_n(z) \to 0$ for all z outside of E.

It is evident that if, corresponding to a set M of natural numbers, we delete from $\{T_{mj}\}$ all elements for which $m \in M$, then every sequence formed from the remaining transformations converges at each z_m with $m \in M$. This proves the following theorem (and thus settles Problem 2 in [1]).

THEOREM 3. *If* E *is a denumerable* SD, *then every subset of* E *is an* SD.

REFERENCE

1. G. Piranian and W. J. Thron, *Convergence properties of sequences of linear fractional transformations*, Michigan Math. J. 4 (1957), 129-135.

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