A CONSTRUCTION OF GRAPHS WITHOUT TRIANGLES HAVING PRE-ASSIGNED ORDER AND CHROMATIC NUMBER

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1. *Introduction and statement of result.*

The chromatic number $\chi(\Gamma)$ of a combinatorial graph Γ is the least cardinal number a such that the set of nodes of Γ can be divided into a subsets so that every edge of Γ joins nodes belonging to different subsets. It is known† that corresponding to every finite a there exists a finite graph Γ_a without triangles satisfying $\chi(\Gamma_a) = a$. In [**1**], Theorem 2, we have extended this result to transfinite values of a. For every graph Γ the order $\phi(\Gamma)$, *i.e.* the cardinal of the set of nodes of Γ, satisfies $\phi(\Gamma) \geqslant \chi(\Gamma)$. The construction used in [**1**] was of considerable complexity and did not allow us to prove that it was most economical, *i.e.* that it leads to a graph Γ_a such that $\phi(\Gamma_a) = a$. This equation was only established ([**1**], Theorem 3) when essential use was made of a form of the general continuum hypothesis.

In the present note we describe a much simpler construction of such a graph Γ_a and we shall at the same time prove, without using the continuum hypothesis, that our new graph Γ_a satisfies $\phi(\Gamma_a) = \chi(\Gamma_a) = a$. Trivially, for instance by adding isolated nodes to the graph, we can make its order equal to any given cardinal b such that $b \geqslant a$, without changing the chromatic number or introducing any triangles.

Theorem. *Given $a \geqslant \aleph_0$, there is a graph Γ_a without triangles such that*

$$\phi(\Gamma_a) = \chi(\Gamma_a) = a.$$

The proof depends on some lemmas, each a special case of a more general proposition. An essential part is played by Lemma 4, which is an adaptation of a result due to Specker [**2**].

2. *Notation.*

We use the notation set out in [**1**], §2. Every small letter, unless the contrary is stated, denotes an ordinal. The order type of an ordered set A is denoted by $\operatorname{tp} A$. If $A, B, \ldots$ are elements of an ordered set then the symbol $\{A, B, \ldots\}_<$ denotes the set $\{A, B, \ldots\}$ and at the same time expresses the fact that $A < B < \ldots$. For a cardinal r, the partition relation‡

$$\alpha \to (\beta_0, \beta_1, \ldots, \hat{\beta}_n)^r \tag{1}$$

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† [**3**], [**4**], [**5**].

‡ The obliteration operator ^ removes from a well-ordered sequence the term above which it is placed.

expresses the fact that whenever $\operatorname{tp} A = \alpha$; $[A]^r = \Sigma(\nu < n) K_\nu$ there is a subset B of A and an ordinal $\nu < n$ such that $\operatorname{tp} B = \beta_\nu$; $[B]^r \subset K_\nu$. If $\theta_0 = \ldots = \hat{\beta}_n = \beta$ we write (1) also in the form

$$\alpha \to (\beta)^r_{|n|}.$$

The logical negation of (1) is denoted by

$$\alpha \nrightarrow (\beta_0, \ldots, \hat{\beta}_n)^r.$$

3. *Lemmas.*

Throughout Lemmas 1–5 we denote by α a fixed ordinal such that either $\alpha = \omega_0$ or α is of the form $\omega_{\lambda+1}$. In the proofs of Lemmas 2, 3, 5 only the case $\alpha = \omega_{\lambda+1}$ is considered. The case $\alpha = \omega_0$ can be dealt with by making the obvious modifications and is easier.

Lemma 1. *Let β be an ordinal and c a cardinal such that*

$$\alpha \to (\alpha)_c^1; \ \beta \to (\beta)_c^1.$$

Then
$$\alpha\beta \to (\alpha\beta)_c^1.$$

Proof. Let $S = \{(y, x) : x < \alpha; \ y < \beta\}$, and order S lexicographically. Then $\operatorname{tp} S = \alpha\beta$. Let $|N| = c$; $S = \Sigma(\nu \in N) S_\nu$. Choose any $y < \beta$. Put $A_\nu(y) = \{x : (y, x) \in S_\nu\}$ $(\nu \in N)$. Then, since every $x < \alpha$ is a member of some $A_\nu(y)$, $[0, \alpha) = \Sigma(\nu \in N) A_\nu(y)$, and by $\alpha \to (\alpha)^1_{|N|}$ there is an element $\nu(y)$ of N with $\operatorname{tp} A_{\nu(y)}(y) \geqslant \alpha$. Put $B_\nu = \{y : \nu(y) = \nu\}$ $(\nu \in N)$. Then, since y can take any value less than β, $[0, \beta) = \Sigma(\nu \in N) B_\nu$, and by $\beta \to (\beta)^1_{|N|}$ there is $\nu_0 \in N$ such that $\operatorname{tp} B_{\nu_0} \geqslant \beta$. Then $\operatorname{tp} A_{\nu_0}(y) \geqslant \alpha$ $(y \in B_{\nu_0})$, and the set $D = \{(y, x) : y \in B_{\nu_0}; \ x \in A_{\nu_0}(y)\}$ satisfies

$$D \subset S_{\nu_0}; \quad \operatorname{tp} S_{\nu_0} \geqslant \operatorname{tp} D = \alpha\beta.$$

This proves Lemma 1.

Lemma 2. $\alpha^3 \to (\alpha^3)_p^1$ *for every cardinal p such that $p < |\alpha|$.*

Proof. We need only consider the case $\alpha = \omega_{\lambda+1}$; $p = \aleph_\lambda$. Let $[0, \alpha) = \Sigma(\nu < \omega_\lambda) S_\nu$. If for all $\nu < \omega_\lambda$ we have $|S_\nu| \leqslant \aleph_\lambda$ then the contradiction $\aleph_{\lambda+1} \leqslant \Sigma(\nu < \omega_\lambda) |S_\nu| \leqslant \aleph_\lambda^2 = \aleph_\lambda$ follows. Hence there is $\nu_0 < \omega_\lambda$ with $|S_{\nu_0}| = \aleph_{\lambda+1}$, and so $\operatorname{tp} S_{\nu_0} = \alpha$. This proves $\alpha \to (\alpha)^1_{\aleph_\lambda}$, and Lemma 2 follows by two applications of Lemma 1.

Lemma 3. *Let $k < \omega_0$, and let V be a set of vectors $(x_0, \ldots, \hat{x}_k)$ with $x_0, \ldots, \hat{x}_k < \alpha$, ordered lexicographically. Let $\operatorname{tp} V = \alpha^k$. Then there are sets $T_\nu(x_0, \ldots, \hat{x}_\nu) \subset [0, \alpha)$ with $\operatorname{tp} T_\nu(x_0, \ldots, \hat{x}_\nu) = \alpha$ $(\nu < k; \ x_0, \ldots, \hat{x}_\nu < \alpha)$ such that the relations $x_\nu \in T_\nu(x_0, \ldots, \hat{x}_\nu)$ $(\nu < k)$ imply $(x_0, \ldots, \hat{x}_k) \in V$.*

Proof. Let $\alpha = \omega_{\lambda+1}$. The assertion holds for $k = 0$. Let $k \geqslant 1$, and use induction with respect to k. Put

$$f(x_0) = \{(x_1, \ldots, \hat{x}_k) : (x_0, x_1, \ldots, \hat{x}_k) \in V\} \quad (x_0 < \alpha).$$

Then $\quad \operatorname{tp} f(x) \leqslant \alpha^{k-1}\ (x<\alpha);\ \operatorname{tp} V = \Sigma(x<\alpha)\operatorname{tp} f(x).$

Put $\quad T_0 = \{x : \operatorname{tp} f(x) = \alpha^{k-1}\}.$

Assume that $\quad \operatorname{tp} T_0 < \alpha.$

Then $\operatorname{tp} T_0 < \omega_{\lambda+1};\ |T_0| \leqslant \aleph_\lambda$, and T_0 is not cofinal in $[0, \alpha)$. There is $\beta < \alpha$ with $T_0 \subset [0, \beta)$. If $k = 1$ then the contradiction

$$\alpha = \operatorname{tp} V = \Sigma(x<\beta)\operatorname{tp} f(x) \leqslant \beta$$

follows. Now let $k \geqslant 2$. Then $\operatorname{tp} f(x) \leqslant \alpha^{k-2}\delta(x)$ where

$$\delta(x) < \alpha;\ \ |\delta(x)| \leqslant \aleph_\lambda \quad (\beta \leqslant x < \alpha).$$

If $\beta \leqslant \gamma < \alpha$ then

$$|\delta(\beta)+\ldots+\hat{\delta}(\gamma)| \leqslant \aleph_\lambda |\gamma| \leqslant \aleph_\lambda;\quad \delta(\beta)+\ldots+\hat{\delta}(\gamma) < \omega_{\lambda+1} = \alpha.$$

Hence $\sigma = \delta(\beta)+\ldots+\hat{\delta}(\alpha) \leqslant \alpha$, and we obtain the contradiction

$$\operatorname{tp} V \leqslant \Sigma(x<\beta)\,\alpha^{k-1} + \Sigma(\beta \leqslant x < \alpha)\,\alpha^{k-2}\delta(x) = \alpha^{k-1}\beta + \alpha^{k-2}\sigma$$
$$\leqslant \alpha^{k-1}(\beta+1) < \alpha^k.$$

Hence the assumption is false, and $\operatorname{tp} T_0 = \alpha$.

Let $x_0 \in T_0$. By induction hypothesis, applied to $f(x_0)$, there are sets

$$T_\nu(x_0, \ldots, \hat{x}_\nu) \subset [0, \alpha) \quad (1 \leqslant \nu < k;\ x_1, \ldots, \hat{x}_\nu < \alpha)$$

with $\quad \operatorname{tp} T_\nu(x_0, \ldots, \hat{x}_\nu) = \alpha \quad (1 \leqslant \nu < k;\ x_1, \ldots, \hat{x}_\nu < \alpha)$

such that whenever

$$x_\nu \in T_\nu(x_0, \ldots, \hat{x}_\nu) \quad (1 \leqslant \nu < k)$$

then $(x_1, \ldots, \hat{x}_k) \in f(x_0)$. Put

$$T_\nu(x_0, \ldots, \hat{x}_\nu) = [0, \alpha) \quad (1 \leqslant \nu < k;\ x_0 \in [0, \alpha) - T_0;\ x_1, \ldots, \hat{x}_\nu < \alpha).$$

Then the sets T_ν $(\nu < k)$ satisfy the assertion of Lemma 3.

LEMMA 4. $\alpha^3 \not\to (3, \alpha^3)^2$.

Proof. Put $S = \{(x, y, z) : x, y, z < \alpha\}$ and order S lexicographically. Then $\operatorname{tp} S = \alpha^3;\ [S]^2 = K_0 + K_1;\ K_0 K_1 = \varnothing$,

$$K_0 = \Big\{\{(a_0, a_1, a_2), (b_0, b_1, b_2)\}_< : a_1 < b_0 < a_2 < b_1 < \alpha\Big\}.$$

If ordinals a_ν, b_ν, c_ν satisfy

$$[\{(a_0, a_1, a_2), (b_0, b_1, b_2), (c_0, c_1, c_2)\}_<]^2 \subset K_0$$

then the contradiction $a_2 < b_1 < c_0 < a_2$ follows.

If, on the other hand, a subset V of S satisfies $\operatorname{tp} V = \alpha^3;\ [V]^2 \subset K_1$ then there are sets T_ν which have, for $k = 3$, the properties mentioned in Lemma 3. Then there are ordinals a_ν, b_ν such that

$$a_0 \in T_0;\quad a_1 \in T_1(a_0) - [0, a_0+1);\quad b_0 \in T_0 - [0, a_1+1),$$
$$a_2 \in T_2(a_0, a_1) - [0, b_0+1);\quad b_1 \in T_1(b_0) - [0, a_2+1);\quad b_2 \in T_2(b_0, b_1).$$

But then the contradiction $\{(a_0, a_1, a_2), (b_0, b_1, b_2)\}_< \varepsilon K_0[V]^2 = \emptyset$ follows. This proves Lemma 4.

LEMMA 5. *There is a graph* Γ *without triangles such that, if* $\chi(\Gamma) = e$,

$$\phi(\Gamma) = |\alpha|; \quad \alpha^3 \nrightarrow (\alpha^3)_e^1.$$

Proof. Let $\alpha = \omega_{\lambda+1}$; $\text{tp}\, S = \alpha^3$. By Lemma 4 there is a partition $[S]^2 = K_0 + K_1$ such that (i) there is no $A \subset S$ such that $\text{tp}\, A = 3$; $[A]^2 \subset K_0$, (ii) there is no $B \subset S$ such that $\text{tp}\, B = \alpha^3$; $[B]^2 \subset K_1$. Put $\Gamma = (S, K_0)$, Then Γ has no triangle, and $\phi(\Gamma) = |S| = |\alpha^3| = \aleph_{\lambda+1}$. Let $|N| = \chi(\Gamma)$. Then there is a function g from S into N such that $g(x) = g(y)$ implies $\{x, y\} \notin K_0$. Then $S = \Sigma(\nu \varepsilon N)\, S_\nu$, where $S_\nu = \{x : g(x) = \nu\}$ $(\nu \varepsilon N)$. Let $\nu \varepsilon N$. If $x, y \varepsilon S_\nu$, then $g(x) = \nu = g(y)$; $\{x, y\} \notin K_0$. Hence $[S_\nu]^2 \subset K_1$; whence by (ii) above $\text{tp}\, S_\nu < \alpha^3$. This proves $a^3 \nrightarrow (\alpha^3)_{|N|}^1$ and completes the proof of Lemma 5.

Proof of the Theorem.

Case 1. $a = \aleph_0$. By Lemma 5, with $\alpha = \omega_0$, there is a graph Γ without triangles such that $\phi(\Gamma) = \aleph_0$; $\omega_0{}^3 \nrightarrow (\omega_0{}^3)_e{}^1$, where $e = \chi(\Gamma)$. By Lemma 2 it follows that $e \geqslant \aleph_0$. Hence $\aleph_0 \leqslant \chi(\Gamma) \leqslant \phi(\Gamma) = \aleph_0$, and we may put $\Gamma_a = \Gamma$.

Case 2. $a > \aleph_0$. Put $M = \{b^+ : \aleph_1 \leqslant b^+ \leqslant a\}$, where b^+ denotes the next larger cardinal to the cardinal b. Then $\aleph_1 \varepsilon M$; $|M| \leqslant a$. Let $c = b^+ \varepsilon M$. Then $b = \aleph_\lambda$ for some λ. Put $\alpha = \omega_{\lambda+1}$. By Lemma 5 there is a graph Γ_c' without triangles such that $\phi(\Gamma_c') = \aleph_{\lambda+1}$; $\alpha^3 \nrightarrow (\alpha^3)_e{}^1$, where $e = \chi(\Gamma_c')$. Then, by Lemma 2, $e \geqslant c$. We can arrange that $\Gamma_c' = (A_c, B_c)$, where $A_{c_0} A_{c_1} = \emptyset$ $(\{c_0, c_1\}_< \subset M)$. Put

$$\Gamma_a = \Big(\Sigma(c \varepsilon M)\, A_c,\ \Sigma(c \varepsilon M)\, B_c\Big).$$

Then $\chi(\Gamma_a) \geqslant \chi(\Gamma'_{\aleph_1}) \geqslant \aleph_1$. If $\chi(\Gamma_a) = d < a$, then $\aleph_2 \leqslant d^+ \leqslant a$; $d^+ \varepsilon M$, and we obtain the contradiction $\chi(\Gamma_a) \geqslant \chi(\Gamma'_{d^+}) \geqslant d^+$. Hence

$$a \leqslant \chi(\Gamma_a) \leqslant \phi(\Gamma_a) = |\Sigma(c \varepsilon M)\, A_c| \leqslant \Sigma(c \varepsilon M)\, a = a|M| \leqslant a,$$

and the theorem is proved.

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