ON SOME EXTREMUM PROBLEMS IN ELEMENTARY GEOMETRY

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(Received May 20, 1960)

Dedicated to the memory of Professor L. FEJÉR

1. Let S denote a set of points in the plane, $N(S)$ the number of points in S. More than 25 years ago we have proved [2] the following conjecture of ESTHER KLEIN—SZEKERES:

There exists a positive integer $f(n)$ with the property that if $N(S) > f(n)$ then S contains a subset P with $N(P) = n$ such that the points of P form a convex n-gon.

Moreover we have shown that if $f_0(n)$ is the smallest such integer then $f_0(n) \le \binom{2n-4}{n-2}$ and conjectured that $f_0(n) = 2^{n-2}$ for every $n \ge 3$.[1] We are unable to prove or disprove this conjecture, but in § 2 we shall construct a set of 2^{n-2} points which contains no convex n-gon. Thus

$$2^{n-2} \le f_0(n) \le \binom{2n-4}{n-2}.$$

A second problem which we shall consider is the following: It was proved by SZEKERES [3] that

(i) In any configuration of $N = 2^n + 1$ points in the plane there are three points which form an angle ($\le \pi$) greater than $(1 - 1/n + 1/nN^2)\pi$.

(ii) There exist configurations of 2^n points in the plane such that each angle formed by these points is less than $(1 - 1/n)\pi + \varepsilon$ with $\varepsilon > 0$ arbitrarily small.

The first statement shows that for sufficiently small $\varepsilon > 0$ there are no configurations of $2^n + 1$ points which would have the property (ii). Hence in a certain sense this is a best possible result; but it does not determine the exact limiting value of the maximum angle for any given $N(S)$.

Let $\alpha(m)$ denote the greatest positive number with the property that in every configuration of m points in the plane there is an angle β with

$$\beta \geqq \alpha(m). \tag{1}$$

* This paper was written while P. ERDŐS was visiting at the University of Adelaide.

[1] The conjecture is trivial for $n = 3$; it was proved by Miss KLEIN, for $n = 4$ and by E. MAKAI and P. TURÁN for $n = 5$.

From (i) and (ii) above it follows that $\alpha(m)$ exists for every $m \geq 3$ and that for $2^n < m \leq 2^{n+1}$,

(2) $$[1 - 1/n + 1/n(2^n + 1)^2]\,\pi \leq \alpha(m) \leq [1 - 1/(n+1)]\,\pi.$$

Two questions arise in this connection:

1. What is the exact value of $\alpha(m)$.
2. Can the inequality (1) be replaced by

(3) $$\beta > \alpha(m)$$

For the first few values of m one can easily verify that

$$\alpha(3) = \frac{1}{3}\pi,\ \alpha(4) = \frac{1}{2}\pi,\ \alpha(5) = \frac{3}{5}\pi,\ \alpha(6) = \alpha(7) = \alpha(8) = \frac{2}{3}\pi,$$

and that the strict inequality (3) is true for $m = 7$ and 8. For $3 \leq m \leq 6$ the regular m-gons represent configurations in which the maximum angle is equal to $\alpha(m)$; but we know of no other cases in which the equality sign would be necessary in (1).

In § **4** we shall prove

THEOREM 1. *Every plane configuration of 2^n points ($n \geqq 3$) contains an angle greater than* $(1 - 1/n)\pi$.

The theorem shows in conjunction with (ii) above that for $n \geq 3$, $\alpha(2^n) = (1 - 1/n)\pi$ and that the strict inequality (3) holds for these values of m. The problem is thus completely settled for $m = 2^n$, $n \geq 2$.

It is not impossible that $\alpha(m) = (1 - 1/n)\pi$ for $2^{n-1} < m < 2^n$, $n \geq 4$, and that (3) holds for every $m > 6$. However, we can only prove that for $0 < k < 2^{n-1}$, $\alpha(2^n - k) \geq (1 - 1/n)\pi - k\pi/2(2^n - k)$ (Theorem 2).

Finally we mention the following conjecture of P. ERDŐS : Given $2^n + 1$ points in n-space, there is an angle determined by these points which is greater than $\frac{1}{2}\pi$. For $n = 2$ the statement is trivial, for $n = 3$ it was proved by **N. H. KUIPER** nad A. H. BOERDIJK.[2] For $n > 3$ the answer is not known; and it seems that the method of § 4 is not applicable to this problem. **

2. In this section we construct a set of 2^{n-2} points in the plane which contains no convex n-gon. For representation we use the Cartesian (x, y) plane. All sets to be considered are such that no three points of the set are collinear.

A sequence of points

$$(x_\nu, y_\nu),\ \nu = 0, 1, \ldots, k, \qquad x_0 < x_1 < \ldots < x_k$$

is said to be *convex*, of length k, if

$$\frac{y_\nu - y_{\nu-1}}{x_\nu - x_{\nu-1}} < \frac{y_{\nu+1} - y_\nu}{x_{\nu+1} - x_\nu} \qquad \text{for } \nu = 1, \ldots, k-1;$$

[2] Unpublished.

** (*Added in proof:* This conjecture was recently proved by DANZER and GRÜNBAUM in a surprisingly simplex way.)

concave, of length k, if the same is true with the inequality sign reversed. It was shown in [2] that a set of more than $\binom{k+l-2}{k-1}$ points must contain either a concave sequence of length k or a convex sequence of length l. We have also stated, without proof, that there exists a set S_{kl} of $f(k, l) = \binom{k+l-2}{k-1}$ points which contains no concave sequence of length k and no convex sequence of length l. We shall first construct an explicit example of such a set. S_{kl} consists of points

$$[x, g_{kl}(x)], \; x = 1, \ldots, \binom{k+l-2}{k-1}$$

where $g_{kl}(x)$ is defined inductively as follows:

(a) $$g_{k1}(1) = g_{1l}(1) = 0.$$

(b) If $k > 1$, $l > 1$, then

$$g_{kl}(x) = g_{k,l-1}(x) \qquad \text{for } 1 \leq x \leq \binom{k+l-3}{k-1},$$

$$g_{kl}(x) = g_{k-1,l}\left[x - \binom{k+l-3}{k-1}\right] + c_{kl} \text{ for } \binom{k+l-3}{k-1} < x \leq \binom{k+l-2}{k-1}$$

where

$$c_{kl} = \text{Max}\left\{\binom{k+l-2}{k-1} g_{k,l-1}\left(\binom{k+l-3}{k-1}\right), \; \binom{k+l-2}{k-1} g_{k-1,l}\left(\binom{k+l-3}{k-2}\right)\right\}.$$

Clearly the construction is such that $g_{kl}(x)$ is monotone increasing and every slope in S_{kl} is positive.

Now if A denotes the set of the first $f(k, l-1) = \binom{k+l-3}{k-1}$ points of S_{kl} and B the set of the last $f(k-1, l)$ points then a concave sequence in S_{kl} which contains two points in A cannot contain a point in B and a convex sequence in S_{kl} which contains two points in B cannot contain a point in A. For the maximum slope in A is $< g_{k,l-1}\left(\binom{k+l-3}{k-1}\right)$ and the maximum slope in B is $< g_{k-1,l}\left(\binom{k+l-3}{k-2}\right)$ so that B is entirely above any line connecting points of A and A is entirely below any line connecting points of B, by the definition of c_{kl}. Hence the maximum concave sequence in S_{kl} has length $k-1$ and the maximum convex sequence has length $l-1$.

To construct a set S of 2^{n-2}points which contains no convex n-gon, we proceed as follows: Let

$$a_k = 2\operatorname{Max}\left\{\left(n-k-\frac{1}{2}\right)g_{k,n-k}\left(\binom{n-2}{k-1}\right)+\binom{n-2}{k-1},\right.$$
$$\left.\left(n-k+\frac{1}{2}\right)g_{k+1,n-k-1}\left(\binom{n-2}{k}\right)+\binom{n-2}{k}\right\}+1$$

($k = 1, \ldots, n-2$), and define S_k, $k = 1, \ldots, n-1$ as follows:

Set $S_1 = S_{1,n-1}$,

$$S_{k+1} = S_{k,n-k} + \left(\sum_{i=1}^{k}(n-i)\,a_i, -\sum_{i=1}^{k} a_i\right),\quad k = 1,\ \ldots,\ n-2.$$

Then

$$S = \bigcup_{k=1}^{n-1} S_k$$

has the required property.

The number of points in S is

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1} = 2^{n-2},$$

so we only have to show that every convex polygon in S has less than n sides. Note that S_1 consists of the point (1,0) alone and $(x, y) \in S_k$, $k > 1$ implies $x > 0$, $y < 0$. Also

$$\frac{a_k - g_{k+1,n-k-1}\left(\binom{n-2}{k}\right)}{(n-k)\,a_k + \binom{n-2}{k}} > \frac{1}{n-k+\frac{1}{2}},$$

$$\frac{a_k - g_{k,n-k}\left(\binom{n-2}{k-1}\right)}{(n-k)\,a_k - \binom{n-2}{k-1}} < \frac{1}{n-k-\frac{1}{2}} \qquad (k = 1,\ \ldots,\ n-2),$$

so that the slope of any line connecting S_k and S_{k+1} is less than $-1\Big/\left(n-k+\frac{1}{2}\right)$ and greater than $-1\Big/\left(n+k-\frac{1}{2}\right)$. Therefore the slope of any line connecting S_k and S_l, $1 \le k < l \le n-1$ is less than $-1\Big/\left(n-k+\frac{1}{2}\right)$ and greater than $-1\Big/\left(n-l+\frac{1}{2}\right)$.

Suppose now that P_i, $(i = 1, \ldots, r)$ is a non-empty subset of S_{k_i}, $1 \leq k_1 < \ldots < k_r \leq n-1$ and such that $P = \bigcup_{i=1}^{r} P_i$ forms a convex polygon. Since the slope of lines within each S_{k_i} is positive, P_i for $1 < i < r$ consists of a single point and P_1 must form a concave sequence, P_r a convex sequence. But then the total number of points in P is at most

$$k_1 + (k_r - k_1 - 1) + (n - k_r) = n - 1.$$

3. The proof of Theorem 1 requires a refinement of the method used in [3], and in this section we set up the necessary graph-theoretical apparatus.

We denote by $C^{(N)}$ the complete graph of order N, i. e. a graph with N vertices in which any two vertices are joined by an edge. If G is a graph, then $S(G)$ shall denote the set of vertices of G. If A is a subset of $S(G)$, then $G|A$ denotes the restriction of G to A. An even (odd) circuit of G is a closed circuit containing an even (odd) number of edges.

A *partition* of G is a decomposition $G = G_1 + \ldots + G_n$ into subgraphs G_i with the following property: Each G_i consists of all vertices and some edges of G such that each edge of G appears in one and only one G_i (G_i may not contain any edge at all). We call a partition $G = G_1 + \ldots + G_n$ *even* if it has the property that no G_i contains an odd circuit.

LEMMA 1. *If a graph G contains no odd circuit then its vertices can be divided in two classes, A and B, such that every edge of G has one endpoint in A and one in B.*

This is well-known and very simple to prove, e. g. [1], p 170.

LEMMA 2. *If* $C^{(N)} = G_1 + \ldots + G_n$ *is an even partition of the complete graph* $C^{(N)}$ *into n parts then* $N \leq 2^n$.

This Lemma was proved in [3]; for the sake of completeness we repeat the argument. Since G_1 contains no odd circuit we can divide $S(C^{(N)})$ in classes A and B, containing N_1 and N_2 vertices respectively, such that each edge of G_1 connects a point of A with a point of B. But then $G_1 + \ldots + G_n$ induces an even partition $G_2' + \ldots + G_n'$ of $G' = C^{(N)} | A$ and since G' is a complete graph of order N_1, we conclude by induction that $N_1 \leq 2^{n-1}$. Similarly $N_2 \leq 2^{n-1}$ hence $N = N_1 + N_2 \leq 2^n$.

To prove Theorem 1 we shall need more precise information about the structure of even partitions of $C^{(N)}$, particularly in the limiting case of $N = 2^n$. The following Lemmas have some interest of their own; they are formulated in greater generality than actually needed for our present purpose.

LEMMA 3. *Let* $N = 2^n$ *and* $C^{(N)} = G_1 + \ldots + G_n$ *an even partition of* $C^{(N)}$ *into n parts. Then the total number of edges emanating from a fixed vertex* $p \in S(C^{(N)})$ *in* $G_{j_1} + \ldots + G_{j_i}$, *where* $1 \leq j_1 < j_2 < \ldots < j_i \leq n$, *is at least* $2^i - 1$ *and at most* $2^n - 2^{n-i}$.

Clearly the order in which the components G_i are written is immaterial, therefore we can assume in the proof that $j_\nu = \nu$, $\nu = 1, \ldots, i$. We also note that the first statement follows from the second one by applying the latter to the complementary partition $G_{i+1} + \ldots + G_n$ and by noting that the number of edges from p in $G_1 + \ldots + G_n$ is $2^n - 1$. We shall prove the second statement in the form that there are at least 2^{n-i} vertices in $S(C^{(n)})$ (including p itself) which are not joined with p in $G_1 + \ldots + G_i$. The statement is trivial for $n = 1$; we may therefore assume the Lemma for $n - 1$.

By assumption, G_1 contains no odd circuit. Therefore by Lemma 1 we can divide its vertices into two classes, A and B, such that no two points of A (or of B) are joined in G_1. Both classes A and B contain 2^{n-1} vertices; for otherwise one of them, say A, would contain more than 2^{n-1} vertices and $C^{(N)}| A$ would have an even partition $G_2' + \ldots + G_n'$ into $n - 1$ parts, contrary to Lemma 2.

Let A be the class containing p. Hence there are at least 2^{n-1} vertices with which p is not joined in G_1. This proves the Lemma for $i = 1$. Suppose $i > 1$ and consider the partition $G_2' + \ldots + G_n'$ of $C^{(N)}| A$, induced by $G_1 + G_2 + \ldots + G_n$. Since the order of $C^{(N)}| A$ is 2^{n-1}, we find by the induction hypothesis that there are at least $2^{n-1-(i-1)} = 2^{n-i}$ vertices in A to which p is not joined in $G_2' + \ldots + G_i'$. But p is not joined with any vertex of A in G_1, therefore it is not joined with at least 2^{n-i} vertices in $G_1 + \ldots + G_i$.

In the special case of $i = 1$ we obtain

Lemma 3.1. *Let $N = 2^n$ and $C^{(N)} = G_1 + \ldots + G_n$ an even partition. Then every p is an endpoint of at least one edge in every G_i.*

If $N < 2^n$, then Lemma 3.1 is no longer true, but the number of vertices for which it fails cannot exceed $2^n - N$. More precisely we shall prove

Lemma 4. *Let $N = 2^n - k$, $0 \le k < 2^n$, and $C^{(N)} = G_1 + \ldots + G_n$ an even partition of $C^{(N)}$ into n parts. Denote by $\nu(p)$, $p \in S = S(C^{(N)})$ the number of graphs G_i in which there is no edge from p. Then*

$$\sum_{p \in S} (2^{\nu(p)} - 1) \le k.$$

Proof. For $n = 1$ the statement is trivial, therefore assume the Lemma for $n - 1$. Let $q_1, \ldots, q_j$ be the "exceptional" vertices in G_1 from which there are no edges in G_1 and denote by Q the union of vertices q_i, $i = 1, \ldots, j$. (Q may be empty). By Lemma 1, S is the union of disjoint subsets A, B and Q such that every edge in G_1 has one endpoint in A and one in B. Denote by A_1 the union of A and Q, by B_1 the union of B and Q. Let a and b be the number of vertices in A and B respectively. Then $a + b + j = 2^n - k$ and $a + j \le 2^{n-1}$, $b + j \le 2^{n-1}$ by Lemma 2, applied to $C^{(N)}|A_1$ and $C^{(N)}| B_1$. Write $a = 2^{n-1} - j - k_1$, $b = 2^{n-1} - j - k_2$ so that $k_1 \ge 0$, $k_2 \ge 0$ and $2^n - j - k_1 - k_2 = 2^n - k$,

$$k = j + k_1 + k_2. \tag{4}$$

By applying the induction hypothesis to $C^{(N)}|A_1$ and to the partition $G_2' + \ldots + G_n'$ induced by $G_2 + \ldots + G_n$, we find

$$\sum_{p \in A} (2^{\mu(p)} - 1) + \sum_{p \in Q} (2^{\mu(p)-1} - 1) \le k_1$$

for some $\mu(p) \ge \nu(p)$. Therefore *a fortiori*

$$\sum_{p \in A} (2^{\nu(p)} - 1) + \sum_{p \in Q} (2^{\nu(p)-1} - 1) \le k_1$$

and similarly

$$\sum_{p \in B} (2^{\nu(p)} - 1) + \sum_{p \in Q} (2^{\nu(p)-1} - 1) \leq k_2.$$

Hence

$$\sum_{p \in S} (2^{\nu(p)} - 1) - j \leq k_1 + k_2 = k - j$$

by (4), which proves the Lemma.

4. Before proving Theorem 1 we introduce some further notations and definitions. To represent points in the Euclidean plane E we shall sometimes use the complex plane which will also be denoted by E. If q_1, p, q_2 are points in E, not on one line and in counterclockwise orientation, the angle ($< \pi$) formed by the lines pq_1 and pq_2 will be denoted by $A(q_1 \, p \, q_2)$.

A set of points S in E is said to have the *property* P_n if the angle formed by any three points of S is not greater than $(1 - 1/n)\,\pi$. We shall briefly say that S is P_n or not P_n according as it has or has not this property.

A direction α in E is a vector from 0 to $e^{i\alpha}$ on the unit circle. An *n-partition* of E with respect to the direction α is a decomposition of $E - \{0\}$ into sectors T_k, $k = 1, \ldots, 2n$ where T_k consists of all points

$$z = r\, e^{i(\alpha+\varphi)},\ r > 0,\ (k-1)\frac{\pi}{n} \leq \varphi < k\,\frac{\pi}{n}.$$

With every set of points $S = \{p_1, \ldots, p_N\}$ and every n-partition of E with respect to some direction α we associate a partition $C^{(N)} = G_1 + \ldots + G_n$ of $C^{(N)}$ in n parts according to the following rule: p_μ, p_ν are joined in G_i if and only if the vector from p_μ to p_ν is in one of the sectors T_i, T_{n+i}.

The following Lemma was proved in [3].

Lemma 5. *If the set S is P_n then the partition $C^{(N)} = G_1 + \ldots + G_n$ associated with any given n-partition of E is necessarily even.*

We shall also need

Lemma 6. *If $p_1\, p_2 \ldots p_n$ are consecutive vertices of a regular n-gon P and q is a point distinct from the centre and inside P then there is a pair of vertices (p_i, p_j) such that $A(p_i q\, p_j) > (1 - 1/n)\,\pi$.*

The proof is quite elementary; if p_i is a vertex nearest to q and if q is in the triangle $p_i\, p_j\, p_{j+1}$ then at least one of the angles $A(p_j\, q\, p_i)$, $A(p_i\, q\, p_{j+1})$ is $> (1 - 1/n)\,\pi$.[3]

Lemma 5 and Lemma 2 give immediately the result that a set of $2^n + 1$ points in the plane cannot be P_n. Our purpose, however, is to prove Theorem 1 which can be stated as follows:

Theorem 1*. *A set of 2^n points in the plane is not P_n.*

Proof. Let S be a set of $N = 2^n$ $(n > 2)$ points in the plane, $p_1, p_2, \ldots, p_k$ the vertices of the least convex polygon of S, in cyclic order and counterclockwise orientation. We can assume that no three vertices of S are collinear, otherwise S is obviously not P_n. We distinguish several cases.

[3] See Problem 4086, *Amer. Math. Monthly*, **54** (1947), p. 117. Solution by C. R. Phelps.

Suppose first that there is an angle $A(p_{i-1}p_ip_{i+1}) < (1-1/n)\pi$. Let α be the direction p_ip_{i+1} and $\sigma(\alpha)$ the n-partition of E with respect to α. Then there are no points of S in the sectors T_1 and T_{n+1} corresponding to $\sigma(\alpha)$, hence there is no edge from the vertex p_i in the component G_n of the associated partition $C^{(N)} = G_1 + \ldots + G_n$. We conclude from Lemma 3.1 that $G_1 + \ldots + G_n$ is not an even partition, hence by Lemma 5, S is not P_n.

Henceforth we assume that all angles $A(p_{i-1}p_ip_{i+1})$ are equal to $(1-1/n)\pi$ so that the least convex polygon has $2n$ vertices. For convenience let these vertices be (in cyclic order and counterclockwise orientation) $p_1q_1p_2q_2\ldots p_nq_n$, and denote by a_i the side length p_iq_i and by b_i the side q_ip_{i+1}.

Suppose that all angles $A(p_{i-1}p_ip_{i+1})$ and $A(q_{i-1}q_iq_{i+1})$ are equal to $(1-2/n)\pi$. Then an elementary argument shows that the triangles $p_{i-1}q_{i-1}p_i$, $q_{i-1}p_iq_i$, $p_iq_ip_{i+1}$ etc are similar, hence

$$\frac{a_{i-1}}{a_i} = \frac{b_{i-1}}{b_i} = \frac{a_i}{a_{i+1}} = \ldots$$

which implies $a_1 = a_2 = \ldots = a_n$, $b_1 = b_2 = \ldots = b_n$. We conclude that $p_1p_2\ldots p_n$ is a regular n-gon.

Now let q be any point of S inside the least convex polygon. If q is inside the triangle $p_iq_ip_{i+1}$ then clearly $A(p_iqp_{i+1}) > (1-1/n)\pi$. Therefore we may assume that each q inside the least convex polygon is already inside $p_1\ldots p_n$. Since $n \geq 3$, there are at least two such points, hence we may assume that q is not the centre. But then by Lemma 6, $A(p_iqp_j) > (1-1/n)\pi$ for suitable i and j.

The last remaining case to be considered is when not all angles $A(p_{i-1}p_ip_{i+1})$ are equal; then for some i, $A(p_{i-1}p_ip_{i+1}) < (1-2/n)\pi$. Since $A(p_{i-1}q_{i-1}p_i) = A(p_iq_ip_{i+1}) = (1-1/n)\pi$, we may assume that there are no points of S inside the triangles $p_{i-1}q_{i-1}p_i$ and $p_iq_ip_{i+1}$. But then if α is the direction of $p_i\,p_{i+1}$ and $\sigma(\alpha)$ the corresponding n-partition of E, $C^{(N)} = G_1 + \ldots + G_n$ the partition of $C^{(N)}$ associated with $\sigma(\alpha)$, then in $G_{n-1} + G_n$ there are only two edges running from p_i, namely p_iq_{i-1} and p_iq_i. By Lemma 3 (with $i = 2$, $j_1 = n-1$, $j_2 = n$), $C^{(N)} = G_1 + \ldots + G_n$ is not an even partition and by Lemma 5, S is not P_n.

Finally we prove

Theorem 2. *In a plane configuration of* $N = 2^n - k$ *points* $(0 < k < 2^{n-1})$ *there is an angle* $\geq (1 - 1/n - k/2N)\pi$.

Proof. Suppose that all angles formed by the points of S are $\leq (1-1/n)\pi - \frac{1}{2}\delta - \delta'$ where $\delta = k\pi/N$ and $\delta' > 0$. Let $p \in S$ and

$$\varphi = \varphi(p) = A(q_1pq_2) \leq (1-1/n)\pi - \frac{1}{2}\delta - \delta', q_1 \in S,\ q_2 \in S$$

the maximum angle at p. If there are several such angles, make an arbitrary but fixed choice.

Let $\alpha = \alpha(p)$ be the direction of $p q_1$ so that there are no lines $p q$, $q \in S$ in the sectors

$$z = \pm r e^{i(\alpha+\Theta)}, \quad 0 < \Theta < (1 - 1/n)\pi - \frac{1}{2}\delta - \delta'.$$

If S has $N = 2^n - k$ points, then there are N pairs of directions $\pm\alpha(p)$. Hence there is a direction β such that at least $k+1$ directions $\varepsilon_\nu \alpha(p_\nu)$, $\nu = 0, 1, \ldots, k$, $\varepsilon_\nu = \pm 1$ are in the interval

$$\beta < \varepsilon_\nu \alpha(p_\nu) < \beta + \delta + \frac{1}{3}\delta'.$$

By a slight displacement of β we can achieve that all directions in S should be different from the direction $\beta + \frac{3}{4}\delta$.

Consider now the n-partition $\sigma\left(\beta + \frac{3}{4}\delta\right)$ of E with respect to $\beta + \frac{3}{4}\delta$ and construct the associated partition $C^{(N)} = G_1 + \ldots + G_n$ with the following modification: Every $p q$ with direction between $\beta + \frac{3}{4}\delta$ and $\beta + \delta + \frac{1}{3}\delta'$ shall be joined in G_n instead of G_1 and every $p q$ with direction between $\beta + \frac{1}{2}\delta + \frac{\pi}{n}$ and $\beta + \frac{3}{4}\delta + \frac{\pi}{n}$ shall be joined in G_2 instead of G_1. With this modification it is still true that the partition $C^{(N)} = G_1 + \ldots + G_n$ is even if S has no angle greater or equal to $(1 - 1/n)\pi - \frac{1}{2}\delta - \delta'$. But it is easy to see that the vertices p_ν are not joined with any vertex of $C^{(N)}$ in G_1. For if $\beta + \frac{3}{4}\delta < \varepsilon_\nu \alpha(p_\nu) < \beta + \delta + \frac{1}{3}\delta'$ then the edges emanating from p_ν in the sectors T_1, T_{n+1} are counted to G_n; and if $\beta < \varepsilon_\nu \alpha(p_\nu) < \beta + \frac{3}{4}\delta$, the only possible edges from p_ν in the sectors T_1, T_{n+1} are in directions between $\beta + \frac{1}{2}\delta + \frac{\pi}{n}$ and $\beta + \frac{3}{4}\delta + \frac{\pi}{n}$ and these are counted to G_2. Thus we have a contradiction with Lemma 4 and the Theorem is proved.

NOTE ADDED IN PROOF. In the case of $k = 1$, Theorem 2 can be improved as follows.

THEOREM 3. Every plane configuration of $2^n - 1$ points $(n \geq 2)$ contains an angle not less than $(1 - 1/n)\pi$.

The theorem shows in particular that $\alpha(2^n - 1) = (1 - 1/n)\pi$, but we cannot decide whether the strict inequality (3) is valid for $m = 2^n - 1$.

PROOF. Suppose $N(S) = 2^n - 1$ and all angles in S are less than $(1 - 1/n)\pi$. Let q be an interior point of S, that is one which is not on the least convex polygon. Let

$$A(q_1 q q_2) = (1 - 1/n)\pi - \delta, \qquad \delta > 0$$

be the largest angle at q. If β is the direction of $q q_1$, $\alpha = \beta + \frac{1}{2}\delta$, $\sigma(\alpha)$ the corresponding n-partition of E, $C^{(N)} = G_1 + \ldots + G_n$ the partition of $C^{(N)}$ associated with $\sigma(\alpha)$, then clearly there are no edges from q in G_1. We conclude from Lemma 4 and Lemma 5 that from all other vertices of S there is at least one edge in every G_i $(i = 1, \ldots, n)$. We show that this leads to a contradiction.

Let $P = (p_1, \ldots, p_m)$ denote the least convex polygon of S. Since each angle in P is less than $(1 - 1/n)\pi$, we have $m \leq 2n - 1$. Therefore if $T_1, \ldots, T_{2n}$ are the sectors of $\sigma(\alpha)$ there is a p_i such that if $p_{i-1}p_i$ is in T_{k-1} then $p_i p_{i+1}$ is not in T_k. But then there is no edge from p_i in G_k, as easily seen by elementary geometry.

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