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Restricted cluster sets

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Let f be a complex-valued function defined in the upper half-plane H. The *cluster set* $C(f, x)$ at a point x on the real axis is defined as the set of all values w (including possibly $w = \infty$) for which there exists a sequence $\{z_n\} = \{x_n + i\, y_n\}$ such that $x_n \to x$, $y_n \downarrow 0$, and $f(z_n) \to w$. We use the term *restricted cluster set* generically: the word "restricted" indicates either that the sequence $\{z_n\}$ occurring in the definition above is subject to special conditions (for example, that it lie on a line segment), or that the cluster set $C(f, x, *)$ is defined in terms of unions or intersections of certain "primitive" cluster sets.

1. An extension of a theorem of Collingwood

By a STOLZ *angle at the origin* we mean a triangular domain with one vertex at the origin and the other two vertices on a common horizontal line in H. Corresponding to each STOLZ angle Δ at the origin and each real number x, we denote by Δ_x the image of Δ under the translation that carries the origin to the point x. By $C(f, x, \Delta)$ we denote the set of values w for which there exists a sequence $\{z_n\}$ in Δ_x such that $z_n \to x$ and $f(z_n) \to w$. For each function f and each real number x, we use the symbol $C(f, x, \delta)$ for the intersection of all sets $C(f, x, \Delta)$.

COLLINGWOOD [2, Theorem 2], has proved that if f is continuous in H, then $C(f, x, \delta) = C(f, x)$, except for a set of values x which is of first category. (The proposition is actually announced for meromorphic functions; but the proof uses only continuity.) We shall now show that the hypothesis of continuity can be dropped from COLLINGWOOD's theorem.

Theorem 1. *If f is a complex-valued function in H, there exists a residual set of values x for which $C(f, x, \delta) = C(f, x)$.*

(*Added January 4, 1960*): We have just learned that E. F. COLLINGWOOD gave a proof of this theorem, in W. HAYMAN's seminar, on December 10, 1958.)

For the sake of notational convenience, we assume, throughout the proof, that f is bounded; the proof becomes valid for the general case provided the metric of the plane is replaced by the metric of the RIEMANN sphere.

Let $\varepsilon > 0$; let w_0 denote a fixed complex number, and Δ a fixed STOLZ angle at the origin. We denote by $E(f, \Delta, w_0, \varepsilon)$ the set of values x for which the disk $|w - w_0| < \varepsilon$ meets the set $C(f, x)$ while the closed disk $|w - w_0| \leq \varepsilon$ does not meet the set $C(f, x, \Delta)$.

Lemma. *$E(f, \Delta, w_0, \varepsilon)$ is of first category.*

Suppose that the lemma is false. Then there exists a positive number h and a subset E_h of $E(f, \Delta, w_0, \varepsilon)$ which is dense in some interval I and satisfies the following condition: if $x \in E_h$, $z \in \Delta_x$, and $\Im z < h$, then $|f(z) - w_0| > \varepsilon$. Since the union of the domains Δ_x $(x \in E_h)$ is a trapezoid with the base I, the disk $|w - w_0| < \varepsilon$ can not meet the set $C(f, x)$. This in turn implies that $E(f, \Delta, w_0, \varepsilon)$ does not meet the interior of the interval I, and therefore the lemma is true.

Let $\{w_n\}$ be a set of N numbers such that for each z in H the distance between $f(z)$ and the set $\{w_n\}$ is less than $\varepsilon/2$. By the lemma, the union of the N sets $E(f, \Delta, w_n, \varepsilon)$ is of first category. This implies that the set of points x for which $C(f, x)$ contains a point at a distance greater than 2ε from $C(f, x, \Delta)$ is of first category.

Next we assign to ε successively the values 1, 1/2, 1/4, ..., and we see that $C(f, x, \Delta) = C(f, x)$, except on a residual set. Finally, we can select a sequence $\{\Delta^{(m)}\}$ of STOLZ angles at the origin such that each STOLZ angle at the origin contains one of the $\Delta^{(m)}$, and the proof of the theorem is complete.

Let λ denote a JORDAN arc which lies in H except for one endpoint at the origin; let λ_x denote the image of λ under the translation that carries the origin to x; and let $C(f, x, \lambda)$ denote the cluster set of f at x along λ_x. COLLINGWOOD [2, Theorem 1] proved that if f is continuous in H, then $C(f, x, \lambda) = C(f, x)$ for all x in some residual set. We point out that this result can be obtained by a slight modification of our proof: since f is uniformly continuous, in each compact subset of H, there exists a domain Δ^* that contains all points of λ except the origin, and such that, with obvious notation, $C(f, x, \Delta^*) = C(f, x, \lambda)$ for each x (in cases where the continuity of f deteriorates rapidly near the x-axis, the domain Δ^* is very narrow near the origin). If the role of the fixed STOLZ angle Δ chosen at the beginning of the proof of Theorem 1 is assigned to Δ^*, the proof can be carried out as before.

Our next result shows that in the conclusion of Theorem 1, no assertion concerning the exceptional set can be made, except that it is of first category. Notation: $C(f, x, S)$ is the union of all sets $C(f, x, \Delta)$.

Theorem 2. *If E is a set of first category on the real axis, there exists a function f in H such that $C(f, x, S) = \{0\}$ for each point x in E, while $C(f, x)$ is the extended plane, for each real x.*

If E is of first category, it is contained in the union of disjoint closed sets F_j, each nowhere dense [3]. Let Δ_1 denote a triangular region, symmetric with respect to the y-axis, and with an angle $\pi - 1$ at the origin; and let

$$D_1 = \bigcup_{x \in F_1} \Delta_{1,x}\,.$$

When $D_1, D_2, \ldots, D_{n-1}$ have been defined, we choose a triangular region Δ_n, symmetric with respect to the y-axis, and with an angle $\pi - 1/n$ at the origin. Clearly, if Δ_n is small enough, then the set

$$D_n = \bigcup_{x \in F_n} \Delta_{n,x}$$

meets none of the sets $D_1, D_2, \ldots, D_{n-1}$. If $f(z) = 0$ in each of the sets D_n, then $C(f, x, S) = \{0\}$ for each x in E. Since the complement of the union of the sets D_n meets every neighborhood of each real point x, the function f can be defined so that $C(f, x)$ is the extended plane, for each real x.

We point out further that in Collingwood's theorem on cluster sets along families of congruent Jordan arcs, the hypothesis of continuity can not be omitted. Indeed, let $\{z_n\}$ be a sequence in H which does not contain any three collinear points but has each point of the real axis as a limit point. If $f(z) = 0$ when $z \notin \{z_n\}$, and if the sequence $\{f(z_n)\}$ is appropriately chosen, then every "segmental" cluster set consists of the origin, while each of the sets $C(f, x)$ consists of the extended plane.

2. Intersections of segmental cluster sets

Corresponding to each line segment L lying in H and terminating at x, let $C(f, x, L)$ denote the set of values w for which there exists a sequence $\{z_n\}$ such that $z_n \in L$, $z_n \to x$, and $f(z_n) \to w$. There exists a function f in H with the property that each real point x is the endpoint of three segments L_j such that the set

$$\bigcap_{j=1,2,3} C(f, x, L_j)$$

is empty [1]. We shall extend this result.

Theorem 3. *There exists a function f in H such that each point x on the real axis is the common endpoint of a family $\{L\}_x$ of rectilinear segments in H with the following properties:*

(i) *$\{L\}_x$ contains $2^{\aleph_0}$ elements, and the set of their directions is a set of second category;*

(ii) *the intersection of the cluster sets of f on any three segments in $\{L\}_x$ is empty.*

To prove this theorem, we shall first construct the families $\{L\}_x$ in such a way that no point of H lies on three of the line segments; the construction of the function f will then be trivial.

Let $\{L_\alpha\}$ $(0 < \alpha < \Omega_c$; here Ω_c denotes the first ordinal of cardinality $2^{\aleph_0})$ be a transfinite sequence of the nonhorizontal lines in the z-plane; let $\{M_\alpha\}$ $(0 < \alpha < \Omega_c)$ be a transfinite sequence of the point sets of type F_σ and of first category in the interval $(0, \pi)$; and let $\{x_\alpha\}$ $(0 < \alpha < \Omega_c)$ be a transfinite sequence (of real numbers) in which each real number occurs $2^{\aleph_0}$ times.

Corresponding to the ordinal $\beta = 1$, we choose the first line in $\{L_\alpha\}$ that passes through the point x_1 and whose angle with the positive real axis does not lie in the set M_1; and we denote it by L^1. Suppose that L^β has been chosen for all β in $0 < \beta < \gamma$. From $\{L^\beta\}$ $(0 < \beta < \gamma)$ we extract the transfinite subsequence of lines that pass through the point x_γ, and we denote by δ the order-type of this subsequence. From $\{L_\alpha\}$ we select as L^γ the first line that passes through x_γ, does not occur in the set $\{L^\beta\}$ $(0 < \beta < \gamma)$, does not pass through the point of intersection of any two lines L_β and $L_{\beta'}$ $(0 < \beta < \beta' < \gamma)$, and whose angle with the positive real axis does not lie in M_δ. The selection is always possible, since the complement of M_δ contains $2^{\aleph_0}$ elements. We see at once that for each real x, the set of lines L^β through the point x has the power $2^{\aleph_0}$. Also, since the set of angles that these lines make with the real axis is not contained in any set of first category, it is of second category.

If z lies on none of the lines L^α $(0 < \alpha < \Omega_c)$, let $f(z) = 0$. To define the function f on the lines L^α, we establish a one-to-one correspondence between the family of lines L^α and the set of lines in the w-plane that have positive slope and are tangent from above to the circle $|w| = 1$. Then, if z lies on two lines L^α and L^β, we define $f(z)$ as the value w that lies on the two corresponding lines in the w-plane; if z lies on precisely one of the lines L^α, we define $f(z)$ as the coordinate of the real point that lies on the corresponding line in the w-plane.

If L is a line through x, and if $L \in \{L^\alpha\}$, then the cluster set $C(f, x, L)$ lies on the corresponding line in the w-plane. Since no point w lies on three lines tangent to the unit circle, no three of the segmental cluster sets $C(f, x, L^\alpha)$ have a common point, and Theorem 3 is proved.

It remains an open question whether the families $\{L\}_x$ can be chosen so that, for each x, the family $\{L\}_x$ contains all nonhorizontal lines through x, or at least a residual set of lines through x.

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