AN EXTREMAL PROBLEM IN THE THEORY OF INTERPOLATION

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1. Let the infinite triangular matrix

$$A=\begin{pmatrix} x_{11} & & & \\ x_{12} & x_{22} & & \\ \vdots & & & \\ x_{1n} & x_{2n} & \cdots & x_{nn} \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

be given, where for $n=1,2,\ldots$ the inequality

$$1\geqq x_{1n}>x_{2n}>\cdots>x_{nn}\geqq -1 \tag{1.1}$$

holds. Putting

$$\omega_n(x,A)=\prod_{j=1}^{n}(x-x_{jn}), \tag{1.2}$$

$$l_{jn}(x,A)=\frac{\omega_n(x,A)}{\omega_n'(x_{jn},A)(x-x_{jn})}, \tag{1.3}$$

the polynomial

$$L_n(x,y_{1n},\ldots,y_{nn},A)=\sum_{j=1}^{n}y_{jn}l_{jn}(x,A), \tag{1.4}$$

the so-called n^{th} Lagrange interpolation polynomial belonging to A, is the only polynomial of degree $\leqq n-1$ having the value y_{jn} at $x=x_{jn}$ for $j=1,2,\ldots,n$. Particularly important is the case when the values y_{jn} are given by

$$y_{jn}=f(x_{jn})\qquad (j=1,2,\ldots,n)$$

where $f(x)$ is a prescribed function continuous in $[-1,+1]$; in this case we shall denote the polynomial in (1.4) more simply by $L_n(x,f,A)$. From the classical investigations of G. FABER[1] and S. BERNSTEIN[2] it follows that *no* matrix A is "effective for the whole class C of functions continuous in

[1] G. FABER [5]. The numbers in brackets refer to the literature quoted at the end of the paper.

[2] S. BERNSTEIN [1].

$[-1, +1]$"; the latter even proved that for every A with (1. 1) there is an $f_0(x) \in C$ and a $-1 \leqq \xi_0 \leqq +1$ such that

$$\overline{\lim_{n\to\infty}} |L_n(\xi_0, f_0, A)| = +\infty,$$

in contrary to everything what was expected since NEWTON.

2. As FEJÉR discovered essentially in 1913, the situation changes completely if instead of the sequence of the Lagrange polynomials $L_n(x, f, A)$ one considers an appropriate special case of the general Hermite interpolation[3] (which HERMITE himself considered only from formal point of view). FEJÉR considered the polynomials $H_n(x, f, A)$ of degree $\leqq 2n-1$ uniquely determined by the requirements

$$\left.\begin{aligned} &H_n(x_{jn}, f, A) = f(x_{jn}), && (2.1)\\ &\left(\frac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = 0 && (2.2)\end{aligned}\right\} \qquad (j = 1, 2, \ldots, n).$$

He proved that choosing e. g. for A the matrix P, the n^{th} row of which consists of the roots α_{jn} of the n^{th} Legendre polynomial

$$\{(x^2-1)^n\}^{(n)},$$

one has, whenever $f \in C$, the relation

$$\lim_{n\to\infty} H_n(x, f, P) = f(x)$$

for $-1 < x < +1$, but not necessarily[4] for $x = \pm 1$. Later he proved[5] that choosing as A the matrix T, the n^{th} row of which consists of the roots β_{jn} of the n^{th} Chebyshev polynomial $T_n(x)$ defined by

$$T_n(\cos\vartheta) = \cos n\vartheta, \tag{2.3}$$

the relation

$$\lim_{n\to\infty} H_n(x, f, T) = f(x) \tag{2.4}$$

[3] L. FEJÉR [6].

[4] As it was shown recently by E. EGERVÁRY and P. TURÁN [2] for the sequence of polynomials $H_n^*(x, f)$ of degree $\leqq 2n-3$, defined by

$$H_n^*(\alpha_{j,n-2}, f) = f(\alpha_{j,n-2}), \quad H_n^*(\pm 1, f) = f(\pm 1),$$

$$\left(\frac{dH_n^*(x, f)}{dx}\right)_{x=\alpha_{j,n-2}} = 0 \qquad (j = 1, 2, \ldots, n-2),$$

the relation

$$\lim_{n\to\infty} H_n^*(x, f) = f(x)$$

holds *uniformly for* $[-1, +1]$.

[5] L. FEJÉR [7].

holds uniformly for $[-1, +1]$. Here, generally, $H_n(x, f, A)$ stands for the polynomial of degree $\leqq 2n-1$ defined by

$$H_n(x_{jn}, f, A) = f(x_{jn}), \tag{2.5}$$

$$\left(\frac{dH_n(x, f, A)}{dx}\right)_{x=x_{jn}} = y'_{jn} \quad (j = 1, 2, \ldots, n) \tag{2.6}$$

where the real numbers y'_{jn} are subject only to the restriction

$$\lim_{n\to\infty} \max_{j=1,\ldots,n} \frac{|y'_{jn}| \log n}{n} = 0. \tag{2.7}$$

3. The relation (2.4) is surprising owing to the great arbitrariness of the slopes y'_{jn}. This raises naturally the question that perhaps choosing another matrix A instead of T this arbitrariness of the slopes can be increased. To give a more exact form to this question we remark that, as easy to see,[6] everything depends upon the expression

$$M_n(A) \overset{\text{def}}{=} \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \tag{3.1}$$

where

$$\mathfrak{h}_{jn}(x, A) = \frac{\omega_n(x, A)^2}{\omega'_n(x_{jn}, A)^2 (x - x_{jn})}. \tag{3.2}$$

Hence it is natural to ask for the "optimal" matrix $A = A^*$ (which is not necessarily unique), i. e. for which

$$M_n(A) = \text{minimal} \tag{3.3}$$

for $n = 1, 2, \ldots$. Since, according to FEJÉR,[7] for arbitrarily small $\varepsilon > 0$ for $n > n_0(\varepsilon)$ the inequality

$$M_n(T) < \left(\frac{2}{\pi} + \varepsilon\right) \frac{\log n}{n} \tag{3.4}$$

holds, we certainly have, denoting[8]

$$\min_A M_n(A) = M_n(A^*) \overset{\text{def}}{=} g(n), \tag{3.5}$$

the inequality

$$\overline{\lim_{n\to\infty}} \frac{n}{\log n} g(n) \leqq \frac{2}{\pi}. \tag{3.6}$$

[6] L. FEJÉR [7].

[7] See L. FEJÉR [7] with a slightly different notation.

[8] It is easy to see that for fixed n the minimum exists.

Now we are going to prove

$$\varliminf_{n\to\infty} \frac{n}{\log n} g(n) \geqq \frac{2}{\pi}, \tag{3.7}$$

i. e.

$$\lim_{n\to\infty} \frac{n}{\log n} g(n) = \frac{2}{\pi}. \tag{3.8}$$

By (3. 7) *our extremal problem is at least asymptotically solved and shown that the choice* $A=T$ *gives essentially the greatest freedom for the choice of the slopes* y'_{jn}. More exactly, we are going to prove the following theorem where c_1 (and later $c_2, c_3, \ldots$) denote positive numerical constants.

THEOREM I. *By whatever choice of the matrix A we have the inequality*

$$\left(M_n(A) \overset{\text{def}}{=}\right) \max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \geqq \frac{2}{\pi n}(\log n - c_1 \log\log n).$$

It would be of interest to determine the exact value of $g(n)$, at least for small n's. A proof of the weaker inequality

$$g(n) \geqq c_2 \frac{\log n}{n} \tag{3.9}$$

could have been proved more briefly; we shall, however, omit this version. Probably also the inequality

$$\int_{-1}^{1} \left\{ \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| \right\} dx > c_3 \frac{\log n}{n} \tag{3.10}$$

holds or even the inequality

$$\sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > c_4 \frac{\log n}{n} \tag{3.11}$$

in $[-1, +1]$ with the exception of a set with measure tending to 0 with $\frac{1}{n}$; we could not prove so far whether or not for all $-1 \leqq a < b \leqq 1$

$$\max_{a\leqq x\leqq b} \sum_{j=1}^{n} |\mathfrak{h}_{jn}(x, A)| > \left(\frac{2}{\pi} - \varepsilon\right) \frac{\log n}{n} \tag{3.12}$$

holds for all $n > n_0(\varepsilon, a, b)$ (or even for $n > n_1(\varepsilon)$).

In our theorem the factor $\log\log n$ can perhaps be replaced by 1; a further refinement, enabling to prove that $g(n)$ is a *convex* function of n, seems to be very difficult.

Our method furnishes mutatis mutandis a proof for the inequality

$$(3.13) \qquad \max_{-1\leq x\leq +1} \sum_{j=1}^{n} |l_{jn}(x, A)| \geq \frac{2}{\pi} \log n - c_5 \log\log n$$

for all matrices A; a somewhat weaker inequality was proved in S. BERNSTEIN's paper [1]. The significance of (3. 13) is given, of course, by the fact that, in conjunction with the fact that for $n > n_1(\varepsilon)$

$$\max_{-1\leq x\leq +1} \sum_{j=1}^{n} |l_{jn}(x, T)| \leq \left(\frac{2}{\pi} + \varepsilon\right) \log n,$$

it solves asymptotically the extremal problem to find the minimum of $\max_{-1\leq x\leq +1} \sum_{j=1}^{n} |l_{jn}(x, A)|$ when A varies. We shall sketch our proof for (3. 13) (Theorem II) and drop the formulation of problems analogous to (3. 10), (3. 11) and (3. 12) with $l_{jn}(x, A)$ instead of $\mathfrak{h}_{jn}(x, A)$.

Since in the proof of our theorem we are always dealing with a large but fixed n, for simplifying the notation we omit n from the indices. Hence for

$$1 \geq x_1 > x_2 > \cdots > x_n \geq -1,$$

$$\omega(x) = \prod_{j=1}^{n} (x - x_j), \quad l_j(x) = \frac{\omega(x)}{\omega'(x_j)(x - x_j)}$$

we have to prove that

$$(3.14) \qquad \max_{-1\leq x\leq +1} \sum_{j=1}^{n} \frac{\omega(x)^2}{\omega'(x_j)^2 |x - x_j|} = \max_{-1\leq x\leq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| =$$

$$= \max_{-1\leq x\leq +1} \sum_{j=1}^{n} |x - x_j| l_j(x)^2 \geq \frac{2}{\pi n} (\log n - c_1 \log\log n).$$

4. We shall need two lemmas.

LEMMA I. *If for a* $0 < b < \frac{1}{2}$ *and* $0 < \eta_1 < 1$ *and a rational polynomial* $J(x)$ *of degree* n *the inequalities*

$$|J(x)| \leq M \quad \textit{for} \quad -1 \leq x \leq +1,$$
$$|J(x)| \leq \eta_1 M \quad \textit{for} \quad -b \leq x \leq +b$$

hold, then for $0 < \eta_2 < \frac{1}{4}$ *and*

$$-(1-\eta_2) b \leq x \leq (1-\eta_2) b$$

the inequality

$$\left|\frac{dJ}{dx}\right| \leq M \left\{(1+b^2)\eta_1 n + \frac{4}{\eta_2^2 b^2}\right\}$$

holds.

For the proof of this lemma we may suppose $M=1$, and consider the pure cosine polynomial

$$J(\cos\vartheta)=J_1(\vartheta). \tag{4. 1}$$

We apply the well-known interpolation formula of M. RIESZ[9] which gives

$$\frac{dJ_1}{d\vartheta}=\frac{1}{2n}\sum_{j=1}^{n}J_1(\vartheta+\vartheta_j)\frac{(-1)^{j+1}}{1-\cos\vartheta_j}$$

where

$$\vartheta_j=\frac{(2j-1)\pi}{2n}.$$

Since our hypothesis amounts to

$$|J_1(\vartheta)|\leqq 1 \quad \text{for} \quad 0\leqq\vartheta\leqq\pi,$$

$$|J_1(\vartheta)|\leqq\eta_1 \quad \text{for} \quad \arccos b\leqq\vartheta\leqq\pi-\arccos b,$$

we get for

$$\arccos(1-\eta_2)b\leqq\vartheta\leqq\pi-\arccos(1-\eta_2)b$$

the estimation

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\frac{\eta_1}{2n}\sum_{\arccos b\leqq\vartheta+\vartheta_j\leqq\pi-\arccos b}^{j}\frac{1}{1-\cos\vartheta_j}+ \tag{4. 2}$$

$$+\frac{\eta_1}{2n}\sum_{\pi+\arccos b\leqq\vartheta+\vartheta_j\leqq 2\pi-\arccos b}\frac{1}{1-\cos\vartheta_j}+\frac{1}{2n}\sum_j{}'\frac{1}{1-\cos\vartheta_j}$$

where the last summation is extended to the ϑ_j's not contained in the previous two. Since

$$\frac{1}{2n}\sum_{j=1}^{n}\frac{1}{1-\cos\vartheta_j}=n,$$

we get

$$\left|\frac{dJ_1}{d\vartheta}\right|\leqq\eta_1 n+\frac{1}{1-\cos(\arccos(1-\eta_2)b-\arccos b)}=\eta_1 n+$$

$$+\frac{1}{1-(1-\eta_2)b^2-\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}=$$

$$=\eta_1 n+\frac{\{1-(1-\eta_2)b^2\}+\sqrt{1-(1-\eta_2)^2b^2}\cdot\sqrt{1-b^2}}{\{1-(1-\eta_2)b^2\}^2-\{1-(1-\eta_2)^2b^2\}(1-b^2)}<$$

$$<\eta_1 n+\frac{2}{1+(1-\eta_2)^2-2(1-\eta_2)}\,\frac{1}{b^2}=\eta_1 n+\frac{2}{\eta_2^2}\,\frac{1}{b^2}.$$

[9] M. RIESZ [9].

Hence for $-(1-\eta_2)b \leqq x \leqq (1-\eta_2)b$

$$\left|\frac{dJ(x)}{dx}\right| = \left|\frac{dJ_1(\vartheta)}{d\vartheta}\right| \frac{1}{\sqrt{1-x^2}} \leqq \left(\eta_1 n + \frac{2}{\eta_2^2 b^2}\right) \frac{1}{\sqrt{1-b^2}} < \eta_1(1+b^2)n + \frac{4}{\eta_2^2 b^2},$$

indeed.

LEMMA II. *Let $J_2(x)$ be a rational polynomial of degree $\leqq m$ which assumes its absolute maximum μ with respect to $[-1, +1]$ at $x=\bar{\xi}$. Then there is an interval I in $[-1, +1]$ of length $\frac{1}{2m^2}$ such that one of its endpoints is $\bar{\xi}$ and in which the inequality*

$$|J_2(x)| \geqq \frac{1}{2}\mu$$

holds.

We choose, namely, as I that one among the intervals

$$\left[\bar{\xi}, \bar{\xi}+\frac{1}{2m^2}\right], \quad \left[\bar{\xi}-\frac{1}{2m^2}, \bar{\xi}\right]$$

which lies in $[-1, +1]$. We may suppose the first. Then using MARKOV's classical theorem[10] we get in I

$$|J_2(x)| = \left|J_2(\bar{\xi}) + \int_{\bar{\xi}}^{x} J_2'(t)\,dt\right| \geqq |J_2(\bar{\xi})| - \int_{\bar{\xi}}^{\bar{\xi}+\frac{1}{2m^2}} \mu m^2\,dt = \mu - \frac{\mu}{2} = \frac{\mu}{2},$$

indeed.

5. We shall employ the following notations. Let

$$M \overset{\text{def}}{=} \max_{-1\leqq x\leqq +1} |\omega(x)|, \tag{5.1}$$

and this should be attained here for $x=\xi$, say. We shall consider the intervals

$$d_\nu:\ -\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu \leqq x \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu \tag{5.2}$$

and

$$d_\nu':\ -\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu\left(1-\frac{1}{\log^3 n}\right) \leqq \tag{5.3}$$
$$\leqq x \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^\nu\left(1-\frac{1}{\log^3 n}\right)$$

for

$$\nu = 0, 1, \ldots, [\log^2 n] \overset{\text{def}}{=} R. \tag{5.4}$$

[10] See MARKOV [8].

We shall use $d'_{\nu+1}-d'_\nu$ and $\bar{d}_\nu$ (the complementary of d_ν with respect to $[-1, +1]$) in the usual sense. We shall denote by ξ_ν one of the values x in d_ν with

$$|\omega(\xi_\nu)| \overset{\text{def}}{=} \max_{x\in d_\nu} |\omega(x)| \overset{\text{def}}{=} M_\nu. \tag{5.5}$$

The intervals d_ν are for $n>c_6$ in $[-1, +1]$ and thus

$$M_0 \leqq M_1 \leqq \cdots \leqq M_R \leqq M. \tag{5.6}$$

6. The proof of our Theorem I is split into three cases.

Case I. There is an index $1\leqq k_0\leqq n$ and a $-1\leqq \xi^* \leqq +1$ such that

$$\max_{-1\leqq x\leqq +1} |l_{k_0}(x)| = |l_{k_0}(\xi^*)| \geqq n^3. \tag{6.1}$$

Applying Lemma II to $l_{k_0}(x)$ we obtain the existence of an interval I in $[-1, +1]$ of length $> \frac{1}{2n^2}$ such that in I the inequality

$$|l_{k_0}(x)| \geqq \frac{1}{2} n^3 \tag{6.2}$$

holds. We choose in I a ξ^{**} as follows. If x_{k_0} is not in I, then let ξ^{**} be the middle-point of I, say; then

$$|\xi^{**}-x_{k_0}| \geqq \frac{1}{4n^2}. \tag{6.3}$$

If x_{k_0} is in I, then ξ^{**} can be chosen in I so that (6.3) holds again. Then we have

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_k(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi^{**})| \geqq |\mathfrak{h}_{k_0}(\xi^{**})| =$$
$$= |\xi^{**}-x_{k_0}| l_{k_0}(\xi^{**})^2 \geqq \frac{1}{4n^2}\frac{1}{4} n^6 > \frac{2}{\pi}\frac{\log n}{n}$$

for $n>c_7$. Hence in this case our theorem is proved and we may suppose in the sequel the inequality

$$\max_{-1\leqq x\leqq +1} |l_k(x)| < n^3 \tag{6.4}$$

for $k=1, 2, \ldots, n$. This last inequality will be used only in the form that it implies[11] upon the x_j's that writing them in the form

$$x_j = \cos \vartheta_j \qquad (0\leqq \vartheta_j \leqq \pi;\ j=1, 2, \ldots, n)$$

[11] See ERDŐS [3]. His proof is an improvement of that contained in ERDŐS—TURÁN [4], esp. p. 548—552.

the ϑ_j's are uniformly distributed in the sense that for $0 \leqq \alpha < \beta < \pi$

$$\left| \sum_{\alpha \leqq \vartheta_j \leqq \beta}^{j} 1 - \frac{\beta - \alpha}{\pi} n \right| < c_8 \log^2 n. \tag{6.5}$$

7. *Case II.* With the notation of **5** we suppose the inequality

$$M_0 < \frac{M}{\log^2 n} \tag{7.1}$$

holds.

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\eta_1 = \frac{1}{\log^2 n}, \quad \eta_2 = \frac{1}{\log^3 n};$$

the assumption (7. 1) assures the applicability of this lemma. This gives for $x \in d_0'$ the estimation

$$|\omega'(x)| \leqq M \left\{ \left(1 + \frac{1}{\log^2 n}\right) \frac{n}{\log^2 n} + 4 \log^8 n \right\} < M \frac{2n}{\log^2 n}$$

roughly, for $n > c_9$. Hence we obtain

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi)| \geqq \frac{1}{2} \sum_{j=1}^{n} \frac{\omega(\xi)^2}{\omega'(x_j)^2} \geqq \frac{M^2}{2} \sum_{x_j \in d_0'}^{j} \frac{1}{\omega'(x_j)^2} \geqq \frac{\log^4 n}{8n^2} \sum_{x_j \in d_0'}^{j} 1.$$

Applying (6. 5), the last sum is (roughly) for $n > c_{10}$

$$> \frac{1}{4} \frac{n}{\log n},$$

i. e.

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \frac{\log^3 n}{32n} > \frac{2}{\pi} \frac{\log n}{n}$$

for $n > c_{11}$. Hence also in this case our theorem is proved and in the sequel we may suppose (Case III)

a) the uniformly dense distribution in (6. 5),

b) the inequality

$$M_0 \geqq \frac{M}{\log^2 n}. \tag{7.2}$$

8. *Case III (and the last).* First we assert that there is an index ν_0 with $0 \leqq \nu_0 \leqq [\log^2 n] = R$ and

$$M_{\nu_0+1} \leqq M_{\nu_0}\left(1 + \frac{1}{\log n}\right). \tag{8.1}$$

For if not, then we should have for all these ν's

$$M_{\nu+1} > M_\nu\left(1+\frac{1}{\log n}\right),$$

i. e. from (5. 6), (7. 2) for $n > c_{12}$ by multiplying we get

$$M \geqq M_R > M_0\left(1+\frac{1}{\log n}\right)^R > M_0\sqrt{n} > \frac{\sqrt{n}}{\log^2 n} M > 2M$$

which is false. Hence (8. 1) is true. With this ν_0 we have, with the notations of **5**,

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| \geqq \sum_{j=1}^{n} |\mathfrak{h}_j(\xi_{\nu_0})| = \tag{8. 2}$$

$$= \sum_{x_j\in d'_{\nu_0}}{}^{j} + \sum_{x_j\in d'_{\nu_0+1}-d'_{\nu_0}}{}^{j} + \sum_{x_j\in \overline{d'_{\nu_0+1}}}{}^{j} \stackrel{\text{def}}{=} S_1+S_2+S_3 \geqq S_1+S_2.$$

To obtain a lower bound for S_1 we use Lemma I with $n > c_{13}$ and

$$\eta_1 = \frac{M_{\nu_0}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0}$ owing to (7. 2) and (5. 6) for $n > c_{14}$

$$|\omega'(x_j)| \leqq M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M}n + 4\log^8 n\right\} <$$

$$< M\left\{\left(1+\frac{25}{\log^2 n}\right)\frac{M_{\nu_0}}{M}n + \left(\frac{M_{\nu_0}}{M}\log^2 n\right)4\log^8 n\right\} =$$

$$= M_{\nu_0}\left\{\left(1+\frac{25}{\log^2 n}\right)n + 4\log^{10} n\right\} < M_{\nu_0}\left(1+\frac{30}{\log^2 n}\right)n,$$

and hence

$$S_1 = \sum_{x_j\in d'_{\nu_0}}{}^{j} \frac{M_{\nu_0}^2}{\omega'(x_j)^2|\xi_{\nu_0}-x_j|} > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2 n^2} \sum_{x_j\in d'_{\nu_0}}{}^{j} \frac{1}{|\xi_{\nu_0}-x_j|}. \tag{8. 3}$$

In order to obtain a lower bound for S_2 we apply again Lemma I with

$$\eta_1 = \frac{M_{\nu_0+1}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(>\frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_0+1}$, as before,

$$|\omega'(x_j)| \leqq M_{\nu_0+1}\left(1+\frac{30}{\log^2 n}\right)n,$$

i. e. by using (8. 1)

$$S_2 = \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{M_{\nu_0}^2}{\omega'(x_j)^2 |\xi_{\nu_0}-x_j|} >$$

$$> \frac{M_{\nu_0}^2}{M_{\nu_0+1}^2} \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^2} \frac{1}{n^2} \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|} >$$

$$> \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4} \frac{1}{n^2} \sum_{x_j \in d'_{\nu_0+1}-d'_{\nu_0}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|}.$$

This and (8. 3) give together for $n > c_{15}$

$$S_1 + S_2 > \frac{1}{\left(1+\frac{30}{\log^2 n}\right)^4 n^2} \sum_{x_j \in d'_{\nu_0+1}}^{j} \frac{1}{|\xi_{\nu_0}-x_j|}. \tag{8. 4}$$

9. Now we use the full force of the uniform distribution in (6. 5). To do so we write first

$$\xi_{\nu_0} = \cos \Theta_{\nu_0}$$

and have

$$-\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0} \leqq \cos \Theta_{\nu_0} \leqq \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0},$$

i. e.

$$\left|\frac{\pi}{2} - \Theta_{\nu_0}\right| < \arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\}; \tag{9. 1}$$

we remark further that the x_j's in (8. 4) are exactly the ϑ_j's with

$$\left|\frac{\pi}{2} - \vartheta_j\right| \leqq \arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\} \stackrel{\text{def}}{=} \alpha. \tag{9. 2}$$

Since

$$\frac{1}{|\xi_{\nu_0}-x_j|} = \frac{1}{|\cos \Theta_{\nu_0} - \cos \vartheta_j|} \geqq \frac{1}{|\Theta_{\nu_0} - \vartheta_j|},$$

we have in the remaining Case III

$$\max_{-1 \leqq x \leqq +1} \sum_{j=1}^{n} |\mathfrak{h}_j(x)| > \left(1-\frac{30}{\log^2 n}\right)^4 \frac{1}{n^2} \sum_{\left|\frac{\pi}{2}-\vartheta_j\right| \leqq \alpha}^{j} \frac{1}{|\Theta_{\nu_0} - \vartheta_j|}. \tag{9. 3}$$

Since from (9. 1) we have

$$\left|\Theta_{\nu_0}-\left(\frac{\pi}{2}\pm\alpha\right)\right| \geqq \arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0+1}\left(1-\frac{1}{\log^3 n}\right)\right\}-$$

$$-\arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^3 n}\right)^{\nu_0}\right\} > \arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\left(1+\frac{1}{2\log^2 n}\right)\right\}-$$

$$-\arcsin\left\{\frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\right\} > \frac{1}{\log n}\left(1+\frac{1}{\log^2 n}\right)^{\nu_0}\frac{1}{2\log^2 n} > \frac{1}{2\log^3 n},$$

the range of summation in (9. 3) is not increased by replacing the original one by

$$|\Theta_{\nu_0}-\vartheta_j| \leqq \frac{1}{2\log^3 n}.$$

Denoting the arcs

$$\Theta_{\nu_0}-(\varkappa+1)\frac{\log^5 n}{n} \leqq \vartheta < \Theta_{\nu_0}-\varkappa\frac{\log^5 n}{n} \qquad \left(\varkappa=0, 1, \ldots, \left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

and

$$\Theta_{\nu_0}+\lambda\frac{\log^5 n}{n} < \vartheta \leqq \Theta_{\nu_0}+(\lambda+1)\frac{\log^5 n}{n} \qquad \left(\lambda=0, 1, \ldots, \left[\frac{3}{2}\frac{n}{\log^8 n}\right]\right)$$

by $U_\varkappa$ and V_λ, respectively, (6. 5) results

$$\sum_{\vartheta_j\in V_\lambda}{}^{j} \frac{1}{|\Theta_{\nu_0}-\vartheta_j|} \geqq \frac{n}{\log^5 n}\frac{1}{\lambda+1}\sum_{\vartheta_j\in V_\lambda}{}^{j} 1 >$$

$$> \frac{n}{\log^5 n}\frac{1}{(\lambda+1)}\left\{\frac{1}{\pi}\log^5 n - c_8\log^2 n\right\} = \frac{n}{\pi}\frac{1}{\lambda+1}\left\{1-\frac{\pi c_8}{\log^3 n}\right\},$$

and similarly for

$$\sum_{\vartheta_j\in U_\varkappa}{}^{j} \frac{1}{|\Theta_{\nu_0}-\vartheta_j|}.$$

Hence from (9. 3) in Case III

$$\max_{-1\leqq x\leqq +1}\sum_{j=1}^{n}|\mathfrak{h}_j(x)| > \left(1-\frac{30}{\log^2 n}\right)^4\frac{2}{\pi n}\left(1-\frac{\pi c_8}{\log^3 n}\right)\sum_{0\leqq\lambda\leqq\left[\frac{n}{\log^8 n}\right]}{}^{\lambda}\frac{1}{\lambda+1} >$$

$$> \frac{2}{\pi}\frac{\log n}{n} - c_{16}\frac{\log\log n}{n}$$

for $n > c_{17}$. Q. e. d.

10. As told we shall sketch the proof of

THEOREM II. *For* $n > c_{18}$ *we have*

$$\max_{-1\leq x\leq +1} \sum_{\nu=1}^{n} |l_\nu(x)| > \frac{2}{\pi}\log n - c_{19}\log\log n.$$

PROOF. Without loss of generality we may suppose the inequality

$$|l_\nu(x)| \leq \log n \tag{10. 1}$$

for $-1 \leq x \leq +1$ and $\nu = 1, 2, \dots, n$, from which the equidistribution (6. 5) follows at once. So we shall have only two cases (keeping the previous notations).

Case I.

$$M_0 < \frac{1}{20\log^2 n} M. \tag{10. 2}$$

We apply Lemma I with

$$J(x) = \omega(x), \quad b = \frac{1}{\log n},$$

$$\eta_1 = \frac{1}{20\log^2 n}, \quad \eta_2 = \frac{1}{\log^3 n};$$

again (10. 2) assures the applicability of this lemma. This gives for $x \in d_0'$ as in **7** for $n > c_{20}$

$$|\omega'(x)| < \frac{M}{10}\frac{n}{\log^2 n}$$

and

$$\max_{-1\leq x\leq +1}\sum_{\nu=1}^{n}|l_\nu(x)| \geq \sum_{\nu=1}^{n}|l_\nu(\xi)| \geq \frac{M}{2}\sum_{x_j\in d_0'}^{j}\frac{1}{|\omega'(x_j)|} > 5\frac{\log^2 n}{n}\sum_{x_j\in d_0'}^{j} 1 > \frac{5}{4}\log n$$

using (6. 5) roughly.

Case II. We may suppose

$$M_0 \geq \frac{M}{20\log^2 n}. \tag{10. 3}$$

Again we have for $n > c_{21}$ an index ν_1 with $0 \leq \nu_1 \leq R$ and

$$M_{\nu_1+1} \leq M_{\nu_1}\left(1 + \frac{1}{\log n}\right); \tag{10. 4}$$

for if not, we should have

$$M > M_0\sqrt{n} > M\frac{\sqrt{n}}{20\log^2 n} > 2M$$

which is false. Again

$$\max_{-1\leqq x\leqq +1} \sum_{j=1}^{n} |l_j(x)| \geqq \sum_{j=1}^{n} |l_j(\xi_{\nu_1})| \geqq \sum_{x_j\in d'_{\nu_1}} {}_j + \sum_{x_j\in d'_{\nu_1+1}-d'_{\nu_1}} {}_j \overset{\text{def}}{=} S'_1 + S'_2.$$

To obtain a lower bound for S'_1 we use Lemma I for $n > c_{22}$ with

$$\eta_1 = \frac{M_{\nu_1}}{M}, \quad \eta_2 = \frac{1}{\log^3 n},$$

$$b = \frac{1}{\log n}\left(1 + \frac{1}{\log^2 n}\right)^{\nu_1} \left(> \frac{1}{\log n}\right).$$

This gives for $x_j \in d'_{\nu_1}$, using also (10.3), for $n > c_{23}$

$$|\omega'(x_j)| \leqq M \left\{ \left(1 + \frac{25}{\log^2 n}\right) \frac{M_{\nu_1}}{M} n + 4 \log^8 n \right\} <$$

$$< M \left\{ \left(1 + \frac{25}{\log^2 n}\right) \frac{M_{\nu_1}}{M} n + \left(\frac{M_{\nu_1}}{M} 20 \log^2 n\right) 4 \log^8 n \right\} < M_{\nu_1}\left(1 + \frac{30}{\log^2 n}\right) n.$$

The further part of the proof runs exactly after the pattern of Theorem I and can be dropped.

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