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Teoria dei numeri. — *On a problem of Sierpiński.* Nota (*) di PAUL ERDŐS, presentata dal Socio straniero W. SIERPIŃSKI.

Let n be a positive integer and denote by $s_n^{(k)}$ the sum of the digits of n written in the k-ary system, and denote by $2 = p_1 < p_2 < \cdots$ the sequence of consecution primes. In a recent paper Sierpiński [1] investigated $s^{(k)}(p_n)$; he proves, among others, that for every k

$$\limsup_{n=\infty} s^{(k)}(p_n) = \infty \tag{1}$$

and immediately deduces from (1) that for infinitely many n

$$s^{(k)}(p_{n+1}) > s^{(k)}(p_n). \tag{2}$$

The question whether for infinitely many n the opposite inequality holds i.e. whether for infinitely many $ns^{(k)}(p_n) > s^{(k)}(p_{n+1})$ remained open. In the present note we shall settle this question of Sierpiński by proving the following

THEOREM. – *For every k there are infinitely many n for which*

$$s^{(k)}(p_n) > s^{(k)}(p_{n+1}).$$

I can not decide if $s^{(k)}(p_n) = s^{(k)}(p_{n+1})$ has infinitely many solutions. Sierpiński [1] deduces this from a conjecture of Schinzel [2]. Presumably

$$\limsup_{n=\infty} (s^{(k)}(p_{n+1}) - s^{(k)}(p_n)) = \infty \text{ and } \liminf_{n=\infty} (s^{(k)}(p_{n+1}) - s^{(k)}(p_n)) = -\infty \tag{3}$$

and even

$$\limsup_{n=\infty} \left((s^{(k)}(p_{n+1})_{/s(p_n)}^{(k)}\right) = \infty \text{ and } \liminf_{n=\infty} \left(s^{(k)}(p_{n+1})_{/s(p_n)}^{(k)}\right) = 0, \tag{4}$$

but I can not prove (3) or (4). In fact I can not disprove

$$|s^{(k)}(p_{n+1}) - s^{(k)}(p_n)| < C$$

and

$$\lim_{n+\infty} \left(s^{(k)}(p_{n+1})_{/s(p_n)}^{(k)}\right) = 1.$$

Put $d_n = p_{n+1} - p_n$ Turán and I [3] proved that $d_{n+1} > d_n$ and $d_n < d_{n+1}$ have both infinitely many solutions and that $\limsup_{n=\infty} d_{n+1/d_n} > 1$, $\liminf_{n=\infty} d_{n+1/d_n} < 1$. But we were unable to exclude the possibility that there is an n_0 so that the following inequalities hold:

$$d_{n_0+1} > d_{n_0} \quad , \quad d_{n_0+2} < d_{n_0+1} \quad , \quad d_{n_0+3} > d_{n_0+2} \quad \text{etc.}$$

(*) Pervenuta all'Accademia il 13 ottobre 1962.

In other words $d_n > d_{n+1} > d_{n+2}$ and $d_n < d_{n+1} < d_{n+2}$ have both only a finite number of solutions. Similarly I can not prove that at least one of the equations $s^{(k)}(p_n) > s^{(k)}(p_{n+1}) > s^{(k)}(p_{n+2})$ and $s^{(k)}(p_n) < s^{(k)}(p_{n+1}) < s^{(k)}(p_{n+2})$ have infinitely many solutions. Sierpiński deduces from the hypothesis of Schinzel that both these inequalities have infinitely many solutions [1].

Proof of the Theorem. I have not been able to find an elementary proof. We have to use the following well known theorem of Hoheisel–Ingham [4]: There exists an absolute constant c_1 so that

$$\pi(x + x^{5/8}) - \pi(x) > c_1 x^{5/8}/\log x \tag{5}$$

($\pi(x)$ denotes the number of primes $\leq x$). Put $s^{(2)}(n) = s(n)$ for sake of simplicity: we will only prove our Theorem for $s(n)$. The proof of the general case is almost identical with the case $k = 2$.

Let $2^k < q_1 < \cdots < q_{t_k} < 2^k + 2^{5k/8}$ be the primes in $(2^k, 2^k + 2^{5k/8})$, further let $2^k - 2^{5k/8} < r_1 < \cdots < r_{s_k} < 2^k$ be the primes in $(2^k - 2^{5k/8}, 2^k)$ (1). By (5) we have

$$t_k > c_2 2^{5k/8}/k \quad , \quad s_k > c_2 2^{5k/8}/k. \tag{6}$$

Now we prove the following

LEMMA. – For all but $o(2^{5k/8}/k)$ primes q_i and r_j we have for every $\varepsilon > 0$ and $k > k_0(\varepsilon)$

$$s(q_i) < (1 + \varepsilon)\frac{5k}{16} \tag{7}$$

and

$$s(r_j) > \frac{3k}{8} + (1 - \varepsilon)\frac{5k}{16} > \frac{11k}{16} - \varepsilon k. \tag{8}$$

Assume that the Lemma is already proved. Then from (6), (7) and (8) it follows that for all sufficiently large k there are primes r_j and q_i satisfying

$$s(r_j) > s(q_i). \tag{9}$$

From (9) and $q_i > r_j$ it clearly follows that for every $k > k_0$ there is a prime p_n satisfying

$$r_j \leqq p_n < q_i$$

and

$$s(p_n) > s(p_{n+1})$$

which proves our Theorem.

Thus we only have to prove our Lemma.

First we prove (7). The primes q_i are all of the form.

$$2^k + \sum_{i=0}^{l} \varepsilon_i 2^i \quad , \quad \varepsilon_i = 0 \text{ or } 1 \quad , \quad l = \left[\frac{5k}{8}\right]. \tag{10}$$

(1) The primes q_i and r_j depend on k, but since there is no danger of confusion we do not indicate this.

If (7) does not hold we clearly must have for $\frac{c_3 2^l}{l}$ primes q_i

$$\sum_{i=0}^{l} \varepsilon_i > \left(\frac{1}{2} + \frac{\varepsilon}{4}\right) l. \tag{11}$$

The number of integers of the form (10) for which (11) holds clearly equals

$$\sum_{r > \left(\frac{1}{2} + \frac{\varepsilon}{4}\right) l} \binom{l}{r}.$$

By a simple and well known computation we obtain ($\eta = \eta(\varepsilon)$ depends only on ε)

$$\sum_{r > \left(\frac{1}{2} + \frac{\varepsilon}{4}\right) l} \binom{l}{r} < 2^{(1-\eta)l} = o\left(\frac{2^l}{l}\right) = o\left(\frac{2^{5k/8}}{l}\right)$$

which proves (7).

The primes r_j are all of the form

$$2^{k-1} + 2^{k-2} + \cdots + 2^{l+1} + \sum_{i=0}^{l} \varepsilon_i 2^i$$

and the proof of (8) proceeds as in the proof of (7). Hence the proof of the Lemma and of our Theorem is complete.

References.

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