ON C_1-SUMMABILITY OF SERIES

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1. INTRODUCTION

Let $\Sigma_{j=1}^{\infty} a_j$ be an infinite series of real, non-negative numbers, and let

$$(1) \qquad (\varepsilon) = \{\varepsilon_j\}_{j=1}^{\infty} \qquad (\varepsilon_j = \pm 1)$$

be any sequence of signs. For any given sequence (ε), we denote by

$$(2) \qquad s_n(\varepsilon) = \sum_{j=1}^{n} \varepsilon_j a_j$$

the n-th partial sum of the series $\Sigma \varepsilon_j a_j$, and by

$$(3) \qquad \sigma_n(\varepsilon) = \mathbf{\Gamma}_{j=1}^{n} \varepsilon_j a_j = \frac{1}{n} \sum_{j=1}^{n} s_j(\varepsilon)$$

the n-th partial C_1-sum of this series. If $s_n(\varepsilon)$, or $\sigma_n(\varepsilon)$, converges, we call

$$(4) \qquad s(\varepsilon) = \lim_{n \to \infty} s_n(\varepsilon) = \sum_{j=1}^{\infty} \varepsilon_j a_j$$

an *attainable point* of Σa_j, or, respectively

$$(5) \qquad \sigma(\varepsilon) = \lim_{n \to \infty} \sigma_n(\varepsilon) = \mathbf{\Gamma}_{j=1}^{\infty} \varepsilon_j a_j$$

a C_1-*attainable point* of Σa_j.

The *attainable set* $S\{a_j\}$ is the set of all attainable points $s(\varepsilon)$ of Σa_j, and the C_1-*attainable set* $SC\{a_j\}$ is the set of all C_1-attainable points $\sigma(\varepsilon)$ of Σa_j.

The sets $S\{a_j\}$ (and more generally the sets $S\{c_j\}$, Σc_j being a series of complex numbers) have been investigated by Hanani [4] and by Calabi and Dvoretzky [2].

Connected with the sets $S\{a_j\}$ and $SC\{a_j\}$ are the sets $T\{a_n\}$ and $TC\{a_n\}$, $T\{a_n\}$ being a set of all numbers τ for which there exists a reordering Σa_{n_i} of the series Σa_n such that $\tau = \Sigma a_{n_i}$, and $TC\{a_n\}$ being the set of numbers τ' such that Σa_n can be reordered so that the new series shall be C_1-summable to τ'. The sets $T\{a_n\}$ have been investigated by Steinitz [7] and by Lorentz and Zeller [5], and the sets $TC\{a_n\}$ by Mazur [6] and by Bagemihl and Erdös [1].

In this paper we investigate the sets $SC\{a_j\}$.

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2. PRELIMINARY PROPOSITIONS

Let $L(\delta)$, with $\delta > 0$, be the family of series Σa_j such that for every j either $a_j \geq \delta$ or $a_j = 0$; in particular, let $L(\delta, \{n_i\})$ be the family of series Σa_j such that $a_{n_i} \geq \delta$ and $a_j = 0$ for $j \neq n_i$ $(i = 1, 2, \cdots)$.

The following propositions are well known:

PROPOSITION 1. *If* $\Sigma \varepsilon_j a_j$ *converges, then it is also* C_1*-summable to its limit; that is, for every series* Σa_j, $S\{a_j\} \subset SC\{a_j\}$.

PROPOSITION 2. *If* $a_j \to 0$, *then* $SC\{a_j\} = S\{a_j\}$ *is perfect.*

In particular, the following proposition of Riemann is well known.

PROPOSITION 3. *If* $a_j \to 0$ *and* $\Sigma a_j = \infty$, *then* $SC\{a_j\} = S\{a_j\}$ *is the whole line.*

Also the following propositions are evidently true:

PROPOSITION 4. *If* $\Sigma \varepsilon_j a_j = s$ *and* $\Sigma \varepsilon_j' b_j = t$ *are two convergent series, then* $\Sigma (\varepsilon_j a_j + \varepsilon_j' b_j)$ *is also convergent, and* $\Sigma (\varepsilon_j a_j + \varepsilon_j' b_j) = s + t$.

PROPOSITION 5. *If* $\Sigma \varepsilon_j a_j$ *and* $\Sigma \varepsilon_j' b_j$ *are two* C_1*-summable series with*

$$\Gamma_{j=1}^{\infty} \varepsilon_j a_j = \sigma \quad \text{and} \quad \Gamma_{j=1}^{\infty} \varepsilon_j' b_j = \tau,$$

then $\Sigma(\varepsilon_j a_j + \varepsilon_j' b_j)$ *is also* C_1*-summable and* $\Gamma_{j=1}^{\infty} (\varepsilon_j a_j + \varepsilon_j' b_j) = \sigma + \tau$.

PROPOSITION 6. *Every series* Σa_j *either*

1) *has a divergent subseries whose general term tends to zero, or*
2) *is a sum of two subseries, one of which is an* $L(\delta)$ *while the other converges absolutely.*

3. DIVERGENT SUBSERIES WITH $a_j \to 0$

THEOREM 1. *Let* Σa_j *be a series of non-negative terms having a subseries* Σa_{n_i} *such that*

$$\sum a_{n_i} = \infty, \quad a_{n_i} \to 0. \tag{6}$$

If

$$SC\{a_j\} \neq \emptyset, \tag{7}$$

then $SC\{a_j\}$ *is the whole line.*

Proof. By (7), there exists a sequence (1) such that $\Sigma \varepsilon_j a_j$ is C_1-summable. Write $\Gamma \varepsilon_j a_j = \sigma$. By (6), either

$$\sum_{+} \varepsilon_{n_i} a_{n_i} = +\infty \quad \text{or} \quad \sum_{-} \varepsilon_{n_i} a_{n_i} = -\infty,$$

($\sum\limits_{+}$ denotes the sum of the positive terms of the series, and $\sum\limits_{-}$ the sum of the negative terms).

Suppose

$$\sum_{+} \varepsilon_{n_i} a_{n_i} = \infty \tag{8}$$

(the proof for the other case is analogous); then the negative terms of the whole series must diverge:

$$\sum_{-} \varepsilon_j a_j = -\infty, \tag{9}$$

because otherwise $\Sigma \varepsilon_j a_j$ could not be C_1-summable.

Let now τ be any given real number. We shall construct a sequence $\{\varepsilon_j'\}$ such that $\Gamma \varepsilon_j' a_j = \tau$.

If $\sigma < \tau$, we change the sign of as many of the earliest terms of $\sum\limits_{-} \varepsilon_j a_j$ as necessary until for the first time

$$-\sum_{\substack{j=1\\-}}^{n} \varepsilon_j a_j = \kappa \geq \frac{\tau - \sigma}{2},$$

which by (9) is possible, and we write

$$\sum_{j=1}^{\infty} \varepsilon_j'' a_j = \sum_{j=1}^{\infty} \varepsilon_j a_j - 2 \sum_{\substack{j=1\\-}}^{n} \varepsilon_j a_j .$$

By Proposition 5, the series $\Sigma \varepsilon_j'' a_j$ is C_1-summable and $\Gamma \varepsilon_j'' a_j = \sigma + 2\kappa = \sigma'' \geq \tau$.

If $\sigma \geq \tau$, put $\varepsilon_j'' = \varepsilon_j$ and $\sigma'' = \sigma$.

Notice that so far the terms of the subseries (8) have not changed their signs. We now change the sign of those terms of the subseries (8), in the order of their appearance, for which

$$\sum_{\substack{+\\i=1}}^{m}{}' \ \varepsilon_{n_i}'' a_{n_i} \leq \frac{1}{2}(\sigma'' - \tau)$$

is satisfied, Σ' being the sum of the terms whose signs have been changed. It follows from (8) and (6) that there exists a finite or infinite subseries $\sum\limits_{+}{}' \varepsilon_{n_i}'' a_{n_i} = 1/2\,(\sigma'' - \tau)$. Write

$$\sum \varepsilon_j' a_j = \sum \varepsilon_j'' a_j - 2 \sum_{+}{}' \varepsilon_{n_i}'' a_{n_i} .$$

By Proposition 5, the series $\Sigma \varepsilon_j' a_j$ is C_1-summable and $\Gamma \varepsilon_j' a_j = \sigma'' - (\sigma'' - \tau) = \tau$.

Theorem 1 is valid for all totally permanent matrices.

The case of a series having a subseries (6) has thus been settled. According to Proposition 6, any other series Σa_j is a sum of an $L(\delta)$ and an absolutely convergent series. Denoting by P the perfect attainable set of the absolutely convergent series (see Proposition 2) and by R the C_1-attainable set of the $L(\delta)$, we see in accordance with Proposition 5 that

$$SC\{a_j\} = R + P = \{r + p\colon r \in R,\ p \in P\}. \tag{10}$$

4. SERIES FOR WHICH $SC\{a_j\}$ IS THE WHOLE LINE

THEOREM 2. *Let* Σa_j *be a series (of non-negative terms) satisfying*

$$\sum a_j = \infty. \tag{11}$$

If there exists an η_0 *with the property that to each* η *in* $0 < \eta \leq \eta_0$ *there corresponds an*

$$n_0 = n_0(\eta) \tag{12}$$

such that for every $n > n_0$

$$\sum_{i=1}^{[\eta n/a_n]} a_{n+i} > 2a_n + \eta, \tag{13}$$

then $SC\{a_j\}$ *is the whole line.*

Proof. Let σ be any real number. We shall construct a sequence (1) such that

$$\Gamma \varepsilon_j a_j = \sigma. \tag{14}$$

According to (12) there exists for every $\eta = 2^{-i}$ $(i = i_0, i_0 + 1, \cdots)$ a number

$$n_i = n_i(2^{-i}) \tag{15}$$

such that for every $n > n_i$, (13) is satisfied with $\eta = 2^{-i}$.

Choose ε_j $(j = 1, 2, \cdots, n_{i_0} - 1)$ arbitrarily. For

$$n_i \leq j < n_{i+1} \qquad (i = i_0, i_0 + 1, \cdots), \tag{16}$$

we fix the signs ε_j as follows:

(a) if $\sigma_{j-1}(\varepsilon) \leq \sigma$ and

  (aa) if $s_{j-1}(\varepsilon) \leq \sigma + 2^{-i}$, we put $\varepsilon_j = 1$;

  (ab) if $s_{j-1}(\varepsilon) > \sigma + 2^{-i}$, we choose ε_j so as to make $s_j(\varepsilon)$ as small as possible but not less than $\sigma + 2^{-i}$;

(b) if $\sigma_{j-1}(\varepsilon) > \sigma$ and

  (ba) if $s_{j-1}(\varepsilon) \geq \sigma - 2^{-i}$, we put $\varepsilon_j = -1$;

  (bb) if $s_{j-1}(\varepsilon) < \sigma - 2^{-i}$, we choose ε_j so as to make $s_j(\varepsilon)$ as large as possible but not greater than $\sigma - 2^{-i}$.

Either σ_j ultimately approaches σ from one side, or else, by the construction of (ε) and condition (11), $\sigma_j(\varepsilon) - \sigma$ changes sign infinitely often.

Suppose that for some j', $s_{j'}(\varepsilon) < \sigma_{j'}(\varepsilon) < \sigma$. According to our construction, the partial sums $s_j(\varepsilon)$ for $j \geq j'$ must be monotonically increasing until they become greater than $\sigma + 2^{-i}$ (for some i which depends on j according to (16)), and then they remain greater than $\sigma + 2^{-i'}$ $(i' \geq i)$ until $\sigma_j(\varepsilon)$ becomes greater than σ. As long as $s_j(\varepsilon) < \sigma_j(\varepsilon)$, $\sigma_j(\varepsilon)$ is monotonically decreasing, but at the moment the increasing sequence $s_j(\varepsilon)$ overtakes the sequence $\sigma_j(\varepsilon)$, the sequence $\sigma_j(\varepsilon)$ begins to increase, and it increases until it reaches a peak value greater than σ. It follows easily that the sequence $\sigma_j(\varepsilon)$ attains alternately minima (say $\sigma_{j_1}(\varepsilon), \sigma_{j_2}(\varepsilon), \cdots$) and maxima $(\sigma_{k_1}(\varepsilon), \sigma_{k_2}(\varepsilon), \cdots)$ with $j_1 < k_1 < j_2 < k_2 < \cdots$, such that

$$\sigma_{j_h}(\varepsilon) \leq \sigma \quad \text{and} \quad \sigma_{k_h}(\varepsilon) > \sigma \quad (h = 1, 2, \cdots).$$

For $j_h \leq j \leq k_h$, the sequence $\{\sigma_j(\varepsilon)\}$ is monotonically increasing, and for $k_h \leq j \leq j_{h+1}$ monotonically decreasing.

For our purpose it is now sufficient to prove that for every $\eta > 0$ there exists an index j^* such that for every $j_h > j^*$

$$0 \leq \sigma - \sigma_{j_h}(\varepsilon) < \eta \tag{17}$$

and for every $k_h > j^*$

$$0 < \sigma_{k_h}(\varepsilon) - \sigma < \eta \tag{18}$$

holds. We shall prove (18), the proof of (17) being analogous.

Let t be an integer such that

$$2^{-t} < \eta/6, \tag{19}$$

and let n_t be the corresponding index fixed by (15). Further, let h be an integer such that $k_{h-1} > n_t$, and m the greatest index $m \leq k_h$ such that $\varepsilon_m = 1$. According to our construction

$$\sigma_{m-1}(\varepsilon) \leq \sigma \tag{20}$$

and

$$\sigma < s_j(\varepsilon) < \sigma + 2^{-t} + 2a_m \quad (m \leq j \leq k_h). \tag{21}$$

By definition of $\sigma_{k_h}(\varepsilon)$, we have

$$\sigma_{k_h}(\varepsilon) = \frac{1}{k_h}\left[(m-1)\,\sigma_{m-1}(\varepsilon) + \sum_{j=m}^{k_h} s_j(\varepsilon)\right];$$

and by (20) and (21)

$$\sigma_{k_h}(\varepsilon) \leq \sigma + \frac{1}{k_h}(2^{-t} + 2a_m)(k_h - m + 1). \tag{22}$$

If $a_m \leq 2^{-t}$, (18) now follows immediately from (19). If $a_m > 2^{-t}$, we observe that $\varepsilon_j = -1$ for $m < j \leq k_h$ and therefore by (21),

$$\sum_{j=m+1}^{k_h} a_j < 2^{-t} + 2a_m .$$

Clearly $m > n_t$ and consequently, by (13), $k_h - m < 2^{-t} m/a_m$. From (13), we also evidently have $1 \leq 2^{-t} m/a_m$, and from (22), we obtain

$$\sigma_{k_h}(\varepsilon) \leq \sigma + \frac{1}{k_h} \cdot 3a_m \cdot 2 \cdot 2^{-t} \frac{m}{a_m},$$

which by (19) implies (18).

For series belonging to $L(\delta)$, we obtain from Theorem 2 the following

COROLLARY 1. *Let* $\Sigma a_j \in L(\delta, \{n_i\})$. *If*

(a) $$n_{i+1} - n_i = o(n_i^\gamma) \quad \textit{and} \quad a_{n_i} = O(n_i^{(1-\gamma)/2}) \qquad (0 < \gamma \leq 1),$$

or if

(b) $$n_{i+1} - n_i = O(n_i^\gamma) \quad \textit{and} \quad a_{n_i} = o(n_i^{(1-\gamma)/2}) \qquad (0 \leq \gamma < 1),$$

then $SC\{a_j\}$ *is the whole line.*

Proof. For $n \neq n_i$, (13) is evidently satisfied. It remains to show that, for sufficiently large i, the inequality

$$\sum_{\alpha=1}^{k_i} a_{n_{i+\alpha}} > 2a_{n_i} + \delta \tag{23}$$

holds, for some k_i with

$$n_{i+k_i} - n_i = o(n_i/a_{n_i}). \tag{24}$$

In view of the inequality $a_{n_{i+\alpha}} \geq \delta$, (23) will be satisfied for

$$k_i \geq \frac{2a_{n_i}}{\delta} + 2,$$

that is, for $k_i = [C a_{n_i}]$ with a sufficiently large constant C. Now, if $n_{i+1} - n_i < p_i n_i^\gamma$, where $p_{i+1} \leq p_i$, then

$$n_{i+1} = n_i \left(1 + \frac{n_{i+1} - n_i}{n_i}\right) \leq n_i (1 + p_i n_i^{\gamma-1}),$$

and since $n_{i+1}^{\gamma-1} < n_i^{\gamma-1}$, $n_{i+k_i} \leq n_i (1 + p_i n_i^{\gamma-1})^{k_i}$, hence

$$\log\frac{n_{i+k_i}}{n_i} \leq k_i p_i n_i^{\gamma-1}.$$

Suppose now that $k_i < q_i n_i^{(1-\gamma)/2}$ $(q_{i+1} \leq q_i)$. Then

$$\frac{n_{i+k_i}}{n_i} < \exp p_i q_i n_i^{(\gamma-1)/2} < 1 + 2p_i q_i n_i^{(\gamma-1)/2}$$

for sufficiently large i. Since we may assume that $p_i \to 0$ in case (a), and that $q_i \to 0$ in case (b), it follows that (24) is satisfied.

Putting, in Corollary 1, $\gamma = 1$ and $\gamma = 0$, respectively, we obtain the still more special

COROLLARY 2. *Let* $\Sigma a_j \in L(\delta, \{n_i\})$. *If*

(a) $$n_{i+1} - n_i = o(n_i) \quad \textit{and} \quad a_{n_i} = O(1),$$

or if

(b) $$n_{i+1} - n_i = O(1) \quad \textit{and} \quad a_{n_i} = o(\sqrt{n_i}),$$

then $SC\{a_j\}$ *is the whole line.*

5. SERIES FOR WHICH $SC\{a_j\}$ IS EMPTY

The next theorem will be stated generally for series Σb_j of elements of a Banach space. The definitions given in Section 1 will apply also under these general conditions. In particular, our theorem will be true when the b_j are real positive numbers, and in this case the norm signs may be omitted.

THEOREM 3. *Let* Σb_j *be a series of elements of a Banach space. If there exist a function* $f(n)$ $(1 \leq f(n) \leq n)$ *and a positive number* $\eta > 0$ *such that for every* N *there exists an* $n \geq N$ *satisfying*

(25) $$||b_n|| - \sum_{0<|i|<f(n)} ||b_{n+i}|| > \frac{\eta n}{f(n)},$$

then $SC\{b_j\}$ *is empty.*

Proof. Assume that $SC\{b_j\} \neq \emptyset$. Without loss of generality, we may then assume that $0 \in SC\{b_j\}$, in other words, that there exists a sequence (1) such that $\Gamma \varepsilon_j b_j = 0$. Hence for every $\eta_1 > 0$ and for sufficiently large n we have, for every m $(m < f(n) \leq n)$,

(26) $$\left\|\sum_{i=0}^{m} s_{n+i}(\varepsilon)\right\| < \eta_1 n, \qquad \left\|\sum_{i=1}^{m} s_{n-i}(\varepsilon)\right\| < \eta_1 n.$$

In particular, we may choose

(27) $$\eta_1 = \eta/2.$$

When η_1 has thus been fixed, there exist indices n satisfying both (25) and (26), and we shall assume in the sequel that n is such an index.

In order to simplify notation, we shall denote by f_n the integer satisfying

$$f(n) - 1 \leq f_n < f(n), \tag{28}$$

and for $m \leq f_n$, we put

$$A_m = \sum_{i=1}^{m} \varepsilon_{n+i} b_{n+i}, \quad B_m = \sum_{i=1}^{m} \varepsilon_{n-i} b_{n-i}, \quad A_0 = B_0 = 0. \tag{29}$$

From (26) and (27) follows

$$\left\| \sum_{k=0}^{f_n} (s_{n-1}(\varepsilon) + \varepsilon_n b_n + A_k) \right\| < \eta n/2. \tag{30}$$

On the other hand, considering

$$\|\varepsilon_n b_n + A_k\| \geq \|b_n\| - \sum_{i=1}^{k} \|b_{n+i}\|,$$

we have by (25)

$$\|\varepsilon_n b_n + A_k\| > \|B_{f_n}\| + \eta n/f(n) \qquad (k = 0, 1, \cdots, f_n). \tag{31}$$

We shall now prove that

$$\|s_{n-1}(\varepsilon)\| > \|B_j\| + \frac{\eta}{2}\frac{n}{f(n)} \qquad (j = 0, 1, \cdots, f_n). \tag{32}$$

Suppose that it is not so, in other words, that

$$\|s_{n-1}(\varepsilon)\| \leq \|B_j\| + \frac{\eta}{2}\frac{n}{f(n)} \tag{33}$$

for some j $(0 \leq j \leq f_n)$. We evidently have

$$\left\| \sum_{k=0}^{f_n} (s_{n-1}(\varepsilon) + \varepsilon_n b_n + A_k) \right\| = \left\| (f_n + 1)\varepsilon_n b_n + \sum_{k=0}^{f_n} (s_{n-1}(\varepsilon) + A_k) \right\| \tag{34}$$

$$\geq (f_n + 1)\|b_n\| - \sum_{k=0}^{f_n} \|s_{n-1}(\varepsilon) + A_k\|.$$

By (33) and (25),

$$\|s_{n-1}(\varepsilon) + A_k\| \leq \|s_{n-1}(\varepsilon)\| + \|A_k\| \leq \|B_j\| + \frac{\eta}{2}\frac{n}{f(n)} + \|A_k\|$$

$$< \|b_n\| - \frac{\eta}{2}\frac{n}{f(n)} \qquad (j = 0, 1, \cdots, f_n),$$

and thus, from (34),

$$\left\|\sum_{k=0}^{f_n} (s_{n-1}(\varepsilon) + \varepsilon_n b_n + A_k)\right\| \geq (f_n + 1)\frac{\eta}{2}\frac{n}{f(n)} \geq \eta n/2,$$

which contradicts (30). Consequently (32) is established.

Finally, by (32),

$$\left\|\sum_{i=1}^{f_n} s_{n-i}(\varepsilon)\right\| = \left\|\sum_{i=0}^{f_n-1} (s_{n-1}(\varepsilon) - B_i)\right\| = \left\|f_n s_{n-1}(\varepsilon) - \sum_{i=0}^{f_n-1} B_i\right\|$$

$$\geq f_n\|s_{n-1}(\varepsilon)\| - \sum_{i=0}^{f_n-1}\|B_i\| > \eta n/2,$$

which contradicts (26), and thus our theorem is proved.

Putting $f(n) = 1$, in Theorem 3, we obtain the well-known, almost trivial result:

COROLLARY 3. *Let* Σb_j *be a series of elements of a Banach space. If there exists a positive number* η *such that* $\|b_n\| > \eta n$ *for infinitely many* n, *then* $SC\{b_j\}$ *is empty.*

Further, putting $f(n) = \eta_2\, n/\|b_n\|$, in Theorem 3, we obtain

COROLLARY 4. *Let* Σb_j *be a series of elements of a Banach space. If there exist positive numbers* η_1, η_2 *such that for infinitely many* n

$$\|b_n\| > (1 + \eta_1) \sum_{0<|i|<\eta_2 n/\|b_n\|} \|b_{n+i}\|, \tag{35}$$

then $SC\{b_j\}$ *is empty.*

Corollary 4 enables us to show that, in the case of real series, the conditions of Corollary 1 are, in some respects, the best possible:

COROLLARY 5. *For each* γ $(0 \leq \gamma \leq 1)$, *there exists a sequence* $\{n_i\}$ *and a series* $\Sigma a_j \in L(1, \{n_i\})$ *such that*

$$n_{i+1} - n_i = O(n_i^{\gamma}) \qquad \textit{and} \qquad a_{n_i} = O(n_i^{(1-\gamma)/2}) \tag{36}$$

and $CS\{a_j\}$ *is empty.*

For $m = 1, 2, \cdots$, put

$$a_{2^m} = 2^{m(1-\gamma)/2}, \quad a_{2^m+[k\cdot 2^{m\gamma}]} = 1 \quad (1 \leq k < 2^{m(1-\gamma)}),$$

and let $a_j = 0$ for all other j's. Then apply Corollary 4 with $n = 2^m$ and $\eta_1 = \eta_2 = 1/4$.

There seems to be no hope of finding a reasonably good converse of Theorem 3 or Corollary 4, for elements of a general Banach space or even for complex numbers. We shall return to the case of complex series in Section 7.

For real series the following question arises:

Problem 1. Let Σa_j be a series of real numbers satisfying (11). What is the smallest constant C such that, if for every $\eta > 0$ and sufficiently large n

$$\sum_{i=1}^{[\eta n/a_n]} a_{n+i} > Ca_n + \eta \tag{37}$$

holds, then $SC\{a_n\}$ is the whole line?

From Corollary 4 it follows easily that $C \geq 1$. On the other hand, we were only able to prove (in Theorem 2) that $C \leq 2$. It seems very likely that this is not the best possible result. Perhaps $C = 1$ or $C = 1 + \eta$ would suffice.

Another problem which we were unable to solve is

Problem 2. Let Σa_j be a series of real numbers. Are the conditions stated in Theorem 3 and Corollary 4 for $SC\{a_j\}$ to be empty not only sufficient but also necessary?

6. THE STRUCTURE OF $SC\{a_j\}$

We know very little about the structure of all the possible sets $SC\{a_j\}$. The following remarks are rather trivial:

Remark 1. If $SC\{a_j\} \neq \emptyset$, then the set $SC\{a_j\}$ is infinite.

There exists a sequence (ε) such that $\Sigma \varepsilon_j a_j$ is C_1-summable, and we may change the sign of any finite number of elements of $\Sigma \varepsilon_j a_j$.

Remark 2. There exists a series $\Sigma a_j \in L(1)$ such that $SC\{a_j\}$ contains an interval but does not contain the whole line.

Put

$$a_{2^n} = 2, \quad a_{2^n+1} = 2 + 2^{-n}, \quad a_j = 0 \text{ for } j \neq 2^n, 2^n + 1 \quad (n = 0, 1, \cdots).$$

Remark 3. Even if $\limsup_{i\to\infty} n_{i+1}/n_i$ is as large as we please, we may construct a series $\Sigma a_j \in L(1, \{n_i\})$ such that $SC\{a_j\}$ is the whole line.

Put $n_{2i} = n_{2i-1} + 1$, $a_{n_{2i-1}} = o(n_{2i})$, $a_{n_{2i}} = a_{n_{2i-1}} + b_i$ with $\Sigma |b_i| = \infty$, $b_i \to 0$.

With respect to the sets $SC\{a_j\}$ of series $\Sigma a_j \in L(\delta, \{n_i\})$ for which (36) is satisfied, we conjecture that the following question has an affirmative answer, in the case $0 \leq \gamma < 1$:

Problem 3. If $\Sigma a_j \in L(\delta, \{n_i\})$ and

$$n_{i+1} - n_i = O(n_i^{\gamma}), \quad a_{n_i} = O(n_i^{(1-\gamma)/2}) \quad (0 \leq \gamma < 1),$$

is it true that $SC\{a_j\}$ is either empty or the whole line?

Perhaps even a stronger conjecture holds:

Problem 4. If $\Sigma a_j \in L(\delta, \{n_i\})$ and

(a) $$n_{i+1} - n_i = o(n_i^{\gamma}) \quad \text{and} \quad a_{n_i} = O(n_i^{1-\gamma}) \quad (0 < \gamma \leq 1),$$

or

(b) $$n_{i+1} - n_i = O(n_i^{\gamma}) \quad \text{and} \quad a_{n_i} = o(n_i^{1-\gamma}) \quad (0 \leq \gamma < 1),$$

is it true that $SC\{a_j\}$ is either empty or the whole line?

By Corollary 1, we can give a positive answer for the case (a) with $\gamma = 1$. On the other hand we cannot solve Problem 3 even in the case $\gamma = 0$.

In connection with Problem 4, we can prove the following theorem:

THEOREM 4. *Corresponding to each denumerable set*

(38) $$b_1, b_2, \cdots,$$

each $\delta > 0$, *and each* γ $(0 < \gamma \leq 1)$, *there exists a series* $\Sigma a_j \in L(\delta, \{n_i\})$, *with*

$$n_{i+1} - n_i = O(n_i^{\gamma}) \quad \textit{and} \quad a_{n_i} = O(n_i^{1-\gamma}),$$

such that $SC\{a_j\}$ *is denumerable and* $SC\{a_j\} \supset \{b_h\}_{h=1}^{\infty}$.

Proof. We shall construct a series Σa_j satisfying the conditions of the theorem. Evidently there exists a sequence

(39) $$d_1, d_2, \cdots$$

such that $0 < d_m \leq 2\delta$ $(n = 1, 2, \cdots)$ and such that every term of the set (38) is a sum (or difference) of a finite number of terms of (39). We shall fix the indices n_{4i} by induction as follows: let n_4 be the smallest integer such that $n_4^{\gamma} > 4$ and, n_{4i} being fixed, we choose $n_{4(i+1)} = n_{4i} + [n_{4i}^{\gamma}]$. The indices n_h $(h \not\equiv 0 \pmod 4)$ we fix by the rule

$$n_{4i-3} + 3 = n_{4i-2} + 2 = n_{4i-1} + 1 = n_{4i}.$$

Further, we put

$$a_{n_{4i-3}} = a_{n_{4i-2}} = n_{4i}^{1-\gamma} \quad (i = 1, 2, \cdots),$$

and if i is the smallest integer such that $n_{4i} \geq 2^{m+2}$ for some m, we put

$$a_{n_{4i-1}} = a_{n_{4i}} = n_{4i}^{1-\gamma} + \frac{1}{2} d_m;$$

otherwise we put

$$a_{n_{4i-1}} = a_{n_{4i}} = n_{4i}^{1-\gamma}.$$

In order that this series should be C_1-summable, the sums $\Sigma_{h=0}^{3} \varepsilon_{n_{4i-h}} a_{n_{4i-h}}$ must differ from zero for at most a finite number of values of i; hence $SC\{a_j\}$ is denumerable. It is also easy to see that all the finite sums (and differences) of the sequence (39) are elements of $SC\{a_j\}$, which proves our theorem.

Problem 5. Is $SC\{a_j\}$ a Borel set, and what are the possible Baire classes into which the sets $SC\{a_j\}$ can fall? In particular, is it true that if $SC\{a_j\}$ is of second category in every point of an interval (a, b) then it contains (a, b)? We do not even have an example where $SC\{a_j\}$ is not an F_σ.

7. SERIES WITH COMPLEX TERMS

In this section we shall formulate, without proofs, some general ideas on series Σc_j with complex terms. Here again (as in Section 5), all the definitions of Section 1 retain their meaning. The following theorem, similar to Theorem 2, is true.

THEOREM 5. *Let Σc_j be a series of complex numbers. Assume that there exists, for some fixed $\delta > 0$ and for each $\eta > 0$, an integer $N = N(\eta)$ such that, for every $n > N$ and for every α $(0 \leq \alpha < 2\pi)$,*

$$\sum_{\alpha, h=1}^{[\eta n/|c_n|]} |c_{n+h}| > C(\delta) \cdot |c_n|, \tag{40}$$

where the α indicates that those c_{n+h} are omitted for which $|\arg c_{n+h} - \alpha| < \delta$, and where $C(\delta)$ is some constant depending on δ only. Then $SC\{c_j\}$ is the whole plane.

We shall refrain from giving a full proof of this theorem and shall confine ourselves to giving an outline of the proof. Notice that from the conditions of the theorem it follows that the series Σc_j has at least two directions of divergence (for the definition, see [4]), the angle between them being not less than δ. Consequently the methods of Section 4.1 of [4] may be used.

The general idea of the proof of Theorem 5 is the same as that of Theorem 2, with the difference that instead of keeping $s_n(\varepsilon)$ in a given half-line as in Theorem 2, we must now keep the $s_n(\varepsilon)$ in a given quadrant of the plane. This is done by means of the terms of Σc_j which are near to two chosen directions of divergence by the methods of [4], the sum of all other terms being kept small by the result of [3].

The condition that there should be at least two directions of divergence is essential. We shall show that without this condition, even under otherwise much stronger restrictions than those of Theorem 5, the set $SC\{c_j\}$ may be empty. We shall give an example for the following.

Remark 4. Let Σc_j be a series of complex terms. Even if for every $\eta > 0$

$$\lim_{n\to\infty} \sum_{h=1}^{[\eta n/|c_n|]} |c'_{n+h}| \Big/ |c_n| = \infty,$$

where c'_{n+h} is the projection of c_{n+h} on any fixed direction, the set $SC\{c_j\}$ may be empty.

Our example will be given under the additional restriction that $1 \leq |c_j| < 2$ $(j = 1, 2, \cdots)$. Put $c_{2^k} = i$ $(k = 1, 2, \cdots)$, and for $2^k < m < 2^{k+1}$, put $c_m = 1$ and $c_m = 1 + i2^{-3k/4}$ in alternate intervals of length $[2^{3k/4}]$ each. A simple argument shows that the conditions of our remark are satisfied and that $SC\{c_j\}$ is empty.

We shall now give several examples showing the possible interrelation between $S\{c_j\}$ and $SC\{c_j\}$, for series Σc_j of complex terms with

$$\sum |c_j| = \infty \quad \text{and} \quad c_j \to 0. \tag{41}$$

If $S\{c_j\}$ is the whole plane, then evidently $SC\{c_j\} = S\{c_j\}$. It can easily be seen, however, that (41) does not imply that $S\{c_j\}$ is the whole plane, or even that $S\{c_j\}$ contains a continuum (compare [4]).

The series $\Sigma(n^{-1} + i2^{-n})$ (for which (41) is satisfied) shows that the relation $SC\{c_j\} = S\{c_j\}$ is also possible in the case where $S\{c_j\}$ contains more than one point but is not the whole plane.

A simple example of a series for which $S(c_j)$ constitutes a proper subset of $SC(c_j)$ is $\Sigma(n^{-1/2} + i2^{-n})$. Here the imaginary part of $\sigma(\varepsilon)$ (if $\sigma(\varepsilon)$ exists) lies in $(-1, 1)$ and determines uniquely the sequence ε, hence the real part of $\sigma(\varepsilon)$. If $\varepsilon_n = (-1)^{[\sqrt{n}]}$, then $\sigma(\varepsilon)$ exists, but $s(\varepsilon)$ does not exist; this proves our assertion concerning the relation between the sets $S(c_j)$ and $SC(c_j)$. We note incidentally that the set $S(c_j)$ is dense in the strip between the lines $y = \pm 1$. For if $z = x + iy$ is any point in this strip and η is any positive number, we can choose a finite number of the ε_j in such a way that, regardless of how the remaining elements are chosen, the imaginary part of $s_n(\varepsilon)$ converges to a value which differs from y by less than η; and we can then choose the remaining ε_j in such a way that the real part of $s_n(\varepsilon)$ converges to x.

Finally we give two examples of series Σc_j for which $S\{c_j\}$ is dense in the whole plane and $SC\{c_j\}$ contains $S\{c_j\}$ properly. In one of them,

$$c_j = \frac{1}{n} + \frac{i}{n!} \quad \text{for} \quad \sum_{k=1}^{n-1} k! \leq j < \sum_{k=1}^{n} k! \qquad (n = 2, 3, \cdots),$$

and $SC\{c_j\}$ is the whole plane; in the other,

$$c_j = \frac{1}{n} + \frac{i}{10^{n^2}} \quad \text{for} \quad \sum_{k=1}^{n-1} 10^{k^2} \leq j < \sum_{k=1}^{n} 10^{k^2} \qquad (n = 2, 3, \cdots),$$

and $SC\{c_j\}$ is a proper subset of the plane.

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