THE CONSTRUCTION OF CERTAIN GRAPHS

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1. Introduction. A graph G is called complete if any two of its vertices are connected by an edge; a set of vertices of G are said to be independent if no two of them are connected by an edge. It follows from a well-known theorem of Ramsay **(1)** that for each pair of positive integers k, l there is an integer $f(k, l)$, which we take to be minimal, such that every graph with $f(k, l)$ vertices either contains a complete graph of k vertices or a set of l independent points. Szekeres **(2)** proved that

$$f(k, l) \leqslant \binom{k+l-2}{k-1},$$

and Erdös **(3; 4)** that

$$f(k, k) \geqslant 2^{k/2},$$
$$f(3, l) > l^{1+c_3},$$

for a positive constant c_3.

Clearly

$$f(k, l) \geqslant f(3, l) > l^{1+c_3},$$

for $k \geqslant 4$. Our object is to prove a stronger result. We say that a set S of points of a graph G is m-independent, if there is no complete subgraph of G having m vertices in S. Let $h(k, l)$ be the minimal integer such that every graph of $h(k, l)$ vertices contains either a complete graph of k vertices or a set of l points which are $(k-1)$-independent. Then clearly

$$h(k, l) \leqslant f(k, l)$$

for all k, l. However we can still prove that

$$h(k, l) > l^{1+c_k},$$

for $k \geqslant 3$. This problem is due to A. Hajnal (oral communication).

Our construction is geometric, and is based on a lemma (§2) of some geometric interest.

2. Regular simplices on the surface of a sphere. We define the relative surface area of a set S on the surface of a sphere in n-dimensional euclidean space to be the surface area of S divided by the surface area of the sphere. We prove

Received October 26, 1961.

LEMMA. *Suppose n and k are positive integers ($k \leqslant n$) and that ζ satisfies*

$$0 < \zeta < \sqrt{2},$$
$$k\{1 - (\tfrac{1}{2}\zeta)^2\}^{n/2} < 1.$$

Then, if S is a set on the surface of the unit sphere Σ in n-dimensional space of relative surface area

$$V > \{1 - (\tfrac{1}{2}\zeta)^2\}^{n/2},$$

there is a regular k-simplex, with its vertices each on Σ within a distance ζ of S, and with its centre at the centre of Σ.*

Remark. This lemma shows that in a space of many dimensions even a set of rather small relative surface area on the unit sphere will always contain a k-simplex, which is very nearly a regular k-simplex of unit circum-radius.

Proof. Let C be the minor spherical cap cut from Σ by a plane passing at a distance $\frac{1}{2}\zeta$ from its centre. Since $\zeta < \sqrt{2}$, it is clear that the union of the segments joining the centre O to the points of C is contained in the sphere with radius

$$\{1 - (\tfrac{1}{2}\zeta)^2\}^{1/2}$$

with its centre at the centre of the base of the cap C. Consequently the relative surface area of C is at most

$$\{1 - (\tfrac{1}{2}\zeta)^2\}^{n/2} < V,$$

and so is less than the relative surface area of S. Let C_ζ and S_ζ be the sets of points on Σ within the distance ζ of the points of C and S respectively. Then by a well-known result of Schmidt **(5)** the relative surface area of S_ζ will be at least that of C_ζ. But C_ζ is the major cap cut from Σ by a plane passing at the distance $\frac{1}{2}\zeta$ from 0, and so has relative surface area at least

$$1 - \{1 - (\tfrac{1}{2}\zeta)^2\}^{n/2} > 1 - (1/k).$$

So the relative surface area of the set T of points of Σ not in S_ζ is less than $1/k$.

Consider the space $\mathfrak{S}_k$ of all ordered sets $X = \{x_1, x_2, \ldots, x_k\}$ of k points of Σ forming a regular k-simplex of circum-radius 1 with the metric

$$d(X, Y) = \sqrt{\left\{\sum_{i=1}^{k} |x_i - y_i|^2\right\}}.$$

It is possible to introduce a measure on the Borel sets of $\mathfrak{S}_k$ giving the whole space unit measure and such that, for $i = 1, 2, \ldots, k$, the measure of the set $\mathfrak{T}_i$ of points $X = \{x_1, x_2, \ldots, x_k\}$ with $x_i \in T$ is equal to the relative surface area of T and so is less than $1/k$. Hence we can choose a point X of $\mathfrak{S}_k$ not in

$$\bigcup_{i=1}^{k} \mathfrak{T}_i.$$

*All our distances are measured in the n-dimensional space, not on the surface of the sphere.

The points $x_1, x_2, \ldots, x_k$ form a regular k-simplex of circum-radius 1 in S_ζ and so within distance ζ of S. This proves the lemma.

3. THEOREM. *Let $k \geqslant 3$ be an integer. If c_k is a positive constant less than*

$$\frac{\log 1/\{1 - (\frac{1}{8}\eta_k)^2\}}{2 \log 4/\eta_k},$$

where

$$1/\eta = 1/\eta_k = \tfrac{1}{2}(k-1)^{1/2}(k-2)^{1/2}[\{2(k-1)^2\}^{1/2} + \{2k(k-2)\}^{1/2}],$$

and l is a sufficiently large integer, there is a graph G, with less than

$$l^{1+c_k}$$

vertices, which contains no complete k-gon, but such that each subgraph with l vertices contains a complete $(k-1)$-gon.

Remark. We can take $c_k \sim 1/(512k^4 \log k)$ as $k \to \infty$.

Proof. Let H be the greatest integer less than l^{1+c_k}. Let ϵ be a small positive constant and let n be the nearest integer to

$$(1+\epsilon)\log H \Big/ \log\left[\frac{4}{\eta\sqrt{(1-(\frac{1}{8}\eta)^2)}}\right].$$

We take the vertices of our graph to be a set N of H points on the surface of the sphere Σ in euclidean n-dimensional space with centre at the origin O and with unit radius, and we join each pair whose distance apart exceeds

$$\sqrt{\{2k/(k-1)\}}.$$

Since the unit sphere contains no simplex with k vertices with all its edges exceeding this length our graph contains no complete k-gon. But if $(k-1)$ points of N have mutual distances apart exceeding

$$\sqrt{\{2(k-1)/(k-2)\}} - \eta_k = \sqrt{\{2k/(k-1)\}},$$

they will form a complete $(k-1)$-gon in the graph. Thus to prove the theorem it will suffice to prove that the points of N can be chosen, so that from any set of l points of N a subset of $(k-1)$ points may be chosen with their mutual distances apart exceeding

$$\sqrt{\{2(k-1)/(k-2)\}} - \eta.$$

With each point x of Σ and each ξ with $0 < \xi < 1$ we associate the spherical cap $C(x, \xi)$ of all points of Σ within a distance ξ of x. Now the union of the segments joining O to the points of $C(x, \xi)$ contains a cone with O as vertex of height

$$1 - \tfrac{1}{2}\xi^2,$$

with a $(n - 1)$-dimensional sphere of radius

$$\xi(1 - \tfrac{1}{4}\xi^2)^{1/2}$$

as its base. But the unit sphere is itself contained in a cylinder of height 2 with a $(n - 1)$-dimensional unit sphere as its base. Hence the relative surface area of $C(x, \xi)$ is at least

$$\frac{1}{2n}(1 - \tfrac{1}{2}\xi^2)[\xi(1 - \tfrac{1}{4}\xi^2)^{1/2}]^{n-1} > \frac{1}{4n}[\xi(1 - \tfrac{1}{4}\xi^2)^{1/2}]^n.$$

Since $0 < \eta < 1$ we can choose ξ with $0 < \xi < \frac{1}{4}\eta$ so that the relative surface area V of $C(x, \xi)$ is exactly

$$V = \frac{1}{4n}[\tfrac{1}{4}\eta\{1 - (\tfrac{1}{8}\eta)^2\}^{1/2}]^n.$$

Let S be the union of all the caps $C(x, \xi)$ with x in N. Let h be the integer nearest to H^ϵ. Since

$$\begin{aligned}\log (h + 1) - \log \{(H + 1)V\} &= \epsilon \log H - \log H + n \log [(4/\eta)\{1 - (\tfrac{1}{8}\eta)^2\}^{-1}] + O(\log n)\\ &= 2\epsilon \log H + O(\log \log H),\end{aligned}$$

we have

$$h + 1 > (H + 1)V,$$

provided l is sufficiently large. A simple probability argument, which we have recently used elsewhere **(6)**, shows that, if the H points of the set N are distributed independently uniformly over Σ, then the expectation of the relative surface area of the set F_h of points of Σ which lie in h or more of the caps

$$C(x, \xi) \text{ with } x \text{ in } N$$

is at most

$$\frac{H!}{h!(H - h)!}V^h(1 - V)^{H-h}\frac{(h + 1)(1 - V)}{(h + 1) - (H + 1)V}.$$

So we may suppose that the points of N are chosen so that the relative surface area V_h of the set F_h satisfies

$$V_h < \frac{H!}{h!(H - h)!}V^h(1 - V)^{H-h}\frac{(h + 1)(1 - V)}{(h + 1) - (H + 1)V}.$$

Now

$$h = H^\epsilon + O(1),$$

and

$$\begin{aligned}V &= \frac{1}{4n}[\tfrac{1}{4}\eta\{1 - (\tfrac{1}{8}\eta)^2\}^{1/2}]^n\\ &= \exp\left[-n \log \frac{4}{\eta\sqrt{(1 - (\frac{1}{8}\eta)^2)}} + O(\log n)\right]\\ &= \exp[-(1 + \epsilon)\log H + O(\log \log H)]\\ &= |(\log H)^{O(1)}|\, H^{-1-\epsilon}.\end{aligned}$$

So, using Stirling's formula and making some elementary reductions, we have

$$\begin{aligned}\log V_h - \log \tfrac{1}{2}V &< \log\left[2\frac{H!}{h!(H-h)!}V^{h-1}(1-V)^{H-h}\frac{(h+1)(1-V)}{(h+1)-(H+1)V}\right]\\ &= -2\epsilon H^{\epsilon}\log H + O(H^{\epsilon}\log\log H).\end{aligned}$$

Thus $V_h < \frac{1}{2}V$, when l is sufficiently large.

Let L be a subset of N with l elements. Let $C'(x, \xi)$ be the part of $C(x, \xi)$ not lying in F_h. The relative surface area of $C'(x, \xi)$ is at least

$$V - V_h > \tfrac{1}{2}V.$$

The points of the union S_L of the sets $C'(x, \xi)$ with x in L belong to at most $h - 1$ of the sets $C'(x, \xi)$. So the relative surface area V_L of S_L is at least

$$\tfrac{1}{2}Vl/(h-1).$$

Hence

$$\begin{aligned}&\log V_L - \log[1-(\tfrac{1}{8}\eta)^2]^{n/2}\\ &\geqslant \log\{\tfrac{1}{2}Vl/(h-1)\} - \log[1-(\tfrac{1}{8}\eta)^2]^{n/2}\\ &= \log l - \epsilon\log H - (1+\epsilon)\log H + \tfrac{1}{2}n\log 1/\{1-(\tfrac{1}{8}\eta)^2\} + O(\log\log H)\\ &= (1+c_k)\left[\frac{1}{1+c_k} - (1+\epsilon)\frac{1}{1+[\log 1/\{1-(\frac{1}{8}\eta)^2\}]/[2\log 4/\eta]} - \epsilon\right]\log l\\ &\qquad + O(\log\log l).\end{aligned}$$

Since

$$c_k < \frac{\log 1/\{1-(\frac{1}{8}\eta)^2\}}{2\log 4/\eta},$$

provided ϵ is chosen to be sufficiently small, we have

$$V_L > [1-(\tfrac{1}{8}\eta)^2]^{n/2},$$

for all sufficiently large l.

Since

$$(k-1)\{1-(\tfrac{1}{8}\eta)^2\}^{n/2} < 1,$$

for all sufficiently large l, we can now apply the lemma, with $\zeta = \frac{1}{4}\eta$, to the set S_L. Thus we can choose a regular $(k-1)$ simplex with each of its vertices on Σ within a distance $\frac{1}{4}\eta$ of S_L and with its centre at the centre of Σ. So we can choose $k-1$ points $x_1, x_2, \ldots, x_{k-1}$ of L, each point within a distance $\frac{1}{2}\eta$ of a different vertex of a regular $(k-1)$-simplex of circum-radius 1 and edge-length

$$\sqrt{\{2(k-1)/(k-2)\}}.$$

Since all the edges of the simplex, $x_1, x_2, \ldots, x_{k-1}$ exceed

$$\sqrt{\{2(k-1)/(k-2)\}} - \eta = \sqrt{\{2k/(k-1)\}},$$

the subgraph of G with vertices $x_1, x_2, \ldots, x_{k-1}$ is a complete $(k-1)$-gon, as required. This completes the proof.

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