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TO WACŁAW SIERPIŃSKI
ON HIS 80-TH BIRTHDAY

ON SOME PROPERTIES OF HAMEL BASES

BY

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I dedicate this little note to Professor Wacław Sierpiński since I use in it methods which he used very successfully on so many occasions.

Throughout this paper $\alpha, \beta, \gamma, \ldots$ will denote ordinal numbers, $n_i, n_\alpha, \ldots$ integers, $r_\alpha, \ldots$ rational numbers, $r_\alpha^+, \ldots$ non-negative rationals and $a, a_\alpha, b, \ldots$ real numbers. H will denote a Hamel basis of the real numbers, H^* the set of all numbers of the form $\sum_\alpha n_\alpha a_\alpha$ $(a_\alpha \in H)$ (the sum is finite) and H^+ the set of all numbers of the form $\sum_\alpha r_\alpha^+ a_\alpha$ $(a_\alpha \in H)$. Measure will always be the Lebesgue measure, and (a, b) will denote the set of numbers $a < x < b$.

Sierpiński showed [1] that there are Hamel bases of measure 0 and also Hamel bases which are not measurable.

We are going to prove the following theorems:

THEOREM 1. *H^* is always non-measurable. In fact H^* has inner measure 0 and for every (a, b) the outer measure of $H^* \frown (a, b)$ is $b-a$.*

THEOREM 2. *Assume $c = \aleph_1$. Then there is an H for which H^+ has measure 0.*

Proof of Theorem 1. The sets H^*+1/n, $2 \leqslant n < \infty$, are pairwise disjoint. Thus a simple argument shows that H^* has inner measure 0.

For every x there exists an n_x so that $n_x \cdot x$ is in H^*, or the sets $1/nH^*$, $2 \leqslant n < \infty$, cover the whole interval $(-\infty, +\infty)$. Hence H^* cannot have outer measure 0, and thus by the Lebesgue density theorem it has a point, say x_0, of outer density 1. But then (since H^* is an additive group) every point of x_0+H^* is a point of outer density 1 of H^*. Finally, it is easy to see that H^* is everywhere dense (since, if a and b are rationally independent, the numbers $n_1 a+n_2 b$ are everywhere dense).

Now it is easy to deduce that the outer measure of $H^* \frown (a, b)$ is $b-a$. To see this observe that since H^* has outer density 1 at x_0, for every $\varepsilon > 0$ there exist arbitrarily small values of η, such that the outer measure of $H^* \frown (x_0-\eta, x_0+\eta)$ is greater than $2(1-\varepsilon)\eta$; but consequently the same holds for $H^* \frown (x_0+t-\eta, x_0+t+\eta)$, where t is an arbitrary element of H^*. Since H^* is everywhere dense, a simple argument shows that the outer measure of $H^* \frown (a, b)$ is greater than $(1-\varepsilon)(b-a)-3\eta$. Since this holds for every ε and η, the outer measure is $b-a$, which completes the proof of Theorem 1.

Now we prove Theorem 2. In fact we shall prove a somewhat stronger theorem:

THEOREM 2'. *Assume $c = \aleph_1$. Then there is an H such that H^+ is a Lusin set (see* [2], *p.* 36-37), *i. e. it intersects every nowhere dense perfectset in a set of power $\leqslant \aleph_0$.*

It is well known (and easy to see) that such a set has the property that if ε_k, $1 \leqslant k < \infty$, is any sequence of numbers, it can be covered by intervals I_k of length ε_k $(1 \leqslant k < \infty)$ (see [3] and also [2], p. 37-39).

We shall construct our H by transfinite induction. Let $\{F_a\}$, $1 \leqslant a < \Omega_1$, be the set of all nowhere dense perfect sets (as is well known, there are $c = \aleph_1$ perfect sets) and let x_a, $1 \leqslant a < \Omega_1$, be a well-ordering of the set of all real numbers. Put

$$F^{(a)} = \bigcup_{1 \leqslant \gamma < a} F_\gamma.$$

$F^{(a)}$ is a set of the first category and for $a > \gamma$ we have $F^{(a)} \supset F^{(\gamma)}$.

We shall denote by $\{a_a\}$, $1 \leqslant a < \Omega_1$, the elements of H. Assume that for $a < \beta$ the a_β have already been constructed. We choose a_β and $a_{\beta+1}$ as follows: Let x_δ be the x_a of smallest index which is not of the form $\sum_i r_{a_i} a_{a_i}$, $a_i < \beta$. Put

$$x_\delta = u - v, \tag{1}$$

where u and v have the following properties:

I. $\{u, v, a_a\}$, $1 \leqslant a < \beta$, are rationally independent.

II. The numbers

$$r_1 u + r_2 v + \sum_i r_{a_i} a_{a_i}, \qquad a_i < \beta, \tag{2}$$

are never in $F^{(\beta)}$, unless $r_1 = -r_2 \neq 0$.

Then put $a_\beta = v$ and $a_{\beta+1} = u$. First we show that such values u and v exist.

Put $u = v + x_\delta$. Then II is equivalent to the relation

$$\Big((r_1+r_2)v + r_1 x_\delta + \sum_i r_{a_i} a_{a_i}\Big) \notin F^{(\beta)}$$

for every choice of $r_1 + r_2 \neq 0$ and arbitrary r_{a_i}, a_{a_i}, $a_i < \beta$. Thus v is in none of the sets

$$\Big(F^{(\beta)} - \sum_i r_{a_i} a_{a_i} - r_1 x_\delta\Big)/(r_1 + r_2). \tag{3}$$

Clearly all sets (3) are sets of the first category and there are only $\aleph_0$ of them. Thus their union is also of the first category and hence there exists a set of v's of second category which is not contained in their union and which thus satisfies II. It is easy to see that there exists at most a countable number of choices of v and $u = v + x_\delta$ which do not satisfy I; hence there exist u and v satisfying both I and II.

This construction can clearly be carried out for all ordinal numbers $\beta < \Omega_1$, and, since $c = \aleph_1$, it gives a Hamel-base H. Clearly H^+ is a Lusin-set. To see this it is sufficient to show that $H^+ \cap F^{(a)}$ has for every $a < \Omega_1$ a power not exceeding $\aleph_0$. Let $\sum_{i=1}^{i} r_{\xi_i}^{+} a_{\xi_i}$ $(\xi_1 < \ldots < \xi_t)$ be an element of H^+. Since $c = \aleph_1$, there are only denumerably many elements of H^+ with $\xi_t \leqslant a$. If $\xi_t > a$, then by our construction $\sum_{i=1}^{t} r_{\xi_i}^{+} a_{\xi_i}$ is not in $F^{(a)}$ since, by II, if $\xi_t > a$, then $\sum_{i=1}^{t} r_{\xi_i} a_{\xi_i}$ can be in $F^{(a)}$ only if $\xi_{t-1} + 1 = \xi_t$ and $r_{\xi_{t-1}} = -r_{\xi_t}$, but it is then not in H^+. This completes the proof of Theorem II.

We have really proved the following stronger statement:

There exists a Hamel-base H with a well-ordering $\{a_a\}$ *such that the set of real numbers* $\sum_{i=1}^{t} r_{a_i} a_{a_i}$ *for which*

$$a_{t-1} \neq a_t - 1 \quad or \quad r_{a_t} + r_{a_{t-1}} = r_{a_t} + r_{a_{t-1}} \neq 0$$

is a Lusin set.

Kuczma asked in [4] the following question: Let $f(X+Y) = f(X) + f(Y)$ and assume that $f(Z) < c$ for every $Z \in P$, where P is such a set that every real number can be written in the form $Z_1 - Z_2, Z_1, Z_2 \in P$. Does it follow then that $f(X) = cX$? The answer is negative. To see this let $f(a_a) \leqslant 0$ for every $a_a \in H$, let $f(a_a)$ be non-linear and let us extend $f(X)$ for every real X by $f(u+v) = f(u) + f(v)$. Clearly $f(Z) \leqslant 0$ for every $Z \in H^+$, every real number is of the form $Z_1 - Z_2, Z_1, Z_2 \in H^+$, and $f(X) \neq cX$.

REFERENCES

[1] W. Sierpiński, *Sur les distances des points dans les ensembles de mesure positive*, Fundamenta Mathematicae 1 (1920), p. 93-104.

[2] — *Hypothèse du continu*, Monografie Matematyczne 4, Warszawa-Lwów 1934.

[3] — *Sur un ensemble non dénombrable, dont toute image continue est de mesure nulle*, Fundamenta Mathematicae 11 (1928), p. 302-304.

[4] M. Kuczma, *On the functional equation* $f(x+y) = f(x) + f(y)$, ibidem 50 (1962), p. 387-391.

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