A PROBLEM IN GRAPH THEORY

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A *graph* consists of a finite set of vertices some pairs of which are *adjacent*, i.e., joined by an edge. No edge joins a vertex to itself and at most one edge joins any two vertices. The *degree* of a vertex is the number of vertices adjacent to it. The *complete k-graph* has k vertices and $\binom{k}{2}$ edges.

We shall say that a graph G has property (n, k), where n and k are integers with $2 \leqq k \leqq n$, if G has n vertices and the addition of any new edge increases the number of complete k-graphs contained in G. For example, let $A_k(n)$ denote a graph with n vertices and $n(k-2)-\binom{k-1}{2}$ edges which consist of a complete $(k-2)$-graph each vertex of which is also joined to each of the $n-(k-2)$ remaining vertices. $A_k(n)$ contains no complete k-graphs but it is easily seen that with the addition of any new edge a complete k-graph is formed. Hence, $A_k(n)$ has property (n, k).

We wish to determine the "minimal (n, k) graphs," i.e., those graphs with property (n, k) and with the minimal number of edges. We prove the following result.

THEOREM 1. *For every pair of integers n and k, with* $2 \leq k \leq n$, *the only minimal* (n, k) *graph is* $A_k(n)$.

We will apply Theorem 1 to prove a conjecture of Erdös and Gallai (see [1]). A set of vertices is said to represent the edges of a graph if each edge contains at least one of these vertices. A graph G is said to be *edge p-critical* if the maximal number of vertices necessary to represent all the edges of G is p, but if any edge is omitted the remaining edges can be represented by $p-1$ vertices. For example the complete $(p+1)$-graph is edge p-critical. In [1] it is conjectured that an edge p-critical graph can have at most $\binom{p+1}{2}$ edges. Theorem 1 immediately implies this conjecture. In fact we prove

THEOREM 2. *Every edge p-critical graph has at most* $\binom{p+1}{2}$ *edges and the only edge p-critical graph with* $\binom{p+1}{2}$ *edges is the complete* $(p+1)$*-graph.*

Finally we would like to state a conjecture. A bipartite graph (k, l) is a bipartite graph having k green and l blue vertices. A complete bipartite graph (k, k) is a graph where all green and blue vertices are adjacent. We now say that a bipartite graph (n, m) has property (n, m, k, k) if any new edge increases the number of complete bipartite (k, k) graphs in our graph (we assume $k \leq n$, $k \leq m$).

Problem. Is it true that every (n, m) graph with property (n, m, k, k) has at least $(k-1)(n+m-k+1)$ edges?

A weaker conjecture would be that every bipartite graph (n, m) which contains no complete bipartite (k, k) but which loses this property when any new edge is added has at least $(k-1)(n+m-k+1)$ edges.

One of the difficulties of proving these conjectures may be that the obvious extremal graphs are certainly not unique, which fact may make an induction proof difficult. One can easily formulate the analogous conjecture for property (n, m, k, l), but we leave this to the reader.

Proof of Theorem 1. We first show that $A_k(n)$ is a minimal (n, k) graph and then we show that it is the only one. We begin by establishing the inequality

$$f_k(n) \geq f_k(n-1) + (k-2), \qquad \text{for } n = k+1,\ k+2, \cdots, \tag{1}$$

where $f_k(n)$ denotes the number of edges in a minimal (n, k) graph.

Let G be any minimal (n, k) graph where $n \geq k+1$. There exist nonadjacent vertices in G, say p and q, as the complete n-graph is clearly not a minimal (n, k) graph. Since $G+(p, q)$, the graph obtained from G by adding an edge joining p and q, contains at least one more complete k-graph than G, it must be that p and q are both adjacent to all the vertices of some complete $(k-2)$-graph. Hence, if we let G^* denote the graph obtained from G by removing q and then joining

p by an edge to every vertex which originally was adjacent to q but not to p, it follows that G^* has at least $k-2$ fewer edges than G. We may assert that G^* has property $(n-1, k)$. For if a and b are nonadjacent vertices in G^*, both different from p, then the addition of the edge (a, b) still forms at least one new complete k-graph since none of the complete k-graphs formed by adding (a, b) to G could have contained both p and q and in G^* the vertex p can serve wherever q was required before; in the remaining cases the addition of a new edge to G^* forms the same new complete k-graphs as were formed by the addition of the same edge to G. Since G^* contains at least $f_k(n-1)$ edges, inequality (1) now follows.

It is obvious that $f_k(k)=\binom{k}{2}-1$. This combined with (1) implies that

$$f_k(n) \geqq \binom{k}{2} - 1 + (n-k)(k-2) = n(k-2) - \binom{k-1}{2}, \tag{2}$$

$$\text{for } n = k+1, k+2, \cdots.$$

But $A_k(n)$ is an example of a graph having property (n, k) and with only $n(k-2)-\binom{k-1}{2}$ edges. Therefore, it must be that $A_k(n)$ is a minimal (n, k) graph and that equality holds throughout in (1) and (2).

We now use induction to show that $A_k(n)$ is the only minimal (n, k) graph. For any fixed admissible value of k this is certainly the case when $n=k$. Assume that the assertion is valid whenever $k \leqq n < m$, for some integer m, and consider any minimal (m, k) graph G. From the fact that equality holds in (1) it is not difficult to see that G^*, constructed as before, must be a minimal $(m-1, k)$ graph. Hence, we may suppose that G^* is the same as $A_k(m-1)$.

If in G^* the vertex p, using the same notation as before, is one of the $k-2$ vertices adjacent to every other vertex in G^*, then in G it must be that q is adjacent to all the other $k-3$ such vertices and to one of the remaining vertices. This is so that the addition of the edge (p, q) to G will form at least one new complete k-graph. Each of the other $m-k$ vertices is adjacent to either p or q but not both for otherwise p and q would be mutually adjacent to more than $k-2$ vertices and G would contain more than $f_k(m)$ edges. We may suppose that one such vertex h is not adjacent to p. But it is now easily seen that the addition of the edge (p, h) would not form a new complete k-graph in G, contradicting the definition of G. The only alternative is that p is one of the vertices of degree $k-2$ in G^*. From the definition of G^* it now follows that G differs from G^* only by the presence of the vertex q of degree $k-2$ which is adjacent to the same $k-2$ vertices as in p. This implies that G is the same as $A_k(m)$ which completes the proof of the theorem.

We may restate the above theorem in the following slightly weaker form: Of all graphs with n vertices which contain no complete k-graphs, where $2 \leqq k \leqq n$, but which lose this property when any new edge is added, the graph $A_k(n)$ and only that graph has the minimal number of edges. This statement could be considered as the dual of the theorem of Turán in [2], which treats the corresponding problem of determining those graphs with this property which have the maximal number of edges.

Proof of Theorem 2. If G has n vertices and is edge p-critical then it is easy to see that the maximum number of vertices, no two of which are adjacent, is $n-p$, i.e., the complementary graph of G does not contain a complete $(n-p+1)$-graph. But, since G is edge p-critical, if we add any edge to the complementary graph it will contain a complete $(n-p+1)$-graph. Hence, by Theorem 1, the number of edges in G is at most

$$\binom{n}{2} - \left[n(n-p-1) - \binom{n-p}{2}\right] = \binom{p+1}{2}$$

with equality only for the complete $(p+1)$-graph, which proves Theorem 2.

References

1. P. Erdös and T. Gallai, On the minimal number of vertices representing the edges of a graph, Publ. Math. Inst. Hung. Acad. Sci., 6 (1961) 181–203, see conjecture (3.3) p. 195.
2. P. Turán, On the theory of graphs, Colloq. Math., 3 (1954) 19–30.

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