ON SOME DIVISIBILITY PROPERTIES OF $\binom{2n}{n}$

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L. Moser [3] recently gave a very simple proof that

$$(1) \qquad \binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$$

has no solutions. In the present note we shall first of all prove that for $a > \frac{n}{2}$, $\binom{2a}{a} \nmid \binom{2n}{n}$, which by the fact that there is a prime p satisfying $n < p \leq 2n$ immediately implies that

$$(2) \qquad \binom{2n}{n} = \prod_{i=1}^{r} \binom{2a_i}{a_i}^{\alpha_i}, \quad \alpha_i \geq 1, \quad n > a_i \geq 1$$

has no solutions. It is easy to see on the other hand that

$$(3) \qquad \prod_{i=1}^{r_1} \binom{2a_i}{a_i}^{\alpha_i} = \prod_{i=1}^{r_2} \binom{2b_i}{b_i}^{\beta_i}, \quad a_i \geq 1, \quad b_i \geq 1$$

has infinitely many non-trivial solutions. I do not know if (3) is solvable if $\alpha_i = \beta_i = 1$. I will discuss some further divisibility properties of $\binom{2n}{n}$ and mention some unsolved problems.

THEOREM. Denote by $g(m)$ the smallest integer $n > m$ for which $\binom{2m}{m} \mid \binom{2n}{n}$. For all m we have

$$(4) \qquad g(m) \geq 2m ,$$

and for $m > m_o$

$$m^{1+c} < g(m) < (2m)^{\log m/\log 2} \tag{5}$$

for a certain absolute constant $c > 0$.

First we prove (4). Put $n = m+k$, $0 < k < m$; then

$$\frac{\binom{2n}{n}}{\binom{2m}{m}} = \prod_{i=1}^{2k} (2m+i) \Big/ \left(\prod_{i=1}^{k} (m+i)\right)^2 . \tag{6}$$

By a simple calculation we can show that for $n \leq 11$, (6) is never an integer. Henceforth we can thus assume $n \geq 12$. It is well known that for $n \geq 12$ there always is a prime p satisfying $\frac{2}{3}n < p < n$. Thus if $m \leq \frac{2n}{3}$, (6) cannot be an integer since the denominator is divisible by p^2 and the numerator only by p. Thus we can assume

$$n \geq 12 , \qquad m > \frac{2n}{3} .$$

Miss Faulkner [2] recently proved that $\prod_{i=1}^{k} (m+i)$ always has a prime factor $q > 2k$ if $m + k \geq P$, where P is the least prime $> 2k$, except if $k = 2$, $m = 7$ or $k = 3$, $m = 7$. In our case these exceptions cannot occur since $n > 11$, $m > \frac{2}{3} n > 7$. Also, since $n > 11$ and $m > \frac{2n}{3}$, $k < \frac{n}{3}$ or $2k < \frac{2n}{3}$; hence $m+k = n > P$. Thus by the theorem of Miss Faulkner there is a prime $q > 2k$ which divides $\prod_{i=1}^{k} (m+i)$.

Let $m+j$, $0 < j < k$ be the unique value for which $m+j \equiv 0 \pmod q$ and assume $q^\alpha \| (m+j)$ (i.e., $q^\alpha | (m+j)$, $q^{\alpha+1} \nmid (m+j)$). Since $q > 2k$, $2m+2j$ is the only integer m of the sequence $2m+i$, $0 < i \leq 2k$, which is a multiple of q. Hence $q^\alpha \| \prod_{i=1}^{2k} (2m+i)$,

$q^{2\alpha} \mid \prod_{i=1}^{k} (m+i)^2$, or (6) cannot be an integer, which proves (4).

It can easily be shown that $g(m) > 2m$ for $m > 1$, (i. e., $g(m) = 2m$ holds only for $m = 1$).

Now we prove the first inequality of (5). It is well known and evident that if $2k + 1 < (2n)^{1/2}$, then no prime p satisfying $\frac{2n}{2k+1} < p < \frac{n}{k}$ divides $\binom{2n}{n}$. Further, it follows from the classical theorem of Hoheisel [3] that if $\varepsilon > 0$ is sufficiently small and $k < n^{\varepsilon}$, $n > n_0(\varepsilon)$, then there always is a prime satisfying

$$(7) \qquad \frac{2n}{2k+1} < p < \frac{n}{k} .$$

Now if $c = c(\varepsilon)$ is sufficiently small and $\frac{5}{2}m < n < m^{1+c}$ then there clearly is a $k < n^{\varepsilon}$ for which

$$m < \frac{2n}{2k+1} < \frac{n}{k} < 2m ,$$

or

$$p \Big| \binom{2m}{m} , \qquad p \nmid \binom{2n}{n},$$

which proves $g(m) > m^{1+c}$ (if $2m < n \leq \frac{5}{2}m$ then the interval $(\frac{2}{3}n, 2m)$ contains a prime, thus $\binom{2m}{m} \nmid \binom{2n}{n}$).

It seems very likely that for every k and $m > m_0(k)$, $g(m) > m^k$, but this is perhaps not easy to prove. It seems likely that to every $\varepsilon > 0$ there is an n_0 so that for every $m > n^{\varepsilon}$ there is a prime p, $m < p < 2m$, such that $p \nmid \binom{2n}{n}$. This would of course imply $g(m) > m^k$.

Now we prove the second inequality of (5). L. Moser [4] observed that $\binom{2m}{m} \mid \binom{2n}{n}$ if $n = \binom{2m}{m} - 1$ (i.e. $(n+1) \mid \binom{2n}{n}$), but this only gives $g(m) < c_1^m$.

We will only outline the proof of the upper bound for $g(m)$. In fact we shall show a stronger result than (5). Let $m > m_0(\varepsilon)$ and $x > m^{\log m/\log 2}$. Then the number of integers $n < x$ for which $\binom{2m}{m} \nmid \binom{2n}{n}$ is less than εx.

It is well known that if

$$n = \sum_{i=0}^{k} a_i p^i, \qquad 0 \le a_i < p,$$

is the p-ary expansion of n, then $p^r \| \binom{2n}{n}$, where

$$r = \sum_{a_i \ge p/2} 1. \tag{8}$$

In other words $p \nmid \binom{2n}{n}$ if and only if all the a_i are $< p/2$. Thus by a simple calculation the number of integers $n < p^{k+1}$ for which $p \nmid \binom{2n}{n}$ equals $[\frac{p}{2}]^{k+1}$. Hence if $x > (2m)^{\log m/\log 2}$ and $p < 2m$ then the number of integers $n < x$ for which $p \nmid \binom{2n}{n}$ is less than

$$\frac{x}{2^{\log m/\log 2}} = \frac{x}{m}. \tag{10}$$

Further, a simple combinatorial argument shows that the number of integers $n < p^{k+1}$ for which $p^r \nmid \binom{2n}{n}$ equals

$$[\tfrac{p}{2}]^{k+1} \sum_{i=0}^{r-1} \binom{k+1}{i} < [\tfrac{p}{2}]^{k+1} (k+1)^r. \tag{11}$$

Hence by (11) we obtain by a simple computation, the details of which we suppress, that the number of integers $n < x$ $(x > (2m)^{\log m/\log 2})$ for which

$$p^r \nmid \binom{2n}{n}, \qquad p < (2m)^{1/r}$$

is also less than $\frac{x}{m}$ (as in (10)). Now it is well known and easy to prove that if $p^r \mid \binom{2m}{m}$ then $p^r < 2m$ (or $p < (2m)^{1/r}$). Hence from (10) the number of integers $n < x$ for which

$$\binom{2m}{m} \nmid \binom{2n}{n}$$

is less than

$$x\frac{\pi(2m)}{m} < \varepsilon x$$

for $m > m_0(\varepsilon)$, which completes the proof of (5).

I do not know to what extent our upper bound for $g(m)$ can be improved.

I have not been able to show that there is an infinite sequence $n_1 < n_2 < \ldots$ so that for every $i < j$, $\binom{2n_i}{n_i} \nmid \binom{2n_j}{n_j}$, but it seems certain that such a sequence exists [1].

REFERENCES

1. See P. Erdős, Quelques Problèmes de la Thèorie des Nombres Problème 57, Monographie de L'Enseignement Math. No. 6.

2. Miss Faulkner's proof is not yet published.

3. For the strongest result in this direction see A. E. Ingham, On the difference between consecutive primes, Quarterly Journal of Math. 8 (1937), 255-266.

4. L. Moser, Insolvability of $\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$, Can. Math. Bull. 6 (1963) 167-169.

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