ON TWO PROBLEMS OF S. MARCUS, CONCERNING FUNCTIONS WITH THE DARBOUX PROPERTY

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1. In [1], S. Marcus has proved the following theorem:

"Let f be a continuous real valued function on a real interval I. Suppose that f assumes a maximum at each point of an everywhere dense subset in I. Then, for each interval $J \subset I$ we have one of the following two possibilities: 1° There exists an interval $K \subset J$ where f is constant; 2° There exists a real set $A(J)$ of the first Baire-category with respect to $f(J)$, such that $J \cap \{x; f(x)=t\}$ is not countable for each $t \in f(J) - A(J)$."

The following problem is proposed in [1], p. 268:

Does the above theorem remain valid if instead "a continuous real valued function" one takes "a real valued function with the Darboux property"?

We shall show that the answer to this problem is negative.

Let Ω_C be the first ordinal number of cardinal c. Let A, B and $C_\alpha (1 \leqslant \alpha < \Omega_C)$ disjoint, denumerable and everywhere dense sets of real numbers, such that

$$R = A \cup B \cup (\bigcup_\alpha C_\alpha),$$

where R is the set of all real numbers. Let

$$\{x_\alpha\},\ 1 \leqslant \alpha < \Omega_C$$

be a wellorder of the set $R \cap (0,1)$. We define the function f as follows:

$$f(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \in B \\ x_\alpha, & \text{if } x \in C_\alpha . \end{cases}$$

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This function takes on in every interval each value belonging to [0,1]; thus, f has the Darboux property. On the other side, the condition 1° of the above theorem is not fulfilled, since f is constant in no interval; the condition 2° is also not fulfilled, since the set $\{x; f(x) = t\}$ is denumerable for each $t \in [0,1]$.

2. It is known that every real function f of a real variable is the sum of two functions g and h, having the Darboux property ([4], [2], [3]). S. Marcus posed to me the following question: If f is Lebesgue (Borel) measurable, is it possible to choose g and h Lebesgue (Borel) measurable and with the Darboux property, such that $f = g + h$? We shall show that the answer is affirmative.

To see this, let S_1 and S_2 be two disjoint F_σ sets of measure zero, which have power c in each interval. If x is not in $S_1 \cup S_2$, put $g(x) = 0$, $h(x) = f(x)$. On S_1 we define $g(x)$ in such a way that, for every interval (a, b), $g(x)$ assumes on $S_1 \cap (a, b)$ every value in $(-\infty, \infty)$; this can clearly be done, since $S_1 \cap (a, b)$ has power c for every interval (a, b).

The function h is defined as follows. If $x \in S_1$ then $h(x) = f(x) - g(x)$. On S_2, $h(x)$ is defined so that on $S_1 \cap (a, b)$ it assumes every value in $(-\infty, \infty)$ (for every interval (a, b)) and $g(x) = f(x) - h(x)$.

Clearly $f(x) = g(x) + h(x)$ for every x and both g and h have the Darboux property. If f is Lebesgue measurable, clearly the same holds for g and h, since $f(x) = h(x)$ and $g(x) = 0$ everywhere except on a set of measure zero.

It is easy to see that, if f is Borel measurable, g and h can also be defined to be Borel measurable.

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3. — *Asupra unei teoreme enunțate de A. Lindenbaum și demonstrate de W. Sierpinski.* Comunicările Acad. R.P.R., 1960, **10**, *7*, 547–550.

4. W. SIERPINSKI, *Sur une propriété des fonctions réelles quelconques, définies dans les espaces métriques.* Le Matematiche, Catania, 1963, **8**, 73–78.