Problems and results on diophantine approximations *

by

P. Erdös

The older literature on this subject (until about 1935) is treated in the excellent book of Koksma [1]. The more recent literature is discussed in a very interesting paper of Cigler and Helmberg [2]. Unlike the above authors I by no means aim to cover the literature completely and will mostly discuss only problems on which I myself worked thus a more exact title would have been "Problems and results on diophantine approximation which have interested me". There will be some overlap with my paper "On unsolved problems" [3]. First I discuss some questions on inequalities of distribution and on uniform distribution.

I. Let $x_1, x_2, \cdots$ be an infinite sequence of real numbers in the interval $(0, 1)$. Denote by $N_n(a, b)$ the number of x_i satisfying

$$a \leqq x_i \leqq b, \quad 1 \leqq i \leqq n.$$

We say that $x_1, x_2, \ldots$ is uniformly distributed if for every $0 \leqq a < b \leqq 1$

$$\lim_{n=\infty} \frac{N_n(a, b)}{n} = b-a. \tag{1}$$

The classical result of Weyl (see [1]) states that the necessary and sufficient condition that the sequence $x_1, \ldots$ should be uniformly distributed is that for every integer k, $1 \leqq k < \infty$

$$\lim_{n=\infty} \frac{1}{n} \sum_{j=1}^{n} e^{2\pi i k x_j} = 0. \tag{2}$$

Here I would like to ask a question which I have not yet answered though it is perhaps very simple. Put

$$A_k = \overline{\lim} \left| \sum_{j=1}^{n} e^{2\pi i k x_j} \right|$$

* Nijenrode lecture

A_k can be infinite, but if $x_j \equiv j\alpha \pmod 1$ then A_k is finite for every k. Is it true that lim sup $A_k = \infty$? I expect that the answer is yes. It is easy to see that if B_k is the least upper bound of $|\sum_{j=1}^{n} e^{2\pi i k x_j}|$ then $\overline{\lim}_{k=\infty} B_k = \infty$.

The discrepancy of $x_1, \ldots, x_n$ we define as follows: (This notion as far as I know is due to van der Corput)

$$(3) \qquad D(x_1, \ldots, x_n) = \sup_{0 \leqq a < b \leqq 1} |N_n(a, b) - (b-a)n|.$$

Equidistribution is equivalent to $D(x_1, \ldots, x_n) = o(n)$. Van der Corput conjectured and Mrs. Ardenne-Ehrenfest proved the beautiful result that for every infinite sequence $x_1, x_2, \ldots$

$$\limsup_{n=\infty} D(x_1, \ldots, x_n) = \infty.$$

In fact Mrs. Ardenne-Ehrenfest showed that for infinitely many n

$$D(x_1, \ldots x_n) > c \log\log n / \log\log\log n.$$

Roth sharpened this result by showing that for infinitely many n

$$D(x_1, \ldots, x_n) > c(\log n)^{\frac{1}{2}}.$$

One can express the theorem of Roth also in the following finite form: There is an absolute constant c so that to every sequence $x_1, \ldots, x_n$ there is an m and an $\alpha < 1$ so that

$$|N_m(0, \alpha) - \alpha m| > c(\log n)^{\frac{1}{2}}.$$

Perhaps in Roth's Theorem $c(\log n)^{\frac{1}{2}}$ can be replaced by $c \log n$, this if true is known to be best possible [4].

I would like to ask a few related questions.

Does there exist an infinite sequence $x_1 < x_2 < \ldots$ so that for every $0 \leqq a < b \leqq 1$

$$(4) \qquad \limsup_{n=\infty} N_n(a, b) < \infty?$$

Denote by $f(a, b)$ the upper limit and by $F(a, b)$ the upper bound of $N_n(a, b)$. The fact that $D(x_1, \ldots, x_n)$ is unbounded only implies that $F(a, b)$ cannot be a bounded function of a and b). On the other hand it is not clear to me why $f(a, b)$ could not be a bounded function of a and b, though this seems very unlikely.

Let $|z_\nu| = 1$, $1 \leqq \nu < \infty$ be an infinite sequence of complex numbers on the unit circle. Is it true that

$$\limsup_{n=\infty} \max_{|z|=1} \prod_{i=1}^{n} |z - z_i| = \infty?$$

I would guess that the answer is yes. If this is the case it would be of interest to estimate how fast $\max_{1 \leqq m \leqq n} A_n$, $(A_n = \max_{|z|=1} \prod_{i=1}^{n} |z - z_i|)$ must tend to infinity.

Let $w_1, \ldots w_n$ be any n points on the surface of the unit sphere. Let C be any spherical cap and denote by

$$(4) \quad D(w_1, \ldots, w_n) = \max_{C} (N(C) - n\alpha_C),\ D_n = \min_{n_1, \ldots, w_n} D(w_1, \ldots, w_n)$$

where $N(C)$ denotes the number of w's which are in C and α_C is the ratio of the surface of C with the surface of the sphere, the maximum is to be taken with respect to all spherical caps. One would expect that D_n is an unbounded function of n, in other words: n points cannot be distributed too uniformly on the surface of the sphere (the situation is of course quite different on the circle). Perhaps this can be proved by the method of Roth, who settles in his paper the analogous question for the square [4].

Let $z_1, z_2 \ldots$ be an infinite sequence of points in the plane. Denote by $N(z_0, r)$ the number of z's in the interior of the circle of center z_0 and radius r. Put

$$f(r) = \max(N(z_0, r) - \pi r^2)$$

where the maximum is to be taken over all circles of radius r. Probably $f(r)$ is unbounded for every choice of the z's and one would like to estimate how fast $f(r)$ or $F(r) = \max_{0 \leqq R \leqq r} f(R)$ tends to infinity. The method of Roth will perhaps help here too [4].

Let $f(n)$ be an arbitrary number theoretic function which only assumes the values ± 1. Is it true that to every c_1 there exists a d and an m so that

$$g(m, d) = \left| \sum_{k=1}^{m} f(k, d) \right| > c_1?$$

It is perhaps even true that

$$\max_{\substack{d, m \\ dm \leqq n}} g(m, d) > c_2 \cdot \log n.$$

The well known Theorem of van der Waerden [5] asserts that for every k there exists an arithmetic progression $a, a+d, \ldots, a+(k-1)d$ for which $f(a) = \ldots = f(a+(k-1)d)$.

Let finally $1 \leqq a_1 \leqq \ldots \leqq a_n$ be n arbitrary integers. Denote

$$M(a_1, \ldots, a_n) = \max_{|z|=1} \prod_{i=1}^{n} |1 - z^{a_i}|,\ f(n) = \min M(a_1, \ldots, a_n)$$

where the minimum is taken over all sequences $a_1, \ldots, a_n$. Szekeres and I [6] proved

$$\lim_{n=\infty} f(n)^{1/n} = 1, \quad f(n) \geqq \sqrt{2n}.$$

Recently Atkinson [7] proved $f(n) < \exp(n^{\frac{1}{2}} \log n)$ $(\exp z = e^z)$. The lower bound has not yet been improved, though we are sure that this is possible, undoubtedly $f(n) > n^k$ for every k and $n > n_0(k)$. Atkinson's result is perhaps not far from being best possible.

Weyl's criterion [2] does not give an estimation of the discrepancy of a sequence. Turán and I [8] proved, sharpening a previous result of van der Corput and Koksma [1] the following result: Assume that for every k satisfying $1 \leqq k \leqq m$ we have

$$|s_k| = |\sum_{j=1}^{n} e^{2\pi i k x_j}| \leqq \psi(k).$$

Then for a certain absolute constant C

$$D(x_1, \ldots, x_n) < C\left(\frac{n}{m+1} + \sum_{k=1}^{m} \frac{\psi(k)}{k}\right). \tag{5}$$

Koksma and Szüsz independently extended this result for the r-dimensional case [9].

An interesting special case of our Theorem is obtained if we assume

$$|s_k| < k^\lambda \text{ for all } k \leqq n^{1/\lambda}. \tag{6}$$

From (5) we obtain that (6) implies

$$D(x_1, \ldots, x_n) < c_1 n^{\lambda/\lambda+1} \tag{7}$$

We could not decide whether the error term in (7) is best possible.

Another result on the discrepancy of points in the complex plane due to Turán and myself states as follows [10]: Let $f(z) = a_0 + \ldots + a_n z^n$ be a polynomial, denote its roots by

$$z_\nu = r_\nu e^{i\varphi_\nu}, 1 \leqq r \leqq n, M = \max_{|z|=1} |f(z)|.$$

Then for $0 \leqq \alpha < \beta \leqq 2\pi$ we have

$$\left|\sum_{\alpha \leqq \varphi_\nu \leqq \beta} 1 - \frac{\beta-\alpha}{2\pi} n\right| < 16\left(\frac{n \log M}{a_0\, a_n}\right)^{\frac{1}{2}}. \tag{8}$$

It would be interesting to investigate whether (8) remains true if n denotes the number of non vanishing terms of the polynomial

$a_0 + \ldots + a_n z^{k_n}$ (or if (8) does not remain true now does it have to be modified).

The following questions have as far as I know not yet been investigated. Let $w_1, \ldots, w_n$ be n points on the unit sphere chosen in such a way that ($\overline{w_i - w_j}$ denotes the distance of w_i and w_j)

$$\prod_{1 \leqq i < j \leqq n} \overline{w_i - w_j}$$

is maximal. Is it then true that $D_n = o(n)$? (see (4)). Can one improve this estimate?

An analogous question would be the following: Put

$$(9) \qquad A_n = \min_{w_1, \ldots, w_n} \max_{w} \prod_{i=1}^{n} \overline{w - w_i}$$

where the maximum is taken over all points w of the unit sphere and $w_1, \ldots w_n$ varies over all n-tuplets of points on the unit sphere. Two questions can be asked. First of all let $w_1, \ldots, w_n$ be one of the sets for which there is equality in (9). Is it true that for this set $D_n = o(n)$? Secondly assume that $\max_w \prod_{i=1}^{n} \overline{w - w_i}$ is not much larger than A_n, how can one estimate D_n?

II. Now we discuss some questions on uniform distribution. It follows early from (2) that for every k and every irrational number α $(n^k\alpha) = n^k\alpha - [n^k\alpha]$ is uniformly distributed, this beautiful and important result was first proved by Weyl and Hardy-Littlewood [1]. For general sequences $n_1 < n_2 < \ldots$ it is very difficult to decide whether $(n_i\alpha)$ is uniformly distributed e.g. Vinogradoff [11] only recently proved that $(p_n\alpha)$ is uniformly distributed for every irrational $\alpha (p_1 = 2 < p_2 < \ldots$ is the sequence of consecutive primes). Weyl proved that for every sequence of integers $n_1 < n_2 <$ and for almost all α $(n_i\alpha)$ is uniformly distributed. Sharpening previous results Cassels and independently and simultaneously Koksma and I [12] proved that for almost all α the discrepancy of $x_k = (n_k\alpha)$ satisfies for every $\varepsilon > 0$

$$(10) \qquad D(x_1, \ldots, x_N) = o(N^{\frac{1}{2}}(\log N)^{5/2+\varepsilon}).$$

Koksma and I use (5), Cassels's method is more elementary. It would be very interesting to investigate to what extent (10) can be improved. Possible $o(N^{\frac{1}{2}}(\log N)^{5/2+\varepsilon})$ can be replaced by $\sigma(N^{\frac{1}{2}}(\log\log N)^c)$ for a certain constant c. In the special case where the sequence n_i is lacunary i.e. where it satisfies $n_{i+1}/n_i > c > 1$. Gál and I proved this, but our proof which is similar to the one we used to establish the law of the iterated logarithm [13] for lacunary sequences n_i, was not published. It is well known and has been perhaps first obtained by Kac and Steinhaus that for lacunary sequences $(n_i\alpha)$ behaves as if they would be independent. Thus our result with Gál gives no indication what will happen if the condition $n_{i+1}/n_i > c > 1$ is dropped.

The following beautiful conjecture is due to Khintchine [14] Let E be measurable subset of $(0, 1)$ of measure $m(E)$. Denote

$$f_n(\alpha) = \sum_{1 \leqq k \leqq n} 1$$

where the summation is extended over those k's for which $(k\alpha)$ is in E. Then for almost all α and every E

$$\lim_{n=\infty} f_n(\alpha)/n = m(E). \tag{11}$$

Presumably the same result holds if $u_1 < u_2 < \ldots$ is any sequence of integers and $f_n(\alpha)$ denotes the number of indices k for which $(n_k\alpha)$ is in E. This conjecture of Khintchine is very deep, directly or indirectly it inspired several papers. More generally one could ask the following question: Let $n_1 < \ldots$ be an infinite sequence of integers and $f(x)$ is any Lebesgue integrable function in $(0, 1)$. Under what conditions on $f(x)$ and on the sequence $n_1 < \ldots$ is it true that for almost all α $\big((n_k\alpha) = n_k\alpha - [n_k\alpha]\big)$

$$\lim_{N=\infty} \frac{1}{N} \sum_{k=1}^{N} f\big((n_k\alpha)\big) = \int_0^1 f(x)dx. \tag{12}$$

Raikov proved that if $n_k = a^k$, ($a > 1$ integer) then (12) holds for every integrable $f(x)$. A simple proof of this result using ergodic theory was given by F. Riesz [15].

Let $n_{k+1} > (1+c)n_k$, $(c > 0)$ and let $f(x)$ be in L_2 and let $\phi_n(f)$ be the n-th partial sum of the Fourier series of $f(x)$. Sharpening a previous result of Kac, Salem and Zygmund I proved that if

$$\int_0^1 \big(f(x) - \phi_n(f(x))\big)^2 dx = 0\left(\frac{1}{(\log\log n)^{2+\varepsilon}}\right) \tag{13}$$

then (12) holds [16].

Further I constructed [16] a lacunary sequence $n_1 < n_2 < \ldots$ and a function $f(x)$ which is in L_p for every p and for which (12) does not hold. In fact for our $f(x)$ we have for almost all α

$$\limsup_{N=\infty} \frac{1}{N} \sum_{k=1}^{N} f\big((n_k\alpha)\big) = \infty, \tag{14}$$

in fact for our $f(x)$ we have for every $\varepsilon > 0$ and almost all α

$$\limsup_{N=\infty} \frac{1}{N(\log\log N)^{\frac{1}{2}-\varepsilon}} \sum_{k=1}^{N} f((n_k\alpha)) = \infty. \tag{15}$$

On the other hand I can show that if $f(x)$ is in L_2 and $\{n_k\}$ is any lacunary sequence then for almost all α and every $\varepsilon > 0$

$$\lim_{N=\infty} \frac{1}{N(\log N)^{\frac{1}{2}+\varepsilon}} \sum_{k=1}^{N} f((n_k\alpha)) = 0. \tag{16}$$

There is a considerable gap between (15) and (16). I think (15) is closer to the truth but I cannot prove this. I would also think that (13) remains true if $o(1/(\log\log n)^{2+\varepsilon})$ is replaced by $0(1/(\log\log\log n)^{c})$ for a certain $c > 0$, but I have not been able to decide this. It is possible that (12) holds for all bounded functions and every lacunary sequence $\{n_k\}$. It seems impossible to modify my example so that it should become a bounded function.

My lacunary sequence for which (12) does not hold is very special, it would be interesting to try to determine for what lacunary sequences (12) hold for all $f(x)$ in L_2 (or in L_1) and for which lacunary sequences this is not the case, e.g. let $a > 1$ be any real number does (12) hold for the sequence $[a^k]$? (If a is an integer this is the quoted result of Raikov).

Koksma [17] proved the following result: Let $f(x)$ be in L_2 and let $\{c_k\}$ be the sequence of its Fourier coefficients. Assume that

$$\sum_{k=1}^{\infty} c_k^2 \sum_{d|k} \frac{1}{d} < \infty.$$

Then for almost all α

$$\lim_{n=\infty} \frac{1}{n} \sum_{k=1}^{n} f((k\alpha)) = \int_0^1 f(x)dx. \tag{17}$$

I was unable to find an $f(x)$ in L_2 or even in L_1 for which (17) does not hold.

III. A sequence $x_1, x_2, \ldots$ in the interval $(0, 1)$ is said to be well distributed if to every $\varepsilon > 0$ there exists a $k_0 = k_0(\varepsilon)$ so that for every $k > k_0$, $n > 0$ and $0 \leqq a < b \leqq 1$

$$|N_{n,n+k}(a, b)/n-(b-a)| < \varepsilon$$

where $N_{n,n+k}(a, b)$ denotes the number of x_m's, $n < m \leqq n+k$ in the interval (a, b). As far as I know the notion of well distributed sequences was introduced by Hlawka and Petersen [18]. Let

$n_{k+1}/n_k > \lambda > 1$, in contrast to the result of Weyl I proved that for almost all α the sequence $(n_k\alpha)$ is not well distributed.[1] If $n_{k+1}/n_k \to \infty$ it is not difficult to show that the values of α for which $(n_k\alpha)$ is well distributed has the power of the continuum. Further I can prove that there is an irrational number α for which $(p_n\alpha)$ is not well distributed (compare [11]). The proof of these results is not yet published. It seems very probable that $(p_n\alpha)$ is never well distributed (i.e. for no value of α) but I have not been able to show this.

IV. Finally I would like to discuss some results on diophantine approximations. Khintchine [19] proved that if $f(q)$ is monotone decreasing then the condition

$$\sum_{q=1}^{\infty} \frac{f(q)}{q} = \infty \tag{18}$$

is necessary and sufficient that for almost all α the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{f(q)}{q^2} \tag{19}$$

should have infinitely many solutions in integers p and q. It is easy to see that if (18) does not hold (i.e. if $\sum_{q=1}^{\infty} f(q)/q < \infty$) then without any assumption of monotonicity on $f(q)$ it follows that for almost all α (19) has only a finite number of solutions. The question now remains: Does (18) imply (19) without any further assumptions on $f(q)$? Duffin and Schaeffer and Cassels deduced (19) from (18) under much weaker assumptions then monotonicity of $f(q)$, but they both showed (18) does not imply (19) without some condition on $f(q)$ [20].

In his paper [20] Cassels introduced a property of sequences which seems to me to be of interest in itself. Let $n_1 < n_2 < \ldots$ be an infinite sequence of integers. Denote by $\varphi(n_1, \ldots, n_{k-1}; n_k)$ the number of integers $1 \leqq a \leqq n_k$ for which $a/n_k \neq b/n_j$ for every $1 \leqq j < k$. Clearly

$$\varphi(n_1, \ldots, n_{k-1}; n_k) \geqq \varphi(n_k).$$

Cassels calls the sequence $\{n_k\}$ a $\sum$ sequence if

$$\liminf_{k=\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \frac{\varphi(n_1, \ldots, n_{i-1}; n_i)}{n_i}\right) > 0. \tag{20}$$

Cassels shows that there are sequences which are not $\sum$-sequences

[1] Petersen informs me that this was known to him.

(i.e. for which the lim inf. in (20) is 0). I have not succeeded to decide the question whether there is a sequence $n_1 < n_2 < \cdots$ for which

$$\lim_{k=\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \frac{\varphi(n_1, \ldots, n_{i-1}; n_i)}{n_i} \right) = 0. \tag{21}$$

I would guess that such a sequence does not exist. I can only prove that

$$\lim_{i=\infty} \frac{\varphi(n_1, \ldots, n_{i-1}; n_i)}{n_i}$$

cannot be 0. In fact I will outline the proof of a somewhat stronger result: assume that

$$\liminf_{i=\infty} \varphi(n_1, \ldots, n_{i-1}; n_i)/n_i = 0 \tag{22}$$

then

$$\limsup_{i=\infty} \varphi(n_1, \ldots, n_{i-1}; n_i)/n_i = 1. \tag{23}$$

Assume that (22) holds. It immediately follows from $\varphi(n_1, \ldots, n_{i-1}; n_i) \geqq \varphi(n_i)$ that there must be arbitrarily large primes p_j so that $n_i \equiv 0 \pmod{p_j}$ for suitable values of i. Assume now that n_k is the smallest n_i for which $n_i = 0 \pmod{p_j}$. Then if $1 \leqq a < n_k$, $a \not\equiv 0 \pmod{p_j}$ clearly implies $a/n_k \neq b/n_j$ for $1 \leqq j < k$, or $\varphi(n_1, \ldots, n_{k-1}; n_k) \geqq (1-1/p_j)n_k$, which implies (23).

Cassels shows [20] that the necessary and sufficient condition that $n_1 < n_2 < \ldots$ should have the property that the divergence of $\sum_{k=1}^{\infty} f(n_k)/n_k$ implies that for almost all α

$$\left| \alpha - \frac{m_k}{n_k} \right| < \frac{f(n_k)}{{n_k}^2}$$

has infinitely many solutions, is that $n_1 < n_2 < \ldots$ should be a $\sum$-sequence. Cassels also shows that every sequence $n_{k+1} > (1+c)n_k$ $(c > 0)$ is a $\sum$-sequence. It seems likely that a weaker condition will imply that a sequence is a $\sum$-sequence, but as far as I know no such condition is known.

Duffin and Schaeffer [20] made the following beautiful conjecture: Let ε_q, $1 \leqq q < \infty$ be an arbitrary sequence of non-negative numbers. The necessary and sufficient condition that for almost all α the inequality

$$\left| \alpha - \frac{p}{q} \right| < \frac{\varepsilon_q}{q}$$

should have infinitely many solutions in integers $(p, q) = 1$ is that

$$\sum_{q=1}^{\infty} \frac{\varepsilon_q \varphi(q)}{q}$$

diverges. ($\varphi(q)$ is Euler's φ function). It is easy to prove the necessity, the real difficulty is to prove the sufficiency.

I proved the following special case of this conjecture. Let $\varepsilon > 0$ be fixed and let $\varepsilon_q = 0$ or $\varepsilon_q = \varepsilon$. The necessary and sufficient condition that for almost all α.

$$\left|\alpha - \frac{p}{q}\right| < \frac{\varepsilon_q}{q^2}, \ (p, q) = 1$$

has infinitely many solutions is that $\sum_{q=1}^{\infty} \varepsilon_q \varphi(q)/q^2$ diverges. The proof is very complicated and has not yet been published. My proof in fact gives the following slightly sharper result: Let $\varepsilon_q \geqq 0$ be a bounded sequence. Then the necessary and sufficient condition that for almost all α,

$$\left|\alpha - \frac{p}{q}\right| < \frac{\varepsilon_q}{q^2}, \ (p, q) = 1$$

has infinitely many solutions is that $\sum_{q=1}^{\infty} \varepsilon_q \varphi(q)/q^2$ diverges. Due to the great technical difficulties of the proof I am not at present certain whether my method gives the general conjecture of Duffin-Schaeffer.

My result immediately implies the following theorem: Let $n_1 < n_2 < \dots$ be an arbitrary infinite sequence of integers. The necessary and sufficient condition that for almost all α infinitely many of the n_i should be denominators of the convergents of the regular continued fraction of α is that $\sum_{i=1}^{\infty} \varphi(n_i)/n_i^2$ diverges. (i.e. it is well known that if $|\alpha - m_i/n_i| < 1/2n_i^2$, $(m_i, n_i) = 1$ then m_i/n_i are convergent of α). Hartman and Szüsz proved a special case of the above result [21]. Finally I would like to state four unrelated problems on diophantine approximation.

1. Hecke and Ostrowski [22] proved the following theorem: Let α be an irrational number and denote by $N_n(u, v)$ the number of integers $1 \leqq m \leqq n$ for which

$$0 \leqq u \leqq (m\alpha) < v \leqq 1.$$

Then if both u and v are of the form $(k\alpha)$ then

$$N_n(u, v) = n(v-u) + O(1) \tag{24}$$

Szüsz and I conjectured the converse of this theorem, i.e. if (24) holds then $u = (k_1 \alpha)$, $v = (k_2 \alpha)$, unfortunately we had not been able to make any progress with this conjecture.

2. Denote by $S(N, A, c)$ the measure of those α in $(0, 1)$ for which

$$\left| \alpha - \frac{x}{y} \right| < \frac{A}{y^2}, \ (x, y) = 1$$

is solvable for some y satisfying $N \leqq y \leqq cN$.

Szüsz, Turán and I conjectured that [23]

$$\lim_{N=\infty} S(N, A, c) = f(A, c)$$

exists. What is its explicit form?

In our paper [23] we only solved a very special case of this problem. Recently Kesten [24] strengthened our results, but the general problem is still unsolved.

3. Consider $0 < \alpha < 1$

$$f(\alpha, n) = \frac{1}{\log n} \sum_{k=1}^{n} \left((k\alpha) - \tfrac{1}{2} \right).$$

Is it true that $f(\alpha, n)$ has an asymptotic distribution function? In other words is it true that there is a non-decreasing function $g(c)$, $g(-\infty) = 0$, $g(+\infty) = 1$, so that if $m[f(\alpha, n), c]$ denotes the measure of the set in α for which $f(\alpha, n) \leqq c$ then $\lim m[f(\alpha, n), c] = g(c)$. Probably $g(c)$ will be a strictly increasing continuous function. Important recent contributions to this problem have recently been made by Kesten, [25] but as far as I know it is not yet completely solved.

4. The following interesting problem is due to LeVeque: Let $a_1 < a_2 < \ldots$ be an infinite sequence tending to infinity satisfying

$$a_{i+1}/a_i \to 1. \text{ Let } a_i \leqq x_n < a_{i+1}. \text{ Put}$$

$$y_n = \frac{x_n - a_i}{a_{i+1} - a_i}, \ 0 \leqq y_n < 1.$$

We say that the sequence x_n, $1 \leqq n < \infty$ is uniformly distributed mod $a_1, a_2, \ldots$ if y_n, $1 \leqq n < \infty$ is uniformly distributed. Is it true that for almost all α the sequence $n\alpha$, $1 \leqq n < \infty$ is uniformly distributed mod $a_1, \ldots$? LeVeque proved this in some special cases [26].

Added in proof: Since this paper was written the following papers were published on the problem of LeVeque:

H. Davenport, P. Erdös and W. J. LeVeque, On Weyl's criterion for uniform distribution, Michigan Math. Journal 10 (1963), 311–314;

H. Davenport and W. J. LeVeque, Uniform distribution relative to a fixed sequence, ibid 10 (1963), 315–319 and

P. Erdös and H. Davenport, Publ. Math. Inst. Hung. Acad. (1963).

REFERENCES

J. F. Koksma

[1] Diophantische Approximationen, Ergebnisse Math. u. Grenzgeb. Vol. 4 Heft 4, Berlin 1936.

J. Cigler and G. Helmberg

[2] Neuere Entwicklungen der Theorie der Gleichverteilung, Jahresbericht D.M.V. 64 (1961), 1–50.

P. Erdös

[3] Some unsolved problems, Publ. Math. Inst. Hung. Acad. Sci. 6 (1961), 221–254.

T. van Aardenne-Ehrenfest

[4] On the impossibility of a just distribution, Indigationes math. 11 (1949), 264–269.

K. F. Roth

On irregularities of distribution, Mathematika, 1 (1954), 73–79.

B. L. van der Waerden

[5] Beweis einer Baudetschen Vermutung, Archiv Viskunde (2) 15 (1928), 212–216, for further problems and results in this direction see

R. Rado

Studien zur Kombinatorik, Math. Zeitschrift 36 (1933) 424–480.

P. Erdös and G. Szekeres

[6] On the product $\prod_{k=1}^{n}(1-z^{a_k})$, Acad. Serbe des Sci. 13 (1959), 29–34.

F. V. Atkinson

[7] On a problem of Erdös and Szekeres. Canadian Math. Bull. 4 (1961), 7–12.

P. Erdös and P. Turán

[8] On a problem in the theory of uniform distribution I, II. Indigationes math. 10 (1948) 370–378, 406–413.

J. F. Koksma

[9] Some theorems on diophantine inequalities, Math. Centrum Amsterdam Scriptum 5 (1950),

P. Szüsz

On a problem in the theory of uniform distribution (written in Hungarian) C.r. first Hungarian Math. Congress 461–472 (1952).

P. Erdös and P. Turán

[10] On the distribution of roots of polynomials, Annals of Math. 51 (1950), 105–119.

I. M. Vinogradov

[11] Über die Abschätzung trigonometrischer Summen mit Primzahlen, Isvestija Akad. Nauk. SSSR Ser. mat 12 (1948), 225–248.

J. W. S. Cassels

[12] Some metrical theorems in diophantine approximation, Proc. Cambridge Phil. Soc. 46 (1950), 209–218.

P. Erdös and J. F. Koksma

[13] On the uniform distribution mod 1 of sequences $\{f(n, \theta)\}$, Indig. Math. 11 (1949), 299–302.

P. Erdös and I. S. Gál

[13] On the law of the iterated logarithm, Indig. Math. 17 (1955) 65–84.

A. J. Khintchine

[14] Ein Satz über Kettenbruche mit arithmetischen Anwendungen, Math. Zeitschrift 18 (1923), 289–306, see p. 303–304.

F. Riesz

[15] Sur la théorie ergodique, Comment. Math. Helv. 17 (1945), 221–239.

P. Erdös

[16] On the strong law of large numbers, Trans. Amer. Math. Soc. 67 (1949), 51–56.

J. F. Koksma

[17] Estimations de fonctions à l'aide d'intégrales de Lebesgue, Bull. Soc. Math. Belg. 6 (1953–54) 4–13.

E. Hlawka

[18] Zur formalen Theorie der Gleichverteilung in kompakten Gruppen, Rend. Cisc. Mat. Palermo 4 (1955), 33–47.

G. M. Petersen

Almost convergence and uniformly distributed sequences, quarterly J. of Math. Oxford II Ser. 7 (1956), 188–191.

See also F. R. Keogh, B. Lawton and G. M. Petersen, Well distributed sequences, Canad. J. Math. 10 (1958), 572–576.

A. J. Khintchine

[19] Einige Sätze über Kettenbrüche mit Anwendungen auf die Theorie der Diophantischen Approximationen Math. Ann. 92 (1924), 115–125.

R. J. Duffin and A. C. Schaeffer

[20] Khintchine's problem in metric diophantine approximation, Duke Math. J. 8 (1941), 243–255, J. W. S. Cassels, Some metrical theorems in diophantine approximation I. Proc. Cambridge Phil. Soc. 46 (1950), 209–218.

S. Hartman and P. Szüsz

[21] On congruence classes of denominators of convergents, Acta Arith. 6 (1960), 179–184. See also

P. Szüsz

Verallgemeinerung und Anwendungen eines Kusminschen Satzes, ibid. 7 (1962), 149–160, and P. Süsz, Über die metrische Theorie der Diophantischen Approximationen (will soon appear in Acta Arith.).

E. Hecke

[22] Über analytische Funktionen und die Verteilung der Zahlen mod 1, Abh. math. Semin. Hamburg Univ. 1 (1922) 54—76,

A. Ostrowski

Miszellen IX and XVI, Notiz zur Theorie der Diophantischen Approximationen und zur Theorie der linearen Diophantischen Approximationen. Jahresbericht der Deutschen Math. Ver. 36 (1927), 178—180 und 39 (1930), 34—46).

P. Erdös, P. Szusz and P. Turán

[23] Remarks on the theory of diophantine approximation, Coll. Math. 6 (1958), 119—126.

H. Kesten

[24] Some probabilistic theorems on diophantine approximations, Trans. Amer. Math. Soc. 103 (1962), 189—217.

H. Kesten

[25] Uniform distribution mod 1, Annals of Math. 71 (1960), 445—471.

W. J. LeVeque

[26] On uniform distribution modulo a subdivision, Pacific J. of Math. 3 (1953), 757—771.

(Oblatum 29-5-63) University College, London.