SOME REMARKS ON SET THEORY, IX. COMBINATORIAL PROBLEMS IN MEASURE THEORY AND SET THEORY

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To the memory of our friend and collaborator, J. Czipszer

1. INTRODUCTION

A well-known theorem of Ramsay [12; p. 264] states that if the k-tuples of an infinite set S are split into a finite number of classes, then there exists an infinite subset of S all of whose k-tuples belong to the same class. (For $k = 1$, this is trivial.)

Suppose that with each element x of an infinite set S there is associated a measurable set $F(x)$ in the interval $[0, 1]$. It is known that if the measure $m(F(x))$ of the sets $F(x)$ are bounded away from zero, then some real number c is contained in infinitely many sets $F(x)$. For the sake of completeness, we prove this.

It clearly suffices to consider the case where S is the set of natural numbers. For each t in S, let

$$G_t = \bigcup_{n=t}^{\infty} F(n) \quad \text{and} \quad G = \bigcap_{t=1}^{\infty} G_t,$$

where $m(F(n)) \geq u > 0$ for $n \in S$. Clearly, $m(G_t) \geq u$ and $G_{t+1} \subset G_t$ $(t = 1, 2, \cdots)$ (throughout the paper, the symbol $\subset$ refers to inclusion in the broad sense). Thus, by a classical theorem of Lebesgue, $m(G) \geq u$. Since each c in G is contained in infinitely many sets $F(t)$, this completes the proof.

Now, in analogy to Ramsay's theorem, one might consider the following problem. Suppose that, for some $u > 0$, there is associated with each k-tuple $X = \{x_1, \cdots, x_k\}$ of elements of an infinite set S a measurable set $F(X)$ of $[0, 1]$ such that $m(F(X)) \geq u$. Does there always exist an infinite subset S' of S such that the sets $F(X)$ corresponding to the k-tuples X of S' have a nonempty intersection? We study this and related questions. In the course of our investigation we are led to a surprising number of unsolved problems.

All of our results concern the case $k = 2$, but we shall state some problems for $k > 2$ as well.

Instead of choosing a measurable subset of $[0, 1]$ for every k-tuple of a set S, we could choose an abstract set having certain properties. Interesting problems of a new type then arise, which we discuss briefly in Section 4. There we investigate some purely graph-theoretical questions, and in particular we give a simple construction of graphs that contain no triangle and have arbitrarily high chromatic numbers.

Received January 9, 1962 and March 18, 1963.

2. NOTATION AND DEFINITIONS

We adopt the following notation:

cardinal numbers: a, b, m, n;

ordinal numbers: $\alpha, \beta, \cdots, \nu, \mu, \cdots$;

nonnegative integers: i, j, k, ℓ, r, s, t;

real numbers in $[0, 1]$: c, u, v, $u_1, u_2, \cdots, \theta$;

abstract sets: S, X, Y;

the cardinal number of S: $\overline{\overline{S}}$;

elements of sets: x, y, $\cdots$;

the least cardinal number greater than n: n^+.

The symbols $[S]^a$ and $[S]^{<a}$ denote the classes of subsets of S that have cardinality a and less than a, respectively. If X and Y are disjoint sets, we write

$$[X, Y] = \{ (x, y) \mid x \in X \text{ and } y \in Y \} .$$

Let S be a set of power m, and let F denote a function that associates a measurable subset of $[0, 1]$ with each $X \in [S]^k$. For brevity, we shall say that F is a set-function on S of type k. (The symbol F will always denote a set-function.) Suppose $0 \leq u \leq 1$. If, for each $x \in [S]^k$, $m(F(X)) \geq u$ or $m(F(X)) > u$, we say that F is of order at least u or of order greater than u, respectively.

Let Z be a subset of $[S]^k$. If

$$\bigcap_{X \in Z} F(X) \neq \emptyset ,$$

we say that Z possesses property $\mathscr{P}$ (with respect to F).

With specific reference to the problems mentioned in Section 1, we introduce the following symbols.

(1) $$(m, k, u) \Rightarrow n \quad \text{and} \quad (m, k, > u) \Rightarrow n$$

represent the respective statements: If $\overline{\overline{S}} = m$ and if F is a set-function on S of type k and of order at least u (of order greater than u), then S has a subset S', of cardinality n, such that $[S']^k$ possesses property $\mathscr{P}$. To say that a statement involving the symbol $\Rightarrow$ is false, we replace $\Rightarrow$ by $\not\Rightarrow$.

The symbolic statement

(2) $$(m, u) \Rightarrow (n_1, n_2)$$

means that if $\overline{\overline{S}} = m$ and F is a set-function on S, of type 2 and of order at least u, then there exist disjoint subsets S_1 and S_2 of S with cardinality n_1 and n_2, respectively, such that $[S_1, S_2]$ possesses property $\mathscr{P}$. Instead of $(m, 2, u) \Rightarrow n$, we often write that S contains a complete graph of power n that has property $\mathscr{P}$ (with respect to F).

The theorems in whose proofs we use the generalized continuum hypothesis are marked by an asterisk: (*).

3. THE CASE $m \leq \aleph_0$

THEOREM 1. *Suppose that* $2 \leq r < \omega$. *Then* $(\aleph_0, 2, u) \Rightarrow r + 1$ *if and only if* $u > 1 - 1/r$.

Proof. First we show the condition that $u > 1 - 1/r$ to be necessary. If $\theta \in (0, 1)$, let

$$\theta = \sum_{t=1}^{\infty} \frac{s_t}{r^t} \qquad (0 \leq s_t < r) \tag{3}$$

be its r-ary expansion with infinitely many positive coefficients s_t. Let S be the set of positive integers. The desired set-function F of type 2 on S is defined as follows. If $1 \leq t_1 \neq t_2 < \omega$, then

$$\theta \in F(\{t_1, t_2\}) \quad \text{if and only if } s_{t_1} \neq s_{t_2} \tag{4}$$

in the r-ary expansion (3) of θ.

Clearly,

$$m(F(\{t_1, t_2\}) = 1 - \frac{1}{r},$$

and thus F is of order no less than $1 - 1/r$. On the other hand, S does not contain a complete graph of power $r + 1$ that has property $\mathscr{P}$. For if $S' = \{t_1, \cdots, t_{r+1}\}$ and $[S']^2$ possesses property $\mathscr{P}$, then there exists a $\theta \in (0, 1)$ such that

$$\theta \in \bigcap_{t_i, t_j \in S';\ i \neq j} F(\{t_1, t_j\}).$$

Therefore, by (4), the numbers $s_{t_1}, \cdots, s_{t_{r+1}}$ are all different, which contradicts (3). This establishes the necessity of our condition.

We complete the proof of the theorem by proving not only the sufficiency of our condition but a stronger result as well; namely, we prove that corresponding to each $u > 1 - 1/r$, there exists an integer k_u such that

$$(k_u, 2, u) \Rightarrow r + 1.$$

Indeed, let k denote a positive integer, let $S = \{0, 1, \cdots, k - 1\}$, and let F be a set-function on S, of type 2 and of order not less than u.

There is no loss of generality in supposing that $m(F(X)) = u$ for each $X \in [S]^2$. For if $m(F(X))$ were greater than u for some of the X, we could replace each of the sets F(X) by a subset $F_1(X)$, of measure u. Clearly, a subset of $[S]^2$ having property $\mathscr{P}$ relative to F_1 would also have property $\mathscr{P}$ relative to F.

Suppose now that every point c of (0, 1) lies in fewer than $u\binom{k}{2}$ of the sets F(X). Then

$$\sum_{X\epsilon[S]^2} m(F(X)) < u\binom{k}{2},$$

contrary to the hypothesis that F has order at least u. Hence some c lies in at least $u\binom{k}{2}$ of the sets F(X). That is, the graph induced by some c has at least $u\binom{k}{2}$ edges, and of course the number h of its vertices is at most k. A special case of a theorem of P. Turán [14; p. 26] asserts that a graph with h vertices and more than $\frac{1}{2}(1 + \varepsilon - 1/r)h^2$ edges contains a complete (r + 1)-gon. It follows that the graph induced by c contains a complete (r + 1)-gon. This completes the proof of Theorem 1.

Let S be the set of natural numbers, and let F be a set-function on S, of type 2 and of order at least u. For each subset S' of S, we write

$$\Pi(S') = \bigcap_{X\epsilon[S']^2} F(X).$$

The "if" part of Theorem 1 asserts that if $u > 1 - 1/r$, then some set S' of r + 1 natural numbers has property $\mathscr{P}$, that is, satisfies the condition $\Pi(S') \neq \emptyset$. The question now arises as to what can be said about the measure of $\Pi(S')$. We prove the following assertion, which provides a sharpening, for the special case r = 2, of Theorem 1.

THEOREM 1(A). *Let* S *be the set of natural numbers, and let* F *be a set-function on* S, *of type* 2 *and of order at least* u $(u > 1/2)$. *Then, for every* $\varepsilon > 0$, *there exists a set* S' *of three natural numbers such that* $m(\Pi(S')) \geq u(2u - 1) - \varepsilon$.

This result is best possible for some special values of u, *in the following sense: If* $u = 1 - 1/k$ $(k = 3, 4, \cdots)$, *then there exist set-functions* F *on* S, *of order* u *and of type* 2, *such that* $m(\Pi(X)) \leq u(2u - 1)$ *for every* $X \subset S$ *with* $\overline{\overline{X}} = 3$.

Remarks. It is obvious that Theorem 1(A) is a generalization of the special case r = 2 of Theorem 1. We do not know whether the positive part of this result is best possible for other values of u. As to the cases $r > 2$, we conjecture that if $u > 1 - 1/r$, then there exists a subset $S' \subset S$ with $\overline{\overline{S'}} = r + 1$ for which

$$m(\Pi(S')) \geq u(2u - 1)(3u - 2) \cdots (ru - (r - 1)) - \varepsilon.$$

Here we also know that the result, if true, is best possible for certain special values of u.

Before proving Theorem 1(A), we state some well-known results that we shall often use in the sequel (see [5] and [9]).

(5) *To each* $\varepsilon > 0$ *and each positive integer* r, *there corresponds an integer* $s_0(\varepsilon, r)$ *with the following property. If* $\{A_k\}$ $(1 \leq k \leq s_0(\varepsilon, r))$ *is a family of measurable subsets of* [0, 1] *and if* $m(A_k) \geq u > 0$ *for all* k, *then there exist* r *integers* $k_1 < k_2 < \cdots < k_r \leq s_0(\varepsilon, r)$ *such that*

$$m\left(\bigcap_{i=1}^{r} A_{k_i}\right) > u^r - \varepsilon .$$

The following is an easy corollary.

(6) *Let* $\{A_k\}$ $(1 \leq k < \infty)$ *be a sequence of measurable subsets of* $[0, 1]$, *let* $m(A_k) \geq u > 0$, *and let* $\varepsilon > 0$. *Then, corresponding to each positive integer* r, *there exists an increasing sequence* $\{h_j\}$ *of integers such that*

$$m\left(\bigcap_{i=1}^{r} A_{k_i}\right) > u^r - \varepsilon$$

for every set $\{k_i\}$ $(1 \leq i \leq r)$ *taken from* $\{h_j\}$.

Now we outline the proof of Theorem 1(A). Let S be the set of natural numbers, let F be a set-function on S satisfying the requirements of Theorem 1(A), and let $\varepsilon > 0$. Without loss of generality, we may assume that $m(F(X)) = u$ for each $X \in [S]^2$.

First we define a partition

$$[S]^3 = J_1 \cup J_2 \cup J_3 \cup J_4$$

as follows. For each $X = \{t_1, t_2, t_3\}$ $(t_1 < t_2 < t_3)$ we put

$$F_1(X) = F(\{t_1, t_2\}) \cap F(\{t_1, t_3\}),$$

$$F_2(X) = F(\{t_1, t_3\}) \cap F(\{t_2, t_3\}),$$

and we write

$$(7) \qquad \begin{cases} X \in J_1 \text{ if } m(F_1(X)) > u^2 - \varepsilon/2 \text{ and } m(F_2(X)) > u^2 - \varepsilon/2, \\ X \in J_2 \text{ if } m(F_1(X)) > u^2 - \varepsilon/2 \text{ and } m(F_2(X)) \leq u^2 - \varepsilon/2, \\ X \in J_3 \text{ if } m(F_1(X)) \leq u^2 - \varepsilon/2 \text{ and } m(F_2(X)) > u^2 - \varepsilon/2, \\ X \in J_4 \text{ if } m(F_1(X)) \leq u^2 - \varepsilon/2 \text{ and } m(F_2(X)) \leq u^2 - \varepsilon/2. \end{cases}$$

If $S_1 \subset S$ and $\overline{\overline{S}}_1 = \aleph_0$, then S_1 contains triplets X_1 and X_2 such that

$$m(F_1(X_1)) > u^2 - \varepsilon/2 \quad \text{and} \quad m(F_2(X_2)) > u^2 - \varepsilon/2 .$$

This is so because by (5) (with $r = 2$) the set S_1 contains no infinite subset all of whose triplets belong to the classes J_i $(i = 2, 3, 4)$.

From Ramsay's theorem (see the beginning of the Introduction) it follows that all triplets of some infinite subset of S belong to J_1. Let $S' = \{t_1, t_2, t_3\}$ $(t_1 < t_2 < t_3)$ be any triplet in J_1. Then, by the assumption that $m(F(\{t_1, t_2\})) = u$ and by the first line of (7),

$$m(\Pi(S')) > u^2 - \frac{\varepsilon}{2} - \left[u - \left(u^2 - \frac{\varepsilon}{2}\right)\right] = u(2u - 1) - \varepsilon .$$

This completes the proof of the first part of Theorem 1(A).

Now we prove the "best possible" part of Theorem 1(A). Let S be the set of natural numbers, and for any $k \geq 3$, consider the k-ary expansion (3) (with k in place of r) of an arbitrary $\theta \in [0, 1]$. Using again the idea of (4), we define $F(\{t_1, t_2\})$ (for $1 \leq t_1 \neq t_2 < \omega$) by the rule

$$\theta \in F(\{t_1, t_2\}) \quad \text{if and only if } s_{t_1} \neq s_{t_2}. \tag{8}$$

Clearly, $m(F(\{t_1, t_2\})) = u = 1 - 1/k$. On the other hand, suppose that $X \in [S]^3$, $X = \{t_1, t_2, t_3\}$ $(t_1 < t_2 < t_3)$. From well-known properties of the expansion (3) and from (8) it follows that

$$m(\Pi(X)) = \frac{k(k-1)(k-2)}{k^3} = (1 - 1/k)(1 - 2/k) = u(2u - 1).$$

This completes the proof of Theorem 1(A).

THEOREM 1(B).

$$\left(\aleph_0, 2, > 1 - \frac{1}{r}\right) \nrightarrow r + 1 \qquad 2 \leq r < \omega.$$

We only outline the proof. First we establish the following result.

(9) *Let* S *be the set of natural numbers. Corresponding to each pair* t_1, t_2 $(1 \leq t_1 \neq t_2 < \omega)$ *and each* $\varepsilon > 0$, *one can define a set function* $F_{\{t_1,t_2\}}$ *on* S, *of type* 2 *and satisfying the following conditions:*

(a) $\Pi(Z) = \emptyset$ *for every* $Z \in [S]^{r+1}$,

(b) $m(F_{\{t_1,t_2\}}(X)) = 1 - \frac{1}{r}$ *for every* $X \in [S]^2$ *except* $X = \{t_1, t_2\}$,

(c) $m(F_{\{t_1,t_2\}}(\{t_1, t_2\})) > 1 - \varepsilon$.

This can be proved, by a slight modification of the construction used in the proof of Theorem 1, as follows.

Let ℓ be an integer, put $k = \ell r$, and for any $\theta \in [0, 1]$, let

$$\theta = \sum_{t=1}^{\infty} \frac{\tau_t}{k^t} \qquad (0 \leq \tau_t < k).$$

For $t \in S$ and $i = 0, \cdots, r - 1$, we now define a set $S_{t,i}$ as follows. If $t \neq t_1$ and $t \neq t_2$, then $S_{t,i}$ is the set of natural numbers s satisfying the condition $\ell i \leq s < \ell(i + 1)$; for the other cases,

$$S_{t_{1,0}} = \{0, 1, \cdots, (\ell - 1)r\},$$

$$S_{t_{1,i}} = \{(\ell - 1)r + i\} \qquad \text{for } i = 1, \cdots, r - 1,$$

$$S_{t_{2,i}} = \{i\} \qquad \text{for } i = 0, \cdots, r - 2,$$

$$S_{t_2,r-1} = \{r - 1, \cdots, \ell r - 1\}.$$

Now we define $F(\{t, t'\})$ for $1 \leq t \neq t' < \omega$ by the stipulation that $\theta \in F(\{t, t'\})$ if and only if s_t and $s_{t'}$ belong to sets $S_{t,i}$ and $S_{t',i'}$ with $i \neq i'$.

F clearly satisfies the requirements (a) and (b) of (9).

On the other hand,

$$m(F_{\{t_1,t_2\}}(\{t_1, t_2\})) \geq \frac{1}{k^2}(k - r)^2 \geq 1 - \frac{2}{\ell} > 1 - \varepsilon$$

if ℓ is sufficiently large.

Now let $\{X_j\}$ $(j < \omega)$ be a well-ordering of type ω of the set $[S]^2$. It follows from (9) that, corresponding to every $j < \omega$, there exists a set-function F_{X_j} on S that satisfies the following conditions:

(10) $$F_{X_j}(X) \subset (2^{-j-1}, 2^{-j}) \quad \text{for every } X \in [S]^2;$$

the set $\Pi(Z)$ (defined with respect to F_{X_j}) is empty for every $Z \in [S]^{r+1}$;

$$m(F_{X_j}(X)) = \left(1 - \frac{1}{r}\right) 2^{-j-1} \quad \text{for every } X \in [X]^2 \text{ except } X_j;$$

$$m(F_{X_j}(X_j)) > (1 - \varepsilon)\, 2^{-j-1}.$$

Next we define the set-function F on S, of type 2, by the condition

(11) $$F(X) = \bigcup_{j<\omega} F_{X_j}(X) \quad \text{for every } X \in [S]^2.$$

We easily see from (10) and (11) that $\Pi(Z) = \emptyset$ for every $X \in [S]^{r+1}$, and that

$$m(F(X_j)) = 1 - \frac{1}{r} + \left(\frac{1}{r} - \varepsilon\right) 2^{-j-1} > 1 - \frac{1}{r}$$

if $\varepsilon < \frac{1}{r}$. Hence F is of order greater than $1 - \frac{1}{r}$, and this proves Theorem 1(B). The idea of the proof is partly due to J. Czipszer.

Let $m_j = m(F(X_j)) - \left(1 - \frac{1}{r}\right)$ for $j < \omega$, and write $m = \Sigma_{j=0}^{\infty} m_j$. In the case of the example just constructed, $m > 1/r - \varepsilon$. We do not know how far this inequality can be improved; we only have some special results which show that if m is sufficiently large for a set-function F on S, of type 2 and of order greater than $1 - 1/r$, then there always exists a complete $(r + 1)$-gon with the property $\mathscr{P}$. We omit the proof of this, and we only mention that questions of this type lead to interesting problems in measure theory.

THEOREM 2. *If* u *is positive, then* $(\aleph_0, u) \Rightarrow (r, \aleph_0)$ *for each nonnegative integer* r.

Proof. We are given a set S with cardinality $\aleph_0$. Without loss of generality we suppose that $S = \{t \mid t < \omega\}$. Let F be a set-function on S, of type 2 and of order at least u. We shall prove that, in fact, to each r and u $(u > 0)$, there corresponds an integer $s = s(u, r)$ with the following property. Amongst any s integers $t_1, \cdots, t_s$, there exist r integers $t_{i_1}, \cdots, t_{i_r}$ such that an infinite subset S' of S exists for which $[\{t_{i_1}, \cdots, t_{i_r}\}, S']$ possesses property $\mathscr{P}$.

If s is a positive integer, we let $Z = \{t_1, \cdots, t_s\}$, and for some t not in Z, we consider the sets $F(\{t_i, t\})$ $(1 \leq i \leq s)$. Let δ be a positive number less than u^r. It follows from (5) that if s is sufficiently large, say $s > s_0(u^r - \delta, r)$, then there exist r vertices $t_{i_1}, \cdots, t_{i_r}$ among the t_i for which

$$m = m\left(\bigcap_{j=1}^{r} F(\{t_{i_j}, t\})\right) > \delta .$$

Since there are infinitely many $t \notin Z$ but only $\binom{s}{r}$ possible choices of indices $i_1, \cdots, i_r$, some set of indices, say $\{i_1, \cdots, i_r\}$, corresponds to infinitely many t. Denote this set of t's by S". Then S" is a subset of S of power $\aleph_0$.

Let

$$E_t = \bigcap_{j=1}^{r} F(\{t_{ij}, t\}) \qquad (t \in S'') .$$

Since $m(E_t) > \delta$, the theorem proved in the Introduction guarantees the existence of a denumerable subset S' such that

$$\bigcap_{t \in S'} E_t \neq \emptyset .$$

But this means that $[\{t_{i_1}, \cdots, t_{i_r}\}, S']$ has property $\mathscr{P}$. This proves Theorem 2.

The question may now be asked: if u is positive, is the statement

$$(\aleph_0, u) \twoheadrightarrow (\aleph_0, \aleph_0)$$

true? We were not, in general, able to answer this question, which is one of the most interesting unsolved problems of our paper. We describe a simple example by means of which J. Czipszer showed that the answer is negative if $u < 1/2$. Let S be the set of natural numbers, and let $2 \leq r < \omega$. Czipszer defined a set-function F_r^* of type 2 on S as follows. If (t_1, t_2) is any pair with $1 \leq t_1 < t_2 < \omega$, and if $\{s_t\}$ denotes the sequence of digits in the nonterminating r-ary expansion of a number θ in $(0, 1]$, then

(12) $$\theta \in F_r^*(\{t_1, t_2\}) \quad \text{if and only if } s_{t_1} > s_{t_2} .$$

Clearly, $m(F_r^*(X)) = \frac{1}{2}\left(1 - \frac{1}{r}\right)$; hence F_r^* is of order at least $\frac{1}{2}\left(1 - \frac{1}{r}\right)$. Since $\frac{1}{2}\left(1 - \frac{1}{r}\right) \to \frac{1}{2}$, we only need to show that if S', S" are disjoint infinite subsets of S, then [S', S"] does not possess property $\mathscr{P}$ with respect to F_r^* for $2 \leq r < \omega$. In fact, if S' and S'' are disjoint infinite subsets of S, then there exists an infinite increasing sequence $\{t_k\}$ of natural numbers such that $t_k \in S'$ if k is odd and $t_k \in S''$ if k is even, and $[S', S'']$ does not possess property $\mathscr{P}$ with respect to F_r^* since the set of edges $\{t_i, t_{i+1}\}$ $(1 \leq i \leq k)$ also fails to possess property $\mathscr{P}$ for $k \geq r$.

Czipszer's example leads to some interesting new questions. First we need some definitions.

Let S be the set of natural numbers, let $T_r = \{t_1, \cdots, t_{r+1}\}$ be a sequence of $r + 1$ natural numbers, and let $T_\infty = \{t_1, \cdots, t_r, \cdots\}$ be an infinite sequence of different natural numbers. Put

$$J_{r+1} = \{\{t_i, t_{i+1}\}\} \ (1 \leq i \leq r), \qquad J_\infty = \{\{t_i, t_{i+1}\}\} \ (1 \leq i \leq \omega).$$

Further, let F be a set-function defined on S, of type 2 and of order at least u. We briefly say that S contains a path J_{r+1} of length $r + 1$ (with property $\mathscr{P}$) or an infinite path J_∞ (with property $\mathscr{P}$) if there exists a T_r or a T_∞ such that the corresponding sets J_{r+1} or J_∞ possess property $\mathscr{P}$ (with respect to F), respectively. If in addition the sequences T_r or T_∞ are increasing, we say that S contains an increasing path of length $r + 1$ or an increasing infinite path, respectively. We do not know under what conditions on u the set S contains an infinite path. Perhaps this is the simplest unsolved problem in our paper.

Now Czipszer's set-functions F_r^* show that for $u < 1/2$ the set S need not contain an infinite increasing path, and more generally, that with respect to a set-function of type 2 and order at least $\frac{1}{2}\left(1 - \frac{1}{r}\right)$, S need not contain an increasing path of length $r + 1$. The question arises whether this is best possible in u. It may be true that if $u \geq 1/2$ then there exists an infinite increasing path, or that if $u > \frac{1}{2}\left(1 - \frac{1}{r}\right)$ then there exists an increasing path of length $r + 1$, respectively. We can prove this only for $r = 2$.

The character of a problem concerning increasing paths is somewhat different from that of the problems treated so far in our paper; for the problem is meaningful only if the basic set S is an ordered set, and the answer depends not only on the power of S, but also on its order type.

Now we give our result concerning the case $r = 2$.

THEOREM 3. *Let* S *be the set of natural numbers, and let* F *be a set-function defined on* S, *of type* 2 *and of order at least* u. *If* $u > 1/4$, *then there exists an increasing path* I_3 *with property* $\mathscr{P}$. *For* $u \leq 1/4$, *this is not necessarily true.*

We do not know what happens in case F is merely required to be of order greater than 1/4.

Proof. The negative part of our theorem is shown by the set-function F_2^* defined in (12). Consider now a set-function satisfying the requirements of Theorem 3.

For $t = 1, 2, \cdots$, define

$$E_t = \bigcup_{t < t' < \omega} F(\{t, t'\}) \quad \text{and} \quad m_t = m(E_t), \tag{13}$$

and let $0 < \varepsilon < u - 1/4$. There exists a real number m and an infinite subset $S' \subset S$ such that $|m - m_t| < \varepsilon/2$ for $t \in S'$.

By (5) and (13), there exist t_1 and t_2 $(1 \leq t_1 < t_2 < \omega,\ t_1, t_2 \in S')$ such that

$$m(E_{t_1} \cap E_{t_2}) > m^2 - \varepsilon/2 .$$

Now $F(\{t_1, t_2\}) \subset E_{t_1}$, and $m + \varepsilon/2 < m^2 - \varepsilon/2 + u$, since $\varepsilon < u - 1/4 < m^2 - m + u$. Hence

$$m(F(\{t_1, t_2\}) \cap E_{t_2}) > 0 ,$$

and therefore

$$F(\{t_1, t_2\}) \cap E_{t_2} \neq \emptyset .$$

Thus, by (13),

$$F(\{t_1, t_2\}) \cap F(\{t_2, t_3\}) \neq 0 \quad \text{for some } t_3 > t_2 .$$

By the definition of a path with property $\mathscr{P}$, this completes the proof of Theorem 3.

Theorem 3 implies immediately that each infinite subset S' of S contains an increasing path J_3. Now there are two kinds of nonincreasing paths J_3: either $t_2 < t_1, t_3$, or else $t_2 > t_1, t_3$. It follows from (5) that each infinite subset S' of S contains nonincreasing paths J_3 of both kinds, for each $u > 0$, and that each infinite subset S' of S contains two elements $X, Y \in [S]^2$, with $X = \{t_1, t_2\}$, $Y = \{t_3, t_4\}$, and $X \cap Y = 0$, such that $F(X) \cap F(Y) \neq \emptyset$ for each prescribed ordering of t_1, t_2, t_3, t_4. With a partition of $[S]^3$ and $[S]^4$ similar to the partition we used in the proof of Theorem 1(A), we can (by applying Ramsay's theorem) prove the following result.

THEOREM 4. *Let* S *be the set of natural numbers, and let* F *be a set-function on* S, *of type* 2 *and of order at least* u *with* $u > 1/4$. *Then there exists an infinite subset* S' *of* S *such that* $F(X) \cap F(Y) \neq \emptyset$ *for every pair* $X, Y \in [S]^2$. *The condition* $u > 1/4$ *is necessary.*

We omit the proof.

Here we may ask the following question. Let S again be the set of natural numbers, and let a system $Z \subset [S]^2$ of edges be called independent if $X \cap Y = \emptyset$ for every pair $X \neq Y$ $(X, Y \in Z)$. Is it true that if F is a set-function on S, of type 2 and of order at least u $(u > 0)$, then S contains an infinite subset S' such that each independent system $Z \subset [S']^2$ possesses property $\mathscr{P}$?

We know that there always exists an infinite subset S' satisfying the weaker condition that every independent system $Z \subset [S]^2$ of edges possesses property $\mathscr{P}$ provided $Z \leq 3$. This can be shown similarly to Theorem 4.

4. THE ABSTRACT CASE

In this section, S always denotes the set of natural numbers.

We say that F is an *abstract set-function of type* 2, provided F associates with each $X \in [S]^2$ a subset F(X) of a fixed set H, and that F possesses property $\mathscr{A}(k)$ if

$$\bigcap_{i=1}^{k} F(X_i) \neq \emptyset$$

for every sequence $\{X_i\}$ $(1 \leq i \leq k)$ in $[S]^2$.

A set-function F of type 2 and of order at least u $(u > 1 - 1/k)$ obviously is an abstract set-function with property $\mathscr{A}(k)$. The following result shows that in the positive theorems proved in Section 3, the assumption that F is of order at least u $(u > 1 - 1/k)$ can not be replaced by the corresponding assumption that F possesses property $\mathscr{A}(k)$. However, some weaker results hold. We state two of them without proof.

THEOREM 5. (a) *Suppose that* F *is an abstract set function with property* $\mathscr{A}(k)$ *for some* k $(3 \leq k < \omega)$. *Then there exists an infinite subset* S' *of* S *such that each nonincreasing path* $I_3 \subset [S']^2$ *has property* $\mathscr{P}$.

(b) *There exists an abstract set-function* F, *possessing property* $\mathscr{A}(3)$, *such that no increasing path* I_3 *of* S *has property* $\mathscr{P}$ *with respect to* F.

We shall now describe some graph-theoretic constructions suggested by these considerations. Let $\mathscr{G}$ be a graph, and let G denote the set of vertices of $\mathscr{G}$. A subset G' of G is said to be a *free subset* of $\mathscr{G}$ if no two vertices belonging to G' are connected by an edge in $\mathscr{G}$. The graph $\mathscr{G}$ is said to have *chromatic number* n provided n is the least cardinal number such that G is the sum of n free subsets.

A well-known result of Tutte [2] states that if n is an integer, then there exists a finite graph $\mathscr{G}$ that contains no triangle and has chromatic number n. Several other authors have constructed such graphs and have given estimates for the minimal number of vertices of $\mathscr{G}$ (see [4, p. 346] and [11]). In our next theorem, we shall give a construction for such graphs that we believe to be simpler than the previous ones; unfortunately, it does not give a very good estimate for the minimal number of vertices of $\mathscr{G}$.

It is sufficient to construct a graph $\mathscr{G}$ that has chromatic number $\aleph_0$ and contains no triangle, since, by a theorem of N. G. de Bruijn and P. Erdös (see [1]), if every finite subgraph of a graph $\mathscr{G}$ is r-chromatic, then $\mathscr{G}$ is also r-chromatic. (In place of this argument, we could also use Ramsay's theorem.)

THEOREM 6. *Let* $G = [S]^2$ $(S = \{1, 2, \cdots\})$, *and let the graph* $\mathscr{G}$ *with the set* G *of vertices be defined by the rule that two distinct vertices* $X = \{s_1, s_2\}$ *and* $Y = \{t_1, t_2\}$ $(1 \leq s_1 < s_2 < \omega;\ 1 \leq t_1 < t_2 < \omega)$ *are connected if and only if either* $s_2 = t_1$ *or* $t_2 = s_1$. *Then* $\mathscr{G}$ *contains no triangle, and its chromatic number is* $\aleph_0$.

Proof. The first statement is trivial. Suppose that the second is false. Then $G = G_1 \cup \cdots \cup G_k$, where k is finite and $G_1, \cdots, G_k$ are free sets in $\mathscr{G}$. Considering that $G = [S]^2$, we see from Ramsay's theorem that there exists an $S' \subset S$ $(\overline{\overline{S'}} = \aleph_0 > 3)$ such that $[S']^2 \subset G_i$ for some i $(1 < i < k)$. Let $t_1, t_2, t_3 \in S'$. Then $X = \{t_1, t_2\}$, $Y = \{t_2, t_3\} \in G_i$, and X and Y are connected in $\mathscr{G}$, contrary to the assumption that G_i is a free set. This completes the proof of Theorem 6.

Generalizing Tutte's theorem, P. Erdös and R. Rado proved [8, p. 445] that if n is an infinite cardinal number, then there exists a graph $\mathcal{G}$ that contains no triangle and has chromatic number n. Moreover, the graph constructed by them has n vertices. Their construction is not quite simple. Using the same idea as in the proof of Theorem 6 and applying a generalization of Ramsay's theorem, we can now give a very simple proof for a part of this result. Namely, we can similarly construct a graph $\mathcal{G}$ that contains no triangle and has chromatic number n; but the set of vertices of this graph is of power greater than n.

P. Erdös proved [3, p. 34-35] the following generalization of Tutte's theorem. If k and n are positive integers, then there exists a graph $\mathcal{G}$, of chromatic number at least n, that contains no circuit of length i for $3 \leq i < k$.

One could have believed that, in analogy with Tutte's theorem, this theorem also could be generalized for $n > \aleph_0$. Surprisingly, this is not so:

If a graph $\mathcal{G}$ contains no circuit of length 4, then its chromatic number is at most $\aleph_0$.

We shall publish the proof of this theorem in a forthcoming paper in which we shall also try to determine what kinds of subgraphs a graph $\mathcal{G}$ of chromatic number greater than $\aleph_0$ must contain. A typical result: $\mathcal{G}$ must contain an infinite path and an even graph $[S_0, S_1]$, where $\overline{\overline{S}}_0 = r$, $\overline{\overline{S}}_1 = \aleph_1$.

On the other hand, we prove the following generalization of the theorem of Erdös and Rado cited above.

THEOREM 7. *Let* k *be a positive integer, and let* n *be an infinite cardinal number. Then there exists a graph $\mathcal{G}$ that has chromatic number at least* n *and contains no circuit of length* $2i + 1$ *for* $1 \leq i \leq k$.

In our construction, the set of vertices of $\mathcal{G}$ is of power greater than n. We do not know whether there exist such graphs $\mathcal{G}$ with $\overline{\overline{G}} = n$. (*Added in proof:* Recently, we proved that such graphs exist for every n.)

We only outline the proof of Theorem 7. Let m be a cardinal number greater than n, and let ϕ denote the initial number of m. To define $\mathcal{G}$, we put $Z = \{\nu\}$ $(\nu < \phi)$ and $G = [Z]^{k+1}$, and for arbitrary different elements

$$X = \{\nu_1, \cdots, \nu_{k+1}\} \quad \text{and} \quad Y = \{\mu_1, \cdots, \mu_{k+1}\}$$

$$(\nu_1 < \cdots < \nu_{k+1};\ \mu_1 < \cdots < \mu_{k+1})$$

of G, we let X and Y be connected in $\mathcal{G}$ if and only if either

$$\nu_2 = \mu_1,\ \nu_3 = \mu_2,\ \cdots,\ \nu_{k+1} = \mu_k$$

or

$$\mu_2 = \nu_1,\ \mu_3 = \nu_2,\ \cdots,\ \mu_{k+1} = \nu_k\,.$$

The fact that $\mathcal{G}$ contains no circuit of length $2i + 1$ for $1 \leq i \leq k$ is assured by a simple and essentially finite combinatorial theorem, which we omit.

Suppose now that the chromatic number of $\mathcal{G}$ is less than n. Then $G = \bigcup_{\alpha < \psi} G_\alpha$, where $\overline{\overline{\psi}} < n$ and where G_α is a free subset of $\mathcal{G}$ for every $\alpha < \psi$. If m is chosen to be greater than

$$2^{2^{\cdot^{\cdot^{\cdot^{2^n}}}}} \quad \text{(k symbols 2)},$$

then, as a corollary of a generalization of Ramsay's theorem that was proved by P. Erdös and R. Rado (see [7, p. 567]), there exist a subset S' of S ($\overline{\overline{S'}} = k + 2$) and an $\alpha < \phi$ such that $[S']^{k+1} \subset G_\alpha$.

Let $S' = \{\nu_1, \cdots, \nu_{k+1}, \nu_{k+2}\}$ $(\nu_1 < \cdots < \nu_{k+2})$. Then

$$X = \{\nu_1, \cdots, \nu_{k+1}\}, \; Y = \{\nu_2, \cdots, \nu_{k+2}\} \in G_\alpha,$$

and X, Y are connected in $\mathscr{G}$. This contradicts the assumption that G_α is free.

5. THE CASE $m = \aleph_1$ OR $m = 2^{\aleph_0}$

In this section we shall often refer to the partition symbol $m \rightarrow (b_\nu)_c^r$ introduced by P. Erdös and R. Rado [6, p. 428]. For the convenience of the reader we restate the definition.

Let m and c $(c \geq 2)$ be cardinal numbers, let r be an integer $(r \geq 1)$, let ϕ denote the initial number of c, and let (b_ν) $(\nu < \phi)$ be a sequence of type ϕ of cardinal numbers.

The implication $m \rightarrow (b_\nu)_c^r$ means that if S is a set of power m and (J_ν) $(\nu < \phi)$ is a partition of the set $[S]^r$ (that is, $[S]^r = \bigcup_{\nu < \phi} J_\nu$), then there exist a subset S' of S and a $\nu_0 < \phi$ such that $\overline{\overline{S'}} = b_{\nu_0}$ and $[S']^r \subset J_{\nu_0}$. The expression $m \not\rightarrow (b_\nu)_c^r$ means that the above statement is false.

Several results concerning the symbolic statement $m \rightarrow (b_\nu)_c^r$ are proved in [6] and [7]. A forthcoming paper by P. Erdös, A. Hajnal, and R. Rado [5] will contain an almost complete discussion of the symbol.

Note that the problem of proving that $m \rightarrow (b_\nu)_c^r$ is a generalization of the problem settled by Ramsay's theorem. Indeed, Ramsay's theorem (see Section 1) asserts that if c is finite, then $\aleph_0 \rightarrow (\aleph_0, \cdots, \aleph_0)_c^r$ (or, more precisely, that $\aleph_0 \rightarrow (b_\nu)_c^r$ provided c is finite and $b_\nu = \aleph_0$ for every $\nu < \phi$.)

Now we turn to our original problems. First we prove the following negative result.

(*) THEOREM 8. *If* S *is a set of power* $2^{\aleph_0}$, *then there exists a set-function* F *on* S, *of type* 2 *and of order at least* 1, *such that no* $Z_1 \subset [S]^2$ *with cardinality greater than* $\aleph_0$ *possesses property* $\mathscr{P}$; *that is, if* $m = 2^{\aleph_0}$, *there need not exist a graph that has at least* $\aleph_1$ *vertices and possesses property* $\mathscr{P}$.

Proof. Let $\{u_\nu\}$ $(\nu < \omega_1)$ and $\{X_\nu\}$ $(\nu < \omega_1)$ be well-orderings of type ω_1 of the sets [0, 1] and $[S]^2$, respectively. For each $\nu < \omega_1$, we define

$$F(X_\nu) = \{u_\mu\} \qquad (\nu \leq \mu < \omega_1).$$

Since each $F(X_\nu)$ has a denumerable complement, $m(F(X_\nu)) = 1$ if $\nu < \omega_1$, and thus F is of order at least 1. On the other hand, the intersection of any $\aleph_1$ of the sets $F(X_\nu)$ is obviously empty. This completes the proof.

A corollary of Theorem 8: if $0 \leq u \leq 1$, then $(2^{\aleph_0}, 2, u) \not\Rightarrow \aleph_1$ provided (*) is assumed. Without the generalized continuum hypothesis, we can prove only the following weaker result.

THEOREM 9. *If* $u \leq 1/2$, *then* $(2^{\aleph_0}, 2, u) \not\Rightarrow \aleph_1$.

Proof. Let S be a set of power $2^{\aleph_0}$. By a result of Sierpiński [13], there exists a partition of $[S]^2$ such that the statements

$$[S]^2 = J_1 \cup J_2, \quad J_1 \cap J_2 = \emptyset, \quad S' \subset S, \text{ and } [S']^2 \subset J_j \quad (j = 1, 2)$$

imply that $\overline{\overline{S'}} \leq \aleph_0$. (In terms of the partition relations implied by $m \to (b_\nu)_c^r$, this means that $2^{\aleph_0} \not\to (\aleph_1, \aleph_1)^2$.

Now we define

$$F(X) = \begin{cases} (0, 1/2) & \text{if } X \in J_1 \\ (1/2, 1) & \text{if } X \in J_2 . \end{cases}$$

This set-function obviously satisfies the requirement of our theorem.

If $u > 1/2$, the argument above is inconclusive. Now, the edges of a complete graph of power $2^{\aleph_0}$ can presumably be split into $2^{\aleph_0}$ disjoint classes in such a way that each subset of S of power $\aleph_1$ contains an edge from each class. This theorem has never been proved, not even for three classes, without the help of the generalized continuum hypothesis (*). A proof using (*) is given in [5]. If we could prove the theorem for r classes $(r > 2)$ without using (*), then by following our proof of Theorem 9, we could clearly show that for each $r < \omega$,

$$(2^{\aleph_0}, 2, 1 - 1/r) \not\Rightarrow \aleph_1 .$$

On the other hand, it is easy to see without using (*) that $(2^{\aleph_0}, 2, > 0) \not\Rightarrow 3$. To prove this, we let S be the interval $[0, 1]$, and we let $F(\{x, y\})$ be the open interval (x, y). Obviously,

$$m(F(\{x, y\})) = |x - y| > 0 ,$$

and no triangle has the property $\mathscr{P}$.

Trees whose longest paths have length at most 2 are unions of stars. It is well known that every complete graph of power $\aleph_1$ is a countable sum of trees, in fact, a countable sum of trees that are unions of stars. Thus, if we assume the continuum hypothesis, then we can construct an $F(X)$ such that $m(F(X)) > 0$ and no graph containing a path of length 3 has property $\mathscr{P}$. For the sake of completeness we remark that if $c = \aleph_1$ and S is a set of power $\aleph_1$, we can construct by the above remark a set-function on S, of positive order and type 2, so that no graph of power $\aleph_1$ and no path of length 3 has property $\mathscr{P}$.

Our only positive result in this section is the next theorem.

THEOREM 10. *If* u *is positive and* $m > \aleph_0$, *then* $(m, 2, u) \Rightarrow \aleph_0$.

Remark. If $u \leq 1/2$, we know by Theorem 9 that this result is best possible if $m \leq 2^{\aleph_0}$. If we assume (*), then, by Theorem 8, Theorem 10 is best possible for each $u \leq m$ and each $m \leq 2^{\aleph_0}$.

Proof. It is sufficient to prove that if $u > 0$, then $(\aleph_1, 2, u) \nrightarrow \aleph_0$.

Let S be a set of power $\aleph_1$; without loss of generality, we suppose that $S = \{\alpha\}$ $(\alpha < \omega_1)$. Let F be a set-function on S, of type 2 and of order at least u. For brevity, we write

$$F(\{\alpha_1, \alpha_2\}) = F_{\alpha_1,\alpha_2} = F_{\alpha_2,\alpha_1}.$$

Let $X \circ Y$ denote the symmetric difference $X \cup Y - (X \cap Y)$ of the sets X and Y. The following theorem is well known (see [10, p. 168]).

THEOREM α. *There exists a denumerable sequence* $\{E_s\}$ $(s < \omega)$ *of measurable subsets of* $[0, 1]$ *such that if* E *is a measurable subset of* $[0, 1]$, *then corresponding to each* $\varepsilon > 0$ *there exists an* $s < \omega$ *for which* $m(E \circ E_s) < \varepsilon$.

Applying Theorem α, we obtain the following result.

THEOREM β. *There exist an* $\alpha_0 < \omega_1$ *and a subset* S_1 *of* S *with cardinality* $\aleph_1$ *such that for each* α *and* α' *in* S_1,

$$m(F_{\alpha_0,\alpha} \circ F_{\alpha_0,\alpha'}) < \varepsilon.$$

Suppose now that $S_k \subset \cdots \subset S_1$ and the elements $\alpha_0, \cdots, \alpha_{k-1}$ are already defined for some k $(0 < k < \omega)$ in such a way that S_k has power $\aleph_1$. Then, if we apply Theorem α $k + 1$ times, we establish the following result.

THEOREM γ. *There exist an* $\alpha_k \in S_k$ $(\alpha_k \neq \alpha_i$ *if* $i < k)$ *and a subset* $S_{k+1} \subset S_k$ *with cardinality* $\aleph_1$ *such that for each* $i \leq k$ *and for each* α *and* α' *in* S_{k+1}

$$m(F_{\alpha_i,\alpha} \circ F_{\alpha_i,\alpha'}) < \varepsilon 2^{-k-1}.$$

Thus by induction on k, α_k and S_{k+1} are defined for every $k < \omega$. Now let

$$G_k = \bigcap_{t=k+1}^{\infty} F_{\alpha_k,\alpha_t}.$$

For each $k < \omega$, it follows from Theorem β and Theorem γ that

$$m(G_k) \geq m(F_{\alpha_k,\alpha_{k+1}}) - \sum_{t=k}^{\infty} \varepsilon 2^{-k-1};$$

hence, if $0 < \varepsilon < u/2$, then

$$m(G_k) \geq u - \varepsilon > u/2.$$

Finally, the theorem proved in the Introduction enables us to conclude that there exists an infinite sequence $\{k_r\}$ $(r < \omega\}$ such that

$$\bigcap_{r=0}^{\infty} G_{k_r} \neq \emptyset .$$

Let $S' = \{\alpha_{k_r}\}$ $(r < \omega)$. Then S' has power $\aleph_0$, and $[S']^2$ possesses property $\mathscr{P}$. The proof of Theorem 10 is now complete.

6. THE CASE $m = \aleph_2$.

Throughout this and the next section we shall assume the generalized continuum hypothesis (*).

(*) THEOREM 11. $(\aleph_2, 2, > 0) \Rightarrow \aleph_0$.

Proof. Let S be a set of power $\aleph_2$, and let F be a set-function on S, of type 2 and of positive order. We split the edges of the complete graph S into countably many classes J_t by stipulating that $X \in J_t$ if and only if for each $t < \omega$ and each $X \in [S]^2$

$$2^{-t-1} < m(F(X)) \leq 2^{-t} .$$

Since the order of F(X) is positive,

$$[S]^2 = \bigcup_{t<\omega} J_t .$$

It follows from a theorem in [5] that $\aleph_2 \to (\aleph_1, \cdots, \aleph_1, \cdots)^2_{\aleph_0}$ (see the definition of $m \to (b_\nu)^r_c$ in Section 5). Hence, at least one of the graphs J_t contains a complete graph of power $\aleph_1$; that is, there exist a subset S' of S of power $\aleph_1$ and a $t_0 < \omega$ such that $[S']^2 \subset J_{t_0}$. Applying Theorem 10, with S' playing the role of S, we obtain the desired conclusion.

Theorem 11 is probably best possible. In fact, it seems likely that even if we were to assume that the order of F is at least 1, we could not deduce the existence of a complete subgraph of power $\aleph_1$ that has property $\mathscr{P}$. This question is connected with the following unsolved problem stated in [5].

Let S be a set of power $\aleph_2$. Does there exist a partition of the complete graph S into disjoint sets J_ν $(\nu < \omega_1)$ such that no countable union of J_ν's contains a complete graph of power $\aleph_1$; that is, such that if S' is a subset of S with cardinality $\aleph_1$, then $[S']^2 \cap J_\nu \neq \emptyset$ for at least $\aleph_1$ sets J_ν?

Probably such a decomposition exists, but we have been unable to construct one. For the sake of the argument, assume that it exists. Let $\{u_\nu\}$ $(\nu < \omega_1)$ be a well-ordering of type ω_1 of the interval [0, 1], and define a set-function of type 2 on S by the condition

$$F(X) = \{u_\mu\} \ (\nu < \mu < \omega_1) \quad \text{for } X \in [S]^2$$

if and only if $X \in J_\nu$ for each $\nu < \omega_1$. Obviously, F is of order at least 1. Moreover, if S' is a subset of power $\aleph_1$ of S, then F assumes $\aleph_1$ distinct values on $[S']^2$; hence, $[S]^2$ does not possess property $\mathscr{P}$.

7. THE CASE $m > \aleph_2$.

Under the assumption that $m > \aleph_2$, the connection between our problems and measure theory becomes tenuous, and the questions become purely set-theoretical. In this section we shall make heavy use of [5].

(*) THEOREM 12. *If* $m = \aleph_{\alpha+1}$ *and* $\mathrm{cf}(\alpha) > 1$, *then* $(m, 2, > 0) \Rightarrow \aleph_\alpha$.

This theorem is a corollary of the following stronger proposition.

(*) THEOREM 12 (A). *If* $m = \aleph_{\alpha+1}$, $\mathrm{cf}(\alpha) > 1$; *and if to each* $X \in [S]^2$ *there corresponds a nonempty subset of* $[0, 1]$, *then there exists a subset* S' *of* S *with cardinality* $\aleph_\alpha$ *and such that* $[S']^2$ *possesses property* $\mathscr{P}$.

Proof. Let $\{u_\nu\}$ $(\nu < \omega_1)$ be a well-ordering of type ω_1 of $[0, 1]$. We define a partition of $[S]^2$ into sets J_ν $(\nu < \omega_1)$ as follows. For each $X \in [S]^2$ and each $\nu < \omega_1$, X is in J_ν if and only if ν is the least ordinal number for which $u_\nu \in F(X)$.

A theorem of [5] states that, under the hypotheses of Theorem 12 (A),

$$\aleph_{\alpha+1} \to (\aleph_\alpha)^2_{\aleph_1}.$$

The next theorem implies that Theorem 12 (A) is best possible even if the complement of each set F(X) consists of one element.

(*) THEOREM 13. *If* $m = \aleph_\beta$ *and if either* β *is of the first kind or* β *is of the second kind and* $\aleph_{\mathrm{cf}(\beta)}$ *is not an inaccessible cardinal greater than* $\aleph_0$, *then*

(a) $(\aleph_\beta, 2, 1) \not\Rightarrow \aleph_\beta$ *if* $\mathrm{cf}(\beta) \neq 0$,

(b) *for each* $u < 1$, $(\aleph_\beta, 2, u) \not\Rightarrow \aleph_\beta$ *if* $\mathrm{cf}(\beta) = 0$.

Moreover, in case (a) *the desired set-function* F *can be chosen so that the complement of each* F(X) *consists of exactly one element.*

Proof. Let S be a set of power $\aleph_\beta$, and consider the case (a). A theorem in [5] implies that there exists a partition of $[S]^2$ into disjoint sets J_ν $(\nu < \omega_1)$ such that if S' is a subset of S with cardinality $\aleph_\beta$, and if $\nu < \omega_1$, then $[S']^2 \cap J_\nu \neq \emptyset$. We let $\{u_\nu\}$ $(\nu < \omega_1)$ be a well-ordering of $[0, 1]$ of type ω_1, and for each $\nu < \omega_1$ and each $X \in [S]^2$ we define

$$F(X) = [0, 1] - \{u_\nu\} \quad \text{if } X \in J_\nu.$$

Clearly, the function F has the desired properties.

We now consider case (b). By virtue of Theorem 1, we may suppose that $\beta > 0$. Doing so, we choose an increasing sequence $\{\beta_t\}$ $(t < \omega)$ of ordinal numbers less than β and cofinal with β, and for each $t < \omega$ we choose a subset S_t of S with cardinality $\aleph_{\beta_t}$ so that the S_t are disjoint and

$$S = \bigcup_{t<\omega} S_t.$$

On the set $S^* = \{S_t\}$ $(t < \omega)$ there exists by Theorem 1 a set-function F^*, of type 2 and of order at least u, such that if $S^{*\prime}$ is a subset of S^* of power $\aleph_0$, then $[S^{*\prime}]^2$ does not possess property $\mathscr{P}$ with respect to F^*.

For any $\{x, y\} \in [S]^2$, suppose that $x \in S_{t_1}$ and $y \in S_{t_2}$, and define $F(\{x, y\})$ as follows:

$$F(\{x, y\}) = (0, 1) \qquad (t_1 = t_2),$$

$$F(\{x, y\}) = F^*(\{S_{t_1}, S_{t_2}\}) \qquad (t_1 \neq t_2).$$

It is easy to verify that F has the desired properties. This completes the proof.

For the case where $\aleph_{cf(\beta)}$ is inaccessible and greater than $\aleph_0$, the problem remains unsolved.

(*) THEOREM 14. *If* $m = \aleph_\beta > \aleph_0$ *is a limit cardinal and if* $n < m$, *then* $(m, 2, > 0) \nrightarrow n$.

It is a theorem in [5] that

$$m \rightarrow (n)^2_{\aleph_1}.$$

Both Theorem 12 and Theorem 14 follow from this. Moreover, just as in Theorem 12 (A), instead of assuming that F is of positive order, we can merely assume that F(X) is nonempty for each $X \in [S]^2$. We omit the details.

The only cases we have not yet discussed are $m = \aleph_{\alpha+1}$, where $\alpha > 1$ and either $cf(\alpha) = 0$ or $cf(\alpha) = 1$.

(*) THEOREM 15. *If* $m = \aleph_{\alpha+1}$, $\alpha > 0$, $cf(\alpha) = 0$, *and* $u > 0$, *then*

$$(m, 2, u) \nrightarrow \aleph_\alpha.$$

Proof. Let S be a set of power $\aleph_{\alpha+1}$. Without loss of generality suppose that $S = \{\nu\}$ $(\nu < \omega_{\alpha+1})$. Let F be a set-function of S, of type 2 and of order at least u. We shall use methods employed in [5].

By the ramification method used there, we know that there exists an increasing sequence $\{\nu_\mu\}$ $(\mu < \omega_\alpha)$ such that if $\mu < \mu' \leq \mu'' < \omega_\alpha$, then

$$F(\{\nu_\mu, \nu_{\mu'}\}) = F(\{\nu_\mu, \nu_{\mu''}\}). \tag{16}$$

We shall write $F_\mu = F(\{\nu_\mu, \nu_{\mu+1}\})$. Let $\{\alpha_t\}$ $(t < \omega)$ be an increasing sequence of ordinal numbers less than α, cofinal with α and such that $\alpha_t > 2$ and $\aleph_{\alpha_t}$ is regular. Since $[0, 1]$ has only $\aleph_2$ subsets, it follows that corresponding to each $t < \omega$, there exist a μ_t and a set Z_t of ordinal numbers

$$\mu \qquad (\omega_{\alpha_{t-1}} < \mu \leq \omega_{\alpha_t};\ \alpha_{-1} = 0)$$

such that Z_t has power $\aleph_{\alpha_t}$, and $F_\mu = F_{\mu_t}$ for each μ in Z_t. If $t < \omega$, then $m(F_{\mu_t}) \geq u > 0$; therefore, we conclude from the theorem proved in the Introduction that there exists an infinite subsequence $\{t_s\}$ $(s < \omega)$ such that

$$\bigcap_{s<\omega} F_{\mu_{t_s}} \neq \emptyset.$$

Let

$$Z = \bigcup_{s<\omega} Z_{t_s}, \qquad S' = \{\nu_\mu\} \ (\mu \in Z).$$

Clearly,

$$\overline{\overline{S'}} = \sum_{s<\omega} \aleph_{\alpha_{t_s}} = \aleph_\alpha ,$$

and, by (16), $[S']^2$ possesses property $\mathscr{P}$. This completes the proof of the theorem.

We note that under the hypotheses of Theorem 15, $(m, 2, >0) \not\twoheadrightarrow \aleph_\alpha$. This is true because of a theorem in [5] which states that if $\mathrm{cf}(\alpha) = 0$, then

$$\aleph_{\alpha+1} \not\to (\aleph_\alpha)^2_{\aleph_0} .$$

That is, if S is a set of power $\aleph_{\alpha+1}$ $(\mathrm{cf}(\alpha) = 0)$, then there exists a partition of $[S]^2$ into disjoint sets J_t $(t < \omega)$ such that if $t < \omega$, $S' \subset S$, and $[S']^2 \subset J_t$, then

$$\overline{\overline{S'}} < \aleph_\alpha .$$

For each $X \in [S]^2$ and each $t < \omega$, we define $F(X) = (2^{-t-1}, 2^{-t})$ if $X \in J_t$. For this F it is obvious that $(m, 2, >0) \not\twoheadrightarrow \aleph_\alpha$.

(*) THEOREM 16. *If* $m = \aleph_{\alpha+1}$, $\mathrm{cf}(\alpha) = 1$, *and* $\alpha > 1$, *then*

(a) $(m, 2, 1) \not\twoheadrightarrow \aleph_\alpha$,

(b) $(m, 2, >0) \twoheadrightarrow n$ $(n < \aleph_\alpha)$.

Note that, in harmony with our remarks in the discussion of the case $m = \aleph_2$, we do not know whether or not $(m, 2, 1) \not\twoheadrightarrow \aleph_\alpha$ is true if the condition $\alpha > 1$ is omitted.

Proof of Theorem 16. The conclusion (b) follows trivially from Theorem 14. To prove (a), we refer to the following theorem in [5]. *Let* S *be a set of power* $\aleph_{\alpha+1}$, *where* $\mathrm{cf}(\alpha) = 1$ *and* $\alpha > 1$. *Then there exists a partition of* $[S]^2$ *into disjoint sets* J_ν $(\nu < \omega_1)$ *such that if* S' *is a subset of* S *of power* $\aleph_\alpha$, *then*

$$[S']^2 \cap J_\nu \neq \emptyset$$

for $\aleph_1$ *sets* J_ν.

We now let $\{u_\nu\}$ $(\nu < \omega_1)$ be a well-ordering of type ω_1 of the interval $[0, 1]$, and we define a set-function F by the condition that for each $\nu < \omega_1$ and each $X \in [S]^2$,

$$F(X) = \{u_\mu\} \ (\nu < \mu < \omega_1) \qquad \text{if } X \in J_\nu .$$

By analogy with the remark made after the proof of Theorem 11, it is easy to see that F has the desired properties.

8. THE CASE $k > 2$

We shall discuss this case only briefly. At present we cannot even settle the following question: Is it true that for each $u > 0$

$$(\aleph_1, 3, u) \Rightarrow 4\,?$$

Let k, m, and n be integers. It is an old problem of P. Turán's to determine the smallest integer $f(k, n, m)$ such that if

$$A_1, \cdots, A_{f(k,n,m)}$$

are k-tuples formed from a set S of m elements, then there always exist n elements of S such that each k-tuple of these n elements is an A_i. As we stated earlier, Turán determined $f(2, n, m)$. For $k > 2$, the problem appears to be quite difficult. It is easy to show that

$$C_{k,n} = \lim_{m\to\infty} \frac{f(k, n, m)}{m^k}$$

exists. The results of Turán [2] imply that

$$0 < C_{k,n} < \frac{1}{k!} \quad \text{and} \quad C_{2,n} = \frac{1}{2}\left(1 - \frac{1}{n-1}\right);$$

but even the value of $C_{3,4}$ is not known.

It is easy to deduce by the methods used to prove Theorem 1 that if $u > k!\,C_{k,n}$, then

$$(\aleph_0, k, u) \Rightarrow n.$$

This is no longer true if $u = k!\,C_{k,n}$.

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