THE AMOUNT OF OVERLAPPING IN PARTIAL COVERINGS OF SPACE BY EQUAL SPHERES

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1. *Introduction.* We say that a system Σ of equal spheres $S_1, S_2, \dots$ covers a proportion θ of n-dimensional space, if the limit, as the side of the cube C tends to infinity, of the ratio

$$\frac{V\left(\prod_{r=1}^{\infty} S_r \cap C\right)}{V(C)}$$

of the volume of C covered by the spheres to the volume of C, exists and has the value θ. We say that such a system Σ has density δ, if the corresponding ratio

$$\frac{\sum_{r=1}^{\infty} V(S_r \cap C)}{V(C)}$$

has the limit δ as the side of the cube C tends to infinity. We confine our attention to systems Σ for which both limits exist. It is clear that $\delta = \theta$, if no two spheres of the system overlap, *i.e.* if we have a packing; and that, in general, the difference $\delta - \theta$ is a measure of the amount of overlapping.

By well-known results of H. Minkowski [**1**] and H. F. Blichfeldt [**2**], the maximum density θ_n of a packing of equal spheres into n-dimensional Euclidean space satisfies

$$\frac{\zeta(n)}{2^{n-1}} \leqslant \theta_n \leqslant \frac{n+2}{2}\left(\frac{1}{\sqrt{2}}\right)^n, \qquad \left(\zeta(n) = \sum_{k=1}^{\infty} k^{-n}\right).$$

These results have been improved, see [**3**], [**4**] and [**5**], but the improvements tell us nothing new about the asymptotic behaviour of $\theta_n^{1/n}$ as $n \to \infty$. If Σ has $\theta > \theta_n$, a general (but not quite trivial) argument shows that $\delta > \theta$; but does not, as far as we can see, give any estimate for $\delta - \theta$.

Our object in this paper is to obtain such an estimate for $\delta - \theta$, for a wide range of values of θ. Our method, which is based on one first used by R. P. Bambah and H. Davenport [**6**] (see also [**7**]), does not work for values of θ approaching θ_n, but it does work for values of θ, which may become exponentially small as n increases. It is also relatively weak, when θ is close to 1, as we do not obtain a result so strong as that of H. S. M. Coxeter, L. Few and C. A. Rogers [**8**] on letting θ tend to 1. It is however the only explicitly known result for

$$\left(\tfrac{4}{5} + o(1)\right)^{n/2} < \theta < 1.$$

[MATHEMATIKA 12 (1964), 171–184]

Our main result is

THEOREM 1. *If $n \geqslant 2^{20}$ and a system Σ of equal spheres covers a proportion θ of n-dimensional space with*

$$\theta > \tfrac{4}{3}(1-4n^{-1/4})^{-n/2}(\tfrac{4}{5})^{n/2}, \tag{1}$$

and has density δ, then

$$\delta \geqslant \theta + \Theta$$

where

$$\Theta = \tfrac{1}{3}[1-\exp(16-\tfrac{1}{2}n^{1/2})][1-4\{(\tfrac{3}{4}\theta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2}. \tag{2}$$

The estimate for $\delta-\theta$ is not quite as simple as one would wish; one gets a better idea of its consequences on noting that, if

$$\frac{1}{n}\log\frac{1}{\theta} = o(1),$$

as $n \to \infty$, *i.e.* if θ does not tend to zero exponentially fast, then

$$\Theta = \tfrac{1}{3}(\tfrac{3}{4}\theta)^{4+o(1)}, \tag{3}$$

as $n \to \infty$; and that, if

$$0 < \liminf_{n\to\infty} \frac{1}{n}\log\frac{1}{\theta} \leqslant \limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{\theta} < \tfrac{1}{2}\log\tfrac{5}{4},$$

i.e. if θ does tend to zero exponentially fast at a rate strictly slower than that of $(\tfrac{4}{5})^{n/2}$, then

$$\Theta = \left(5-4\theta^{-2/n}+o(1)\right)^{n/2}. \tag{4}$$

We remark that Rogers [**9**] (see also [**10**]) has a result (Theorem 1), which implies (on choosing V to satisfy $\theta = V-\tfrac{1}{2}V^2$) that, if $0<\theta\leqslant\tfrac{1}{2}$, then there is a lattice system of spheres, covering the proportion θ of n-dimensional space, and with density at most

$$\theta + \frac{\theta^2}{(1-\theta)+\sqrt{(1-2\theta)}} < \theta + 2\theta^2. \tag{5}$$

This shows that we cannot expect too much overlapping when θ is small.

2. *The approximation of spheres by polyhedra.* In this section we prove a lemma on the volumes of the parts of a convex polyhedron lying inside and outside a sphere. Our result tells us that, if the volume of a polyhedron Π does not greatly exceed the volume of a sphere S, and if Π has not too many faces, then the volume of $\Pi \cap S$ is substantially smaller than that of S.

LEMMA 1. *Let S be the sphere with centre* $\mathbf{o}$ *and radius* 1, *and let* Π *be a convex polyhedron containing* $\mathbf{o}$. *Let* $N(t)$ *be the number of faces of* Π *whose* $(n-1)$*-dimensional planes are within the distance* t *of* $\mathbf{o}$. *Then, provided* $0<h<1<r$, *we have*

$$V(S\cap\Pi)\leqslant r^{-n}\Phi V(\Pi)+$$
$$+\left[1-\Phi+\Phi\int_h^r nN(x)\left(1-\frac{x^2}{r^2}\right)^{(n/2)-1}\frac{x}{r}d\left(\frac{x}{r}\right)\right]V(S), \quad (6)$$

where

$$\Phi=\left\{\frac{r^2(1-h^2)}{r^2-h^2}\right\}^{n/2}. \quad (7)$$

Proof. Let $F_1, F_2, \ldots, F_N$ be the $(n-1)$-dimensional faces of Π and let $h_1, h_2, \ldots, h_N$ be the perpendicular distances from $\mathbf{o}$ to the $(n-1)$-dimensional planes of these faces. We suppose that the faces are named so that

$$h_1\leqslant h_2\leqslant h_3\leqslant\ldots\leqslant h_N.$$

For each i let C_i denote the semi-infinite cone with vertex $\mathbf{o}$ and with F_i as one of its sections, *i.e.* the set of points which can be expressed vectorially in the form $\lambda\mathbf{x}$ with $\lambda\geqslant 0$ and $\mathbf{x}\in F_i$.

Let S^* be the sphere with centre $\mathbf{o}$ and radius r. Let $\mathbf{y}_i$ be the point of F_i nearest to $\mathbf{o}$. If $|\mathbf{y}_i|\geqslant r$ we have

$$V(S^*\cap C_i)\leqslant V(\Pi\cap C_i). \quad (8)$$

If $h\leqslant|\mathbf{y}_i|<r$, the points of $S^*\cap C_i$ not in Π are contained in the sphere S_i with centre $\mathbf{y}_i$ and radius $(r^2-|\mathbf{y}_i|^2)^{1/2}$. In this case

$$V(S^*\cap C_i)\leqslant V(\Pi\cap C_i)+V(S_i)$$
$$=V(\Pi\cap C_i)+(r^2-|\mathbf{y}_i|^2)^{n/2}V(S). \quad (9)$$

If $|\mathbf{y}_i|<h$, we again note that the points $\mathbf{y}$ of $S^*\cap C_i$ satisfying

$$\mathbf{y}.\mathbf{y}_i\geqslant h|\mathbf{y}_i|, \quad (10)$$

are contained in a sphere of radius $(r^2-h^2)^{1/2}$, this time it is the one with centre $h\mathbf{y}_i/|\mathbf{y}_i|$. So the volume of the set of the points $\mathbf{y}$ of $S^*\cap C_i$ satisfying (10) is at most $(r^2-h^2)^{n/2}V(S)$. Now consider the set H_i^* of points $\mathbf{y}^*$ of $S^*\cap C_i$ not in Π, but with

$$\mathbf{y}^*.\mathbf{y}_i<h|\mathbf{y}_i|. \quad (11)$$

With each point $\mathbf{y}^*$ of H_i^* we associate the point

$$\mathbf{y}=\mathbf{y}_i+\phi(\mathbf{y}^*-\mathbf{y}_i),$$

where

$$\phi=\left\{\frac{1-h^2}{r^2-h^2}\right\}^{1/2}.$$

The region $C_i - \Pi$ is convex and contains both $\mathbf{y}^*$ and $\mathbf{y}_i$ in its closure. So $\mathbf{y}$ also lies in the closure of $C_i - \Pi$. Also, for $\mathbf{y}^*$ in H_i^*,

$$\begin{aligned}\mathbf{y}.\mathbf{y} &= \{(1-\phi)\,\mathbf{y}_i + \phi\mathbf{y}^*\}.\{(1-\phi)\,\mathbf{y}_i + \phi\mathbf{y}^*\}\\ &= (1-\phi)^2\,\mathbf{y}_i.\mathbf{y}_i + 2\phi(1-\phi)\,\mathbf{y}^*.\mathbf{y}_i + \phi^2\,\mathbf{y}^*.\mathbf{y}^*\\ &\leqslant (1-\phi)^2 h^2 + 2\phi(1-\phi)\,h^2 + \phi^2 r^2\\ &= h^2 + \phi^2(r^2-h^2) = 1.\end{aligned}$$

Thus the transformation $\mathbf{y}^* \to \mathbf{y}$ transforms the set H_i^* into a sub-set of

$$S \cap C_i - \Pi.$$

Hence

$$V(H_i^*) \leqslant \phi^{-n}\, V\Big((S-\Pi)\cap C_i\Big),$$

and

$$\begin{aligned}V(S^*\cap C_i) &\leqslant V(\Pi\cap C_i) + (r^2-h^2)^{n/2}\,V(S) + V(H_i^*)\\ &\leqslant V(\Pi\cap C_i) + (r^2-h^2)^{n/2}\,V(S) + \phi^{-n}\,V\Big((S-\Pi)\cap C_i\Big).\end{aligned} \tag{12}$$

Summing the results (8), (9) and (12), we obtain

$$V(S^*) \leqslant V(\Pi) + \phi^{-n}\,V(S-\Pi) + N(h)(r^2-h^2)^{n/2}\,V(S) + V(S)\int_h^r (r^2-x^2)^{n/2}\,dN(x).$$

Integrating by parts, we have

$$\int_h^r (r^2-x^2)^{n/2}\,dN(x) = -(r^2-h^2)^{n/2}\,N(h) + \int_h^r nx(r^2-x^2)^{(n/2)-1}\,N(x)\,dx.$$

Hence

$$\begin{aligned}\Phi V(S) &= r^{-n}\,\Phi V(S^*)\\ &\leqslant r^{-n}\,\Phi V(\Pi) + \{V(S) - V(S\cap\Pi)\}\\ &\quad + \Phi V(S)\int_h^r nN(x)\left(1-\frac{x^2}{r^2}\right)^{(n/2)-1}\frac{x}{r}\,d\left(\frac{x}{r}\right);\end{aligned}$$

so (6) follows.

3. *Periodic systems of spheres.* We say that a system of equal spheres $S_1, S_2, \ldots$ is a periodic system with period R, if the spheres of the system have a representation

$$S + \mathbf{a}_i + \mathbf{b}_j, \qquad i = 1, 2, \ldots, M, \quad j = 1, 2, \ldots, \tag{13}$$

where S is a fixed sphere with $\mathbf{o}$ as centre, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M$ is a finite set of points, and where $\mathbf{b}_1, \mathbf{b}_2, \ldots$ is an enumeration of the points of the lattice Λ_R of points whose coordinates are integral multiples of R. We use $\Sigma(S; \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M; R)$ to denote this system (13).

Our next lemma shows us that, if, when we keep S, M and R fixed and vary $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M$ to ensure that $\Sigma(S, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M; R)$ covers the largest possible proportion of space, we find that this proportion is not too close to 1, then not too many of the centres

$$\mathbf{a}_i+\mathbf{b}_j, \qquad i=1, 2, \ldots, M, \quad j=1, 2, \ldots,$$

can lie in spheres of radius $2h$ centred on these points.

LEMMA 2. *Let S be the sphere with centre $\mathbf{o}$ and radius 1 in n-dimensional Euclidean space, with $n \geqslant 4$. Let $R > 2$ be given, Let M be a positive integer. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M$ be chosen so that the system*

$$\Sigma(S; \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M; R)$$

covers the largest possible proportion of the whole space. Suppose that this proportion is θ and that

$$\theta \leqslant 1-(1-n^{-1/2})^{n/2}. \tag{14}$$

Then, for each k with $1 \leqslant k \leqslant M$, and for each h with $0 < h < \frac{1}{2}R$, the number $N_k(h)$ of the centres

$$\mathbf{a}_i+\mathbf{b}_j, \qquad i=1, 2, \ldots, M, \quad j=1, 2, \ldots,$$

within distance $2h$ of $\mathbf{a}_k$ satisfies

$$N_k(h) < (4h^2+1)^{n/2} \exp(2n^{3/4}). \tag{15}$$

Further, if $n \geqslant 2^{20}$, $1 < r^2 < \frac{5}{4}(1-4n^{-1/4})$,

and
$$h^2 = r^2-\tfrac{1}{4}+4n^{-1/4}, \tag{16}$$

then

$$\int_h^r nN_k(x)\left(1-\frac{x^2}{r^2}\right)^{(n/2)-1} \frac{x}{r}\, d\left(\frac{x}{r}\right) < \exp(-2n^{3/4}). \tag{17}$$

Proof. Suppose that $1 \leqslant k \leqslant M$ and $0 < h < \frac{1}{2}R$. Let

$$\mathbf{c}_m = \mathbf{a}_{i(m)}+\mathbf{b}_{j(m)}, \qquad m=1, 2, \ldots, N_k(h),$$

be the points of the system within distance $2h$ of the point $\mathbf{a}_k$. As $h < \frac{1}{2}R$ it follows that the points

$$\mathbf{a}_{i(m)}, \qquad m=1, 2, \ldots, N_k(h),$$

are all distinct.

We take

$$y = \{1-(\tfrac{1}{2})^{2/n}(1-\theta)^{2/n}\}^{1/2}, \tag{18}$$

and consider the sphere Σ with centre $\mathbf{a}_k$ and with radius

$$z=[(2h+y)^2+(1-y^2)]^{1/2}=[4h^2+1+4hy]^{1/2}.$$

It is clear that if the sphere

$$S+\mathbf{c}_m$$

has any point not in Σ, the whole set

$$(S+\mathbf{c}_m)-\Sigma$$

will be contained in the sphere Σ_m with centre

$$\mathbf{a}_k+(2h+y)\frac{\mathbf{c}_m-\mathbf{a}_k}{|\mathbf{c}_m-\mathbf{a}_k|}$$

and with radius

$$(1-y^2)^{1/2}.$$

It follows that the volume of the union

$$\bigcup_{m=1}^{N}(S+\mathbf{c}_m)$$

is at most

$$V(\Sigma)+\sum_{m=1}^{N}V(\Sigma_m)=[(4h^2+1+4hy)^{n/2}+N_k(h)(1-y^2)^{h/2}]\,V(S).$$

Let V_m be the volume of the part of $S+\mathbf{c}_m$ which lies in no other sphere

$$S+\mathbf{a}_i+\mathbf{b}_j$$

with

$$\mathbf{a}_i+\mathbf{b}_j\neq\mathbf{c}_m.$$

Then clearly

$$\begin{aligned}\sum_{m=1}^{N}V_m&\leqslant V\Big(\bigcup_{m=1}^{N}(S+\mathbf{c}_m)\Big)\\&\leqslant V(\Sigma)+\sum_{m}V(\Sigma_m)\\&=[(4h^2+1+4hy)^{n/2}+N_k(h)(1-y^2)^{n/2}]\,V(S).\end{aligned}$$

So we may suppose that m is chosen with $1\leqslant m\leqslant N_k(h)$ so that

$$V_m\leqslant\Big[\big(N_k(h)\big)^{-1}(4h^2+1+4hy)^{n/2}+(1-y^2)^{n/2}\Big]\,V(S). \tag{19}$$

For convenience we may suppose that $i(m)=1$. Let $S'+\mathbf{c}_m$ denote the part of $S+\mathbf{c}_m$ not lying in any of the sets $S+\mathbf{a}_i+\mathbf{b}_j$ with $\mathbf{a}_i+\mathbf{b}_j\neq\mathbf{c}_m$.

Then the sets

$$\bigcup_{j=1}^{\infty}(S'+\mathbf{c}_m+\mathbf{b}_j),\qquad\bigcup_{i=2}^{M}\bigcup_{j=1}^{\infty}(S+\mathbf{a}_i+\mathbf{b}_j)$$

are disjoint and their union is

$$\bigcup_{i=1}^{M} \bigcup_{j=1}^{\infty} (S+\mathbf{a}_i+\mathbf{b}_j).$$

Comparing the densities of these sets, we see that the density of the set

$$\bigcup_{i=2}^{M} \bigcup_{j=1}^{\infty} (S+\mathbf{a}_i+\mathbf{b}_j)$$

is

$$\theta - V_m R^{-n}.$$

So, if $\sigma(\mathbf{x})$ is the characteristic function of the set

$$E = \bigcup_{i=2}^{M} \bigcup_{j=1}^{\infty} (S+\mathbf{a}_i+\mathbf{b}_j).$$

and C is the cube defined by

$$0 \leqslant x_i < R, \qquad i = 1, 2, \ldots, n,$$

we have

$$R^{-n} \int_C \sigma(\mathbf{x})\, d\mathbf{x} = \theta - V_m R^{-n}.$$

Now consider the density $\theta(\mathbf{t})$ of the set

$$E \cup \left\{ \bigcup_{j=1}^{\infty} (S+\mathbf{t}+\mathbf{b}_j) \right\}.$$

If $\rho(x)$ is the characteristic function of S, we have

$$\theta(\mathbf{t}) = R^{-n} \int_C \left[1 - \left(1 - \sum_{j=1}^{\infty} \rho(\mathbf{x}-\mathbf{t}-\mathbf{b}_j) \right) \left(1 - \sigma(\mathbf{x}) \right) \right] d\mathbf{x}.$$

So

$$\begin{aligned} R^{-n} \int_C \theta(\mathbf{t})\, dt &= R^{-2n} \int_C \int_C \left[1 - \left(1 - \sum_{j=1}^{\infty} \rho(\mathbf{x}-\mathbf{t}-\mathbf{b}_j) \right) \left(1 - \sigma(\mathbf{x}) \right) \right] d\mathbf{x}\, d\mathbf{t} \\ &= 1 - R^{-n} \int_C [R^n - V(S)] \left(1 - \sigma(\mathbf{x}) \right) d\mathbf{x} \\ &= 1 - \left(1 - R^{-n} V(S) \right) \left(1 - (\theta - R^{-n} V_m) \right). \end{aligned}$$

But by the original choice of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ to maximize the density we have

$$\theta(\mathbf{t}) \leqslant \theta,$$

$$\theta \geqslant 1 - \left(1 - R^{-n} V(S) \right) (1 - \theta + R^{-n} V_m),$$

and

$$V_m \geqslant \frac{(1-\theta)\, V(S)}{1 - R^{-n} V(S)} > (1-\theta)\, V(S).$$

Substituting this in (19) we have

$$1-\theta<\left(N_k(h)\right)^{-1}(4h^2+1+4hy)^{n/2}+(1-y^2)^{n/2},$$

so that

$$N_k(h)<\frac{(4h^2+1+4hy)^{n/2}}{1-\theta-(1-y^2)^{n/2}}$$

$$\leqslant\frac{(4h^2+1)^{n/2}(1+y)^{n/2}}{(1-\theta)-(1-y^2)^{n/2}}$$

$$=2(1-\theta)^{-1}(4h^2+1)^{n/2}(1+y)^{n/2}.$$

by (18).

Since

$$\theta\leqslant 1-(1-n^{-1/2})^{n/2},$$

and $n\geqslant 4$, we have

$$\begin{aligned}(1-\theta)^{-1}&\leqslant(1-n^{-1/2})^{-n/2}\\&=\exp\{-\tfrac{1}{2}n\log(1-n^{-1/2})\}\\&=\exp\{\tfrac{1}{2}n^{1/2}+\tfrac{1}{2}\cdot\tfrac{1}{2}+\tfrac{1}{2}\cdot\tfrac{1}{3}n^{-1/2}+\dots\}\\&<\exp\{\tfrac{1}{2}n^{1/2}+\tfrac{1}{2}\}.\end{aligned}$$

Also

$$(1-\theta)^{2/n}\geqslant 1-n^{-1/2}$$

and

$$1-(\tfrac{1}{2})^{2/n}=1-\exp\{-(2/n)\log 2\}<(2/n)\log 2<2/n.$$

So

$$\begin{aligned}y\,\{1-(\tfrac{1}{2})^{2/n}(1-\theta)^{2/n}\}^{1/2}&\leqslant\{1-(1-\{1-(\tfrac{1}{2})^{2/n}\})(1-n^{-1/2})\}^{1/2}\\&<\{1-(1-\{2/n\})(1-n^{-1/2})\}^{1/2}\\&<2n^{-1/4}.\end{aligned}$$

Hence

$$\begin{aligned}2(1-\theta)^{-1}(1+y)^{n/2}&<2\exp\{\tfrac{1}{2}n^{1/2}+\tfrac{1}{2}+\tfrac{1}{2}ny\}\\&<2\exp\{\tfrac{1}{2}\mathrm{n}^{1/2}+\tfrac{1}{2}+n^{3/4}\}<\exp\{2n^{3/4}\},\end{aligned}$$

and the result (15) follows.

Now suppose that $n\geqslant 2^{20}$, that

$$1<r^2<\tfrac{5}{4}(1-4n^{-1/4}), \tag{20}$$

and that

$$h^2=r^2-\tfrac{1}{4}+4n^{-1/4}. \tag{21}$$

Then

$$\tfrac{3}{4}+4n^{-1/4}<h^2<1-n^{-1/4}. \tag{22}$$

Also, provided $h \leqslant x \leqslant r$, we have

$$\frac{d}{d(x^2)}\{(4x^2+1)(r^2-x^2)\} = 4r^2-1-8x^2$$
$$\leqslant 4r^2-1-8h^2$$
$$= -4r^2+1-32n^{-1/4} < 0.$$

Hence

$$(4x^2+1)(r^2-x^2) \leqslant (4h^2+1)(r^2-h^2)$$

for $h \leqslant x \leqslant r$, and

$$\int_h^r nN_k(x)\left(1-\frac{x^2}{r^2}\right)^{(n/2)-1} \frac{x}{r}\, d\left(\frac{x}{r}\right)$$
$$\leqslant \int_h^r nr^{-n}[\exp(2n^{3/4})](4x^3+x)[(4x^2+1)(r^2-x^2)]^{(n/2)-1}\,dx$$
$$\leqslant nr^{-n}[\exp(2n^{3/4})](r^4+\tfrac{1}{2}r^2)[(4h^2+1)(r^2-h^2)]^{(n/2)-1}$$
$$= n[\exp(2n^{3/4})]\left[\frac{r^4+\frac{1}{2}r^2}{(4h^2+1)(r^2-h^2)}\right][r^{-2}(4h^2+1)(r^2-h^2)]^{n/2}$$
$$\doteq n[\exp(2n^{3/4})]\left[\frac{r^4+\frac{1}{2}r^2}{(4h^2+1)(\frac{1}{4}-4n^{-1/4})}\right][1-4(4-r^{-2})n^{-1/4} - 64r^{-2}n^{-1/2}]^{n/2}$$
$$\leqslant n[\exp(2n^{3/4})]\left[\frac{(\frac{5}{4})^2+\frac{1}{2}(\frac{5}{4})}{4(\frac{1}{4}-\frac{1}{8})}\right][1-12n^{-1/4}]^{n/2}$$
$$< \exp(2n^{3/4}+\log\tfrac{15}{8}+\log n-6n^{3/4})$$
$$< \exp(-2n^{3/4}).$$

This proves (17).

4. *Proof of Theorem* 1. It is clear from the nature of Theorem 1 that the methods described in Chapter 1 of [**12**] suffice to reduce the general case of Theorem 1 to the special case when the system Σ is a periodic system of the type (13) described in §3 with period $R > 2$. Suppose then that Σ is a periodic system with period $R > 2$, with density δ, and covering a proportion θ of space, with

$$\theta > \tfrac{4}{3}(1-4n^{-1/4})^{-n/2}(\tfrac{4}{5})^{n/2}. \tag{23}$$

Let Σ be the system $\Sigma(\mathrm{S};\ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_M;\ R)$ introduced in §3. Let Σ_0 be a corresponding system

$$\Sigma(S;\ \mathbf{a}_1^{(0)}, \mathbf{a}_2^{(0)}, \ldots, \mathbf{a}_M^{(0)};\ R),$$

with the same S, M, R, but with $\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(0)}, \ldots, \mathbf{a}_M^{(0)}$ chosen to maximize the proportion of space covered by the system. Then the density of Σ_0 is $\delta_0 = \delta$ and Σ_0 covers a proportion $\theta_0 \geqslant \theta$ of space. Write

$$\vartheta = \min\{\theta_0, 1-(1-n^{-1/2})^{n/2}\}. \tag{24}$$

If $\theta_0 > \vartheta$, we can obtain a new periodic system Σ_1, with density $\partial < \delta$, covering precisely the proportion ϑ of space, and satisfying the required maximality condition, by increasing R to a suitable value $R^{(1)}$ and modifying the choice of $\mathbf{a}_1^{(0)}, \mathbf{a}_2^{(0)}, \ldots, \mathbf{a}_M^{(0)}$ appropriately. Let this new system be

$$\Sigma_1 = \Sigma(S;\ \mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}, \ldots, \mathbf{a}_M^{(1)};\ R^{(1)}).$$

If $\theta_0 = \vartheta$, we take $\Sigma_1 = \Sigma_0$ and write $\partial = \delta$.

In either case we arrive at a system

$$\Sigma_1 = \Sigma(S;\ \mathbf{a}_1^{(1)}, \mathbf{a}_2^{(1)}, \ldots, \mathbf{a}_M^{(1)};\ R^{(1)})$$

of density ∂ covering the proportion ϑ of space, with

$$R^{(1)} > 2,$$

$$\tfrac{4}{3}(1-4n^{-1/4})^{-n/2}(\tfrac{4}{5})^{n/2} < \vartheta \leqslant 1-(1-n^{-1/2})^{n/2}, \tag{25}$$

$$\partial \leqslant \delta. \tag{26}$$

We write

$$r = (\tfrac{3}{4}\vartheta)^{-1/n}. \tag{27}$$

Then

$$1 < r^2 < \tfrac{5}{4}(1-4n^{-1/4}). \tag{28}$$

We also write

$$h^2 = r^2 - \tfrac{1}{4} + 4n^{-1/4}. \tag{29}$$

Let $\mathbf{c}_1, \mathbf{c}_2, \ldots$ be an enumeration of the points

$$\mathbf{a}_i^{(1)} + \mathbf{b}_j^{(1)}, \qquad i = 1, 2, \ldots, M, \ \ j = 1, 2, \ldots, \tag{30}$$

where $\mathbf{b}_1^{(1)}, \mathbf{b}_2^{(1)}, \ldots$ is an enumeration of the points whose coordinates are integral multiples of $R^{(1)}$. For each positive integer k, let $\Pi(\mathbf{c}_k)$ be the Voronoi polyhedron of all points $\mathbf{x}$, satisfying

$$|\mathbf{x}-\mathbf{c}_k| \leqslant |\mathbf{x}-\mathbf{c}_l|, \qquad l = 1, 2, \ldots.$$

Let $N_k(x)$ be the number of points of the system (30) within distance $2x$ of the point $\mathbf{c}_k$. Then, as $n \geqslant 2^{20}$, $R^{(1)} > 2$, ϑ satisfies (25) and r and h satisfy (29), we have, by Lemma 2,

$$\int_h^r nN_k(x)\left(1-\frac{x^2}{r^2}\right)^{(n/2)-1} \frac{x}{r}\, d\left(\frac{x}{r}\right) < \Xi, \tag{31}$$

where we write

$$\Xi = \exp(-2n^{3/4}). \tag{32}$$

Writing
$$\Phi = \left\{\frac{r^2(1-h^2)}{r^2-h^2}\right\}^{n/2}$$
$$= \{r^2(5-4r^2)\}^{n/2}\left[\frac{1-\{16/(5-4r^2)\}n^{-1/4}}{1-16n^{-1/4}}\right]^{n/2}$$
$$< \{r^2(5-4r^2)\}^{n/2}, \tag{33}$$

and using Lemma 1, we deduce that, for each integer k,
$$V\Big(\{S+\mathbf{c}_k\}\cap\Pi(\mathbf{c}_k)\Big) \leqslant r^{-n}\Phi V\Big(\Pi(\mathbf{c}_k)\Big)+[1-\Phi+\Phi\Xi]\,V(S). \tag{34}$$

But by the periodicity of the system with period $R^{(1)}$ in each coordinate, we have
$$\sum_{i=1}^{M} V\Big(\{S+\mathbf{a}_i^{(1)}\}\cap\Pi(\mathbf{a}_1^{(1)})\Big) = \vartheta(R^{(1)})^n,$$
$$\sum_{i=1}^{M} V\Big(\Pi(\mathbf{a}_i^{(1)})\Big) = (R^{(1)})^n,$$
$$\sum_{i=1}^{M} V(S) = \partial(R^{(1)})^n.$$

So summing M inequalities of the form (34), and dividing by $(R^{(1)})^n$, we have
$$\vartheta \leqslant r^{-n}\Phi+[1-\Phi+\Phi\Xi]\,\partial.$$

Hence
$$\partial \geqslant \frac{\vartheta-r^{-n}\Phi}{1-\Phi+\Phi\Xi}$$
$$= \vartheta+\frac{\vartheta\Phi+r^{-n}\Phi-\vartheta\Phi\Xi}{1-\Phi(1-\Xi)}$$
$$\geqslant \vartheta+\{\vartheta\Phi-r^{-n}\Phi-\vartheta\Phi\Xi\}$$
$$= \vartheta+\tfrac{1}{4}\vartheta\Phi\{1-4\Xi\}.$$

But
$$\Phi = \left\{\frac{r^2(1-h^2)}{r^2-h^2}\right\}^{n/2}$$
$$= \left\{\frac{r^2(5-4r^2-16n^{-1/4})}{1-16n^{-1/4}}\right\}^{n/2}$$
$$= r^n\left\{1-\frac{4(r^2-1)}{1-16n^{-1/4}}\right\}^{n/2}$$
$$\geqslant r^n\{1-4(r^2-1)(1+32n^{-1/4})\}^{n/2}$$
$$= (\tfrac{3}{4}\vartheta)^{-1}[1-4\{(\tfrac{3}{4}\vartheta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2}.$$

So

$$\partial \geqslant \vartheta+\tfrac{1}{3}\{1-4\exp(-2n^{3/4})\}[1-4\{(\tfrac{3}{4}\vartheta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2}. \qquad (35)$$

When we have $\theta \leqslant \vartheta$ it follows that

$$\delta \geqslant \partial \geqslant \theta+\tfrac{1}{3}\{1-4\exp(-2n^{3/4})\}[1-4\{(\tfrac{3}{4}\theta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2},$$

so that, certainly

$$\delta \geqslant \theta+\tfrac{1}{3}[1-\exp(16-\tfrac{1}{2}n^{1/2})][1-4\{(\tfrac{3}{4}\theta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2}.$$

When $\vartheta < \theta$ we have $\vartheta < \theta_0$ so that

$$\vartheta = 1-(1-n^{-1/2})^{n/2} < \theta.$$

In this case

$$\delta \geqslant \partial \geqslant \eta_0+\alpha[1-4\beta\{(\tfrac{3}{4}\eta_0)^{-2/n}-1\}]^{n/2},$$

where we write

$$\alpha = \tfrac{1}{3}\{1-4\exp(-2n^{3/4})\},$$
$$\beta = 1+32n^{-1/4},$$
$$\eta_0 = 1-(1-n^{-1/2})^{n/2}.$$

Here

$$\tfrac{1}{4} < \alpha < \tfrac{1}{3},$$
$$1 < \beta \leqslant 2,$$
$$0 < 1-\eta_0 < \exp(-\tfrac{1}{2}n^{1/2}).$$

We write

$$f(\phi) = \alpha[1-4\beta\{\phi-1\}]^{n/2},$$

and study $f(\phi)$ in the range

$$(\tfrac{3}{4})^{-2/n} \leqslant \phi \leqslant (\tfrac{3}{4}\eta_0)^{-2/n}.$$

We have

$$f(\phi) \geqslant \tfrac{1}{4}[1-8\{(\tfrac{3}{4}\eta_0)^{-2/n}-1\}]^{n/2}.$$

But

$$\begin{aligned}(\tfrac{3}{4}\eta_0)^{-2/n} &= (\tfrac{4}{3})^{2/n}\exp\left(-\frac{2}{n}\log\{1-(1-n^{-1/2})^{n/2}\}\right)\\ &< (\tfrac{4}{3})^{2/n}\exp\left(\frac{4}{n}(1-n^{-1/2})^{n/2}\right)\\ &< (\tfrac{4}{3})^{2/n}\exp\left(\frac{4}{n}\exp(-\tfrac{1}{2}n^{1/2})\right)\\ &< (\tfrac{4}{3})^{2/n}\left(1+\frac{5}{n}\exp(-\tfrac{1}{2}n^{1/2})\right) \qquad (36)\end{aligned}$$

$$< \left(1+\frac{4}{n}\log\tfrac{4}{3}\right)\left(1+\frac{5}{n}\exp(-\tfrac{1}{2}n^{1/2})\right)$$

$$< 1+\frac{3}{n}. \qquad (37)$$

Hence

$$f(\phi) \geqslant \tfrac{1}{4}\left[1-\frac{24}{n}\right]^{n/2}$$

$$> \tfrac{1}{4}e^{-13} > e^{-15}. \qquad (38)$$

But we also have

$$-f'(\phi) = 4\alpha\beta\tfrac{1}{2}n[1-4\beta\{\phi-1\}]^{(n/2)-1}$$

$$< \frac{4n}{1-8(3/n)}f(\phi) < 5nf(\phi).$$

So, dividing by $f(\phi)$, and integrating over the range

$$(\tfrac{3}{4}\theta)^{-2/n} \leqslant \phi \leqslant (\tfrac{3}{4}\eta_0)^{-2/n},$$

we obtain, on using (36)

$$\log\left[f\left((\tfrac{3}{4}\theta)^{-2/n}\right)\right]-\log\left[f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right)\right]$$

$$< 5n[(\tfrac{3}{4}\eta_0)^{-2/n}-(\tfrac{3}{4}\theta)^{-2/n}]$$

$$\leqslant 5n[(\tfrac{3}{4}\eta_0)^{-2/n}-(\tfrac{3}{4})^{-2/n}]$$

$$< 5n(\tfrac{4}{3})^{2/n}\left(\frac{5}{n}\exp(-\tfrac{1}{2}n^{1/2})\right)$$

$$< 26\exp(-\tfrac{1}{2}n^{1/2}).$$

Hence

$$f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right) > f\left((\tfrac{3}{4}\theta)^{-2/n}\right)\exp\{-26\exp(-\tfrac{1}{2}n^{1/2})\}$$

$$> f\left((\tfrac{3}{4}\theta)^{-2/n}\right)\{1-26\exp(\tfrac{1}{2}n^{1/2})\}.$$

Consequently

$$\delta \geqslant \eta_0+f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right)$$

$$\geqslant \theta-(1-\eta_0)+f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right)$$

$$\geqslant \theta-[\exp(-\tfrac{1}{2}n^{1/2})]e^{15}f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right)+f\left((\tfrac{3}{4}\eta_0)^{-2/n}\right)$$

$$\geqslant \theta+[1-\exp(15-\tfrac{1}{2}n^{1/2})][1-26\exp(-\tfrac{1}{2}n^{1/2})]f\left((\tfrac{3}{4}\theta)^{-2/n}\right)$$

$$\geqslant \theta+\tfrac{1}{3}[1-\exp(16-\tfrac{1}{2}n^{1/2})][1-4\{(\tfrac{3}{4}\theta)^{-2/n}-1\}\{1+32n^{-1/4}\}]^{n/2}.$$

Thus we have the inequality in each case.

References

1. H. Minkowski, "Diskontinuitätsbereich für arithmetische Aequivalenz", *J. reine angew. Math.*, 129 (1905), 220–274.

2. H. F. Blichfeldt, "The minimum value of quadratic forms, and the closest packing of spheres", *Math. Annalen*, 101 (1929), 605–608.

3. H. Davenport and C. A. Rogers, "Hlawka's theorem in the geometry of numbers", *Duke Math. J.*, 14 (1947), 367–375.

4. W. Schmidt, "On the Minkowski-Hlawka Theorem", *Illinois J. of Math.*, 7 (1963), 18–23 and 714.

5. C. A. Rogers, "The packing of equal spheres", *Proc. London Math. Soc.* (3), 8 (1958), 609–620.

6. R. P. Bambah and H. Davenport, "The covering of n-dimensional space by spheres", *Journal London Math. Soc.*, 27 (1952), 224–229.

7. P. Erdős and C. A. Rogers, "The covering of n-dimensional space by spheres", *Journal London Math. Soc.*, 28 (1953), 287–293.

8. H. S. M. Coxeter, L. Few and C. A. Rogers, "Covering space with equal spheres", *Mathematika*, 6 (1959), 147–157.

9. C. A. Rogers, "Lattice coverings of space with convex bodies", *Journal London Math. Soc.*, 33 (1958), 208–212.

10. ———, "Lattice coverings of space: the Minkowski-Hlawka theorem", *Proc. London Math. Soc.* (3), 8 (1958), 447–465.

11. ———, "A note on coverings", *Mathematika*, 4 (1957), 1–6.

12. ———, *Packing and covering* (Cambridge, 1964).

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London.

(*Received on the* 17*th of August*, 1964.)

Printed by C. F. Hodgson & Son, Ltd., Pakenham Street, London, W.C.1