ON TCHEBYCHEFF QUADRATURE

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1. Tchebycheff proposed the problem of finding $n+1$ constants $A, x_1, x_2, \ldots, x_n$ $(-1 \leqslant x_1 < x_2 < \ldots < x_n \leqslant +1)$ such that the formula

$$\int_{-1}^{1} f(x)dx = A \sum_{i=1}^{n} f(x_i) \tag{1}$$

is exact for all algebraic polynomials of degree $\leqslant n$. In this case it is clear that $A = 2/n$. Later S. Bernstein **(1)** proved that for $n \geqslant 10$ not all the x_i's can be real. For a history of the problem and for more references see Natanson **(4)**. However, we know that for suitable A_i, the formula

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} A_i f(\xi_i) \tag{2}$$

is exact for all polynomials of degree $\leqslant 2n-1$ and that all the ξ_i's are real. Indeed the ξ_i's are the zeros of the Legendre polynomials $P_n(x)$ of degree n and all the A_i's are non-negative.

Thus one observes that if one determines $n+1$ constants as in the Tchebycheff case, there exists a number n_0 (in this case $n_0 = 10$) such that not all the x_i's are real for $n > n_0$. However, if we allow ourselves more freedom, as in the Gauss quadrature case of formula (2), there is no number n_0 such that for $n > n_0$ some of the ξ_i's must become imaginary, since in this case all the ξ_i's turn out to be real and lie in $[-1, 1]$.

Two questions arise naturally in this connection. We formulate them as follows:

PROBLEM 1. *Given a fixed integer k, we wish to determine $n+k+1$ $(n \geqslant k+2)$ constants $A_i, y_i (i = 1, 2, \ldots, k)$, $x_j (j = 1, 2, \ldots, n-k)$, and B so that the formula*

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{k} A_i f(y_i) + B \sum_{j=1}^{n-k} f(x_j) \tag{3}$$

is exact for all polynomials of degree $\leqslant n+k$. We require the y_i's and x_j's to be in $[-1, 1]$. Does there exist a number n_0 such that for $n > n_0$ the formula (3) *is no longer valid?*

PROBLEM 2. *If for every n, the formula* (3) *is only required to be valid for all polynomials of degree $m = m(n) < n$, what is the order of $m(n)$?*

The object of this paper is to show that in Problem 2, $m(n) = O(\sqrt{n})$, whence it is clear that the answer to Problem 1 is in the affirmative.

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When $n = k$ or $k + 1$, Problem 1 has a negative answer as is seen by the Gauss quadrature formula. For $k = 0$, the answer to Problem 1 is known and is due to Bernstein. But Problem 2 does not seem to have been formulated even for $k = 0$.

If $k = 1$, one can determine the constants in (3) easily when $n = 2$ or 3. When $n = 2$, one has the system of equations

$$\begin{aligned} A + B &= 2, \\ Ay_1 + Bx_1 &= 0, \\ Ay_1^2 + Bx_1^2 &= \tfrac{2}{3}, \\ Ay_1^3 + Bx_1^3 &= 0, \end{aligned}$$

which have the solution $A = B = 1$, $x_1 = -y_1 = 1/\sqrt{3}$. Also when $k = 1$, $n = 3$, we have the system of equations

$$\begin{aligned} A + 2B &= 2, \\ Ay_1 + B(x_1 + x_2) &= 0, \\ Ay_1^2 + B(x_1^2 + x_2^2) &= \tfrac{2}{3}, \\ Ay_1^3 + B(x_1^3 + x_3^3) &= 0, \\ Ay_1^4 + B(x_1^4 + x_2^4) &= \tfrac{2}{5}, \end{aligned}$$

which have a solution, viz. $y_1 = 0$, $x_1 = -x_2 = \sqrt{\frac{3}{5}}$, $A = \frac{8}{9}$, $B = \frac{5}{9}$.

For larger values of n, the equations become very cumbersome to handle.

2. We shall prove the following:

THEOREM 1. *k being a fixed integer and n a large integer, if the formula* (3) *is exact for all polynomials of degree $\leqslant m = m(n) < n$ for real x_i, y_i, A_i, and B with x_i, y_i in $[-1, 1]$, then $m \leqslant c_k\sqrt{n}$ where c_k depends on k only.*

A consequence of Theorem 1 is the following result.

THEOREM 2. *There exists an integer n_0 such that for $n > n_0$ no formula* (3) *can be valid for every polynomial $f(x)$ of degree $\leqslant n + k$ with real*

$$y_1, y_2, \ldots, y_k, x_1, x_2, \ldots, x_{n-k}$$

in $[-1, 1]$.

We assume in our proof of Theorem 1 that the x_i and y_i are in $[-1, 1]$, but we can also prove it without assuming this. It suffices to assume that they are real. The proof of this stronger statement follows the same lines but is a bit more complicated.

For the proof of Theorem 1, we need the following lemmas.

LEMMA 1. **(2**, p. 529**)**. *For the fundamental polynomials $l_{kn}(x)$ of Lagrange interpolation formed upon any n points $x_1 < x_2 < \ldots < x_n$, we have*

$$l_{kn}(x) + l_{k+1,n}(x) \geqslant 1 \tag{4}$$

for $x_k \leqslant x \leqslant x_{k+1}$.

It follows from this lemma that for every x_0 with $x_k \leqslant x_0 \leqslant x_{k+1}$, we have

$$\text{either } l_{kn}(x_0) \geqslant \tfrac{1}{2} \quad \text{or} \quad l_{k+1,n}(x_0) \geqslant \tfrac{1}{2}. \tag{5}$$

From a theorem of Fejér **(3)**, we know that when $\xi_1, \xi_2, \ldots, \xi_n$ are the Tchebycheff abscissas (zeros of $T_n(x) = \cos n\theta, \cos\theta = x$), we have

$$\sum_{i=1}^{n} l_{in}^{\,2}(x) \leqslant 2 \tag{6}$$

whence

$$|l_{in}(x)| \leqslant \sqrt{2} \qquad (i = 1, 2, \ldots, n; -1 \leqslant x \leqslant 1). \tag{7}$$

LEMMA 2. *Given an integer m sufficiently large and points $x_0, y_1, y_2, \ldots, y_k$ in $[-1, 1]$, such that*

$$x_0 = 1 - c_1/m^2, \qquad |x_0 - y_i| > c_2/m^2 \qquad (i = 1, 2, \ldots, k),$$

c_1, c_2 being some positive constants independent of m, there exist constants c_3, c_4 depending on c_1, c_2, and k, and a polynomial $P_m(x)$ of degree $\leqslant m$, with the following properties:

(i) $0 \leqslant P_m(x) \leqslant \alpha^k$ *for* $-1 \leqslant x \leqslant 1$, α *independent of* m,
(ii) $P_m(x_0) = 1$,
(iii) $P_m(y_i) = 0, i = 1, 2, \ldots, k$,
(iv) $P_m(x) < \frac{1}{2}$ if $|x_0 - x| > c_3/m^2$,

and

(v) $\displaystyle\int_{-1}^{1} P_m(x)dx < c_4/m^2.$

Proof. It is enough to prove the result for $k = 1$. For if $P_{M,i}(x)$ is a polynomial of degree $M = [m/k]$ with properties (ii), (iv), and (v) and with $P_{M,i}(y_i) = 0$ and $0 < P_{M,i}(x) \leqslant \alpha$ for $-1 \leqslant x \leqslant 1$ instead of (i) and (iii), then we consider the polynomial

$$P(x) = \prod_{i=1}^{k} P_{M,i}(x)$$

which is of degree $\leqslant m$. It is clear that $P(x)$ possesses properties (i)–(iv), and since $P_{M,i}(x)$ $(i = 1, 2, \ldots, k)$ are non-negative, we have

$$\begin{aligned} \int_{-1}^{1} P(x) &= \int_{-1}^{1} \prod_{i=1}^{k} P_{M,i}(x)dx \\ &\leqslant \prod_{i=1}^{k-1} \max_{-1\leqslant x\leqslant 1} P_{M,i}(x) \int_{-1}^{1} P_{M,k}(x)dx \\ &\leqslant C_5/M^2 \leqslant C_6/m^2. \end{aligned} \tag{8}$$

We may therefore take $k = 1$ in the lemma. Set

$$P_m(x) = C_7 \frac{(x - y_1)^2}{(x_0 - y_1)^2} (l_{pm}(x))^4, \tag{9}$$

where $l_{pm}(x)$ is the fundamental polynomial of Lagrange interpolation on Tchebycheff abscissas $(-1 < \xi_m < \xi_{m-1} < \ldots \xi_1 < 1)$ given by

$$\xi_j = \cos \frac{2j-1}{2m}\,\pi, \qquad j = 1, 2, \ldots, m.$$

Put $\xi_0 = 1$ and $\xi_{m+1} = -1$. Then

$$l_{pm}(x) = \frac{T_m(x)}{(x-\xi_p)T_m{}'(\xi_p)}. \tag{10}$$

We shall show that $P_m(x)$ is the polynomial required. Since $x_0 = 1 - C_1/m^2$, we may suppose that $\xi_{p+1} \leqslant x_0 \leqslant \xi_p$ for some finite p, p independent of m. By Lemma 1 and the remark following it, either $l_{pm}(x_0) \geqslant \frac{1}{2}$ or $l_{p+1,m}(x_0) \geqslant \frac{1}{2}$. Let $l_{pm}(x_0) \geqslant \frac{1}{2}$, to be precise. Using (7), we can fix a constant $C_8 \leqslant 4$ such that

$$P_m(x_0) = C_8\,(l_{pm}(x_0)) = 1. \tag{11}$$

Thus $P_m(x)$ satisfies (ii) and (iii). To prove that $P_m(x)$ satisfies (i) and (iv), we observe that if $|x - y_1| \leqslant |x_0 - y_1|$, we have

$$P_m(x) \leqslant C_8\,(l_{pm}(x)) \leqslant 16. \tag{12}$$

If $|x - y_1| > |x_0 - y_1|$ we shall still show that $P_m(x)$ is bounded. For if $\xi_{i+1} \leqslant y_1 \leqslant \xi_i$, then from (10), we have for $\xi_{s+1} \leqslant x \leqslant \xi_s$, the inequality

$$\begin{aligned} |l_{pm}(x)| &\leqslant \frac{1}{m|\xi_s - \xi_p|}\cdot\sqrt{(1-\xi_p^2)} \\ &= \frac{\left|\sin\dfrac{2p-1}{2m}\,\pi\right|}{2m\left|\sin\dfrac{s-p}{2m}\,\pi\right|\;\left|\sin\dfrac{s+p-1}{2m}\,\pi\right|} \leqslant \frac{C_9}{(s-p)^2}. \end{aligned} \tag{13}$$

Also for $\xi_{s+1} \leqslant x \leqslant \xi_s$, we have

$$\begin{aligned} \frac{(x-y_1)^2}{(x_0-y_1)^2} &\leqslant \frac{(\xi_{s+1}-\xi_i)^2}{(x_0-y_1)^2} \leqslant \frac{(1-\xi_{s+1})^2}{(C_2/m^2)^2} \\ &= C_{10}\left(m \sin\frac{2s+1}{2m}\,\pi\right)^4 = C_{11}\cdot s^4. \end{aligned} \tag{14}$$

Thus we have for $|x - y_1| > |x_0 - y_1|$,

$$P_m(x) \leqslant 4\left(\frac{C_9}{(s-p)^2}\right)^4\cdot C_{11}\,s^4 = \frac{C_{12}s^4}{(s-p)^8} \leqslant \frac{C_{13}}{(s-p)^4}. \tag{15}$$

We can now prove part (iv) of the lemma. Namely, the constant C_3 can be taken so large that for all x such that $|x_0 - x| > C_3/m^2$ inequality (15) will hold, and with such a large s that the right-hand member of (15) will be $\leqslant \frac{1}{2}$. Also, combining (15) and (12) we prove part (i) of the lemma with $\alpha = \max(16, C_{13})$.

To prove (v) we observe that

$$I = \int_{-1}^{1} P_m(x)dx = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 = \sum_{s=0}^{p-1} \int_{\xi_{s+1}}^{\xi_s} P_m(x)dx, \qquad I_2 = \int_{\xi_{p+1}}^{\xi_p} P_m(x)dx,$$

$$I_3 = \sum_{s=p+1}^{s_0-1} \int_{\xi_{s+1}}^{\xi_s} P_m(x)dx, \qquad I_4 = \sum_{s=s_0}^{m} \int_{\xi_{s+1}}^{\xi_s} P_m(x)dx.$$

Here s_0 is the largest value of s for which $|\xi_s - y_1| < |x_0 - y_1|$. Since

$$|\xi_s - \xi_{s+1}| = \left| \cos \frac{2s-1}{2m} \pi - \cos \frac{2s+1}{2m} \pi \right| \leqslant C_{14} \cdot \frac{s}{m^2},$$

we have, using the definition (9) of $P_m(x)$,

$$\begin{aligned} I_1 &\leqslant C_7 \sum_{s=0}^{p-1} C_9^{\,4} \frac{|\xi_s - \xi_{s+1}|}{(s-p)^8} \cdot \frac{(1-\xi_{i+1})^2}{(C_3/m^2)^2} \\ &\leqslant \frac{C_7\, C_9^{\,4}}{C_3^{\,2}} \cdot \frac{C_{14}}{m^2} \sum_{s=0}^{p-1} \frac{s}{(s-p)^8} \cdot \left(m \sin \frac{2i+1}{2m} \pi \right)^4 \\ &\leqslant \frac{C_{15}}{m^2} \sum_{s=0}^{p-1} \frac{s}{(s-p)^8} \leqslant \frac{C_{16}}{m^2}. \end{aligned}$$

Similarly,

$$I_2 \leqslant 4|\xi_p - \xi_{p+1}| \cdot \frac{(1-\xi_{p+1})^2}{(x_0 - y_1)^2} \leqslant \frac{C_{17}}{m^2},$$

$$I_3 \leqslant C_7 \sum_{s=p+1}^{s_0-1} \frac{|\xi_s - \xi_{s+1}|}{(s-p)^8} \leqslant \frac{C_{18}}{m^2},$$

and

$$I_4 \leqslant C_7 \sum_{s=s_0}^{m} \frac{|\xi_s - \xi_{s+1}|}{(s-p)^8} \cdot \frac{(1-\xi_{s+1})^2}{(x_0 - y_1)^2} \leqslant \frac{C_{19}}{m^2} \sum_{s=s_0}^{m} \frac{s^5}{(s-p)^8} \leqslant \frac{C_{20}}{m^2}.$$

Combining all these estimates for I_1, I_2, I_3, and I_4, we see at once that Property (iv) is verified. This completes the proof of Lemma 2.

LEMMA 3. *If $Q_m(x)$ is a polynomial of degree m, non-negative in $-1 \leqslant x \leqslant 1$, and if $Q_m(x_0) = 1$ for some x_0 in $[-1, 1]$, then*

$$\int_{-1}^{1} Q_m(x)dx \geqslant \frac{1}{2m^2}.$$

This is an immediate consequence of Bernstein's inequality regarding derivatives of a polynomial of degree m.

3. Proof of Theorem 1. We shall show that if we allow $m > Mt$ where $M = [a\sqrt{n}]$, a and t being sufficiently large constants, we arrive at a contradiction.

Taking $f(x)$ to be a polynomial

$$P_{2k}(x) = \prod_{i=1}^{k} (x - y_i)^2,$$

we see at once from (3) that $B > 0$.

Consider now the $k + 1$ intervals

$$\left(1 - \frac{iC}{M^2}, 1 - \frac{(i-1)C}{M^2}\right), \qquad i = 1, 2, \ldots, k + 1,$$

where C is sufficiently large. Denote the ith interval by I_i. Then there is at least one of the intervals, I_j (say), which is free of the k points $y_1, y_2, \ldots, y_k$. Denote the middle half of I_j by I', so that I' is

$$\left(1 - \frac{4j-1}{4M^2}C, 1 - \frac{4j-3}{4M^2}C\right).$$

We consider now two possibilities:

(i) there is no x_i in I',

(ii) there is at least one x_i in I'.

In case (i), we take x_0 to be the middle point of I'. Then one can easily see that there exist constants C_1 and C_2 such that

$$x_0 = 1 - C_1/M^2 \quad \text{and} \quad |x_0 - y_i| > C_2/M^2 \quad \text{for } i = 1, 2, \ldots, k.$$

Then by Lemma 2, there exists a non-negative polynomial $P_M(x)$ of degree M which satisfies the conditions (i)–(v) of Lemma 2. By the quadrature formula (3), we have

$$\int_{-1}^{1} P_M(x)dx = B \sum_{i=1}^{n-k} P_M(x_i) < \frac{C_4}{M^2},$$

where the inequality follows from Lemma 2, (v).

Since $P_M(x_0) = 1$, we have by Lemma 3

$$\int_{-1}^{1} P_M(x)dx > \frac{1}{2M^2},$$

so that for a suitable constant λ between C_4 and $\frac{1}{2}$, we have

$$B \sum_{i=1}^{n-k} P_M(x_i) = \frac{\lambda}{M^2}.$$

Again using (3) and Property (iv) of Lemma 2, we have

$$\int_{-1}^{1} (P_M(x))^t dx = B \sum_{i=1}^{n-k} (P_M(x_i))^t < B \sum_{i=1}^{n-k} P_M(x_i)(\tfrac{1}{2})^{t-1},$$

while Lemma 3 gives

$$\int_{-1}^{1} (P_M(x))^t dx > \frac{1}{2M^2t^2}\,;$$

whence we have

$$\frac{1}{2M^2t^2} < \frac{\lambda}{M^2}\left(\frac{1}{2}\right)^{t-1},$$

which is impossible for t sufficiently large. Thus we cannot have case (i). Thus there is at least one x_i (say x_1) in I', and there exist constants C_1 and C_2 such that

$$x_1 = 1 - C_1/M^2 \quad \text{and} \quad |x_1 - y_i| > C_2/M^2, \qquad i = 1, 2, \ldots, k.$$

Then there exists a polynomial $P_M(x)$ of Lemma 2. As in case (i), we have

$$\int_{-1}^{1} P_M(x)dx = B \sum_{i=1}^{n-k} P_M(x_i) < \frac{C_4}{M^2}.$$

Since by Property (ii) of Lemma 2, $P_M(x_i) = 1$, we have

$$B < C_4/M^2 < C_4/a^2n \text{ (since } M = [a\sqrt{n}]).$$

However, taking

$$f(x) = P_{2k}(x) = \prod_{i=1}^{k} (x - y_i)^2$$

in (3), we have $|P_{2k}(x)| \leqslant 2^{2k}$ in $(-1, 1)$, so that

$$\alpha_k = \int_{-1}^{1} P_{2k}(x)dx = B \sum_{i=1}^{n-k} P_{2k}(x_i) < \frac{C_4}{a^2 n}(n-k)2^{2k} < \frac{C_4}{a^2}\cdot 2^{2k},$$

which is impossible if $a > (C_4\, 2^{2k}/\alpha_k)^{\frac{1}{2}}$.

This contradiction completes the proof of the theorem.

4. By a modification of our method we can show that not all the x_i's can be real if the quadrature formula is to hold. We do not know if the order of m given by Theorem 1 is the best possible. It would be interesting to find a numerical value for the n_0 whose existence is claimed in Theorem 2. Another interesting problem which calls for attention is the study of the modified Tchebycheff quadrature problem when some weight-function is used in formula (3). It would also be interesting to inquire into the nature of n_0 as a function of k.

References

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