REMARKS ON A THEOREM OF ZYGMUND

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To J. E. LITTLEWOOD *on his* 80*th birthday*

A well-known theorem of Zygmund (**6**) states that if $n_1 < n_2 < \ldots$ is a sequence of integers satisfying

$$n_{k+1}/n_k > 1+c \quad (c > 0), \tag{1}$$

and, if $|a_k| \to 0$, $|b_k| \to 0$ is any sequence, then

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x) \tag{2}$$

converges for at least one x; in fact the set of x for which (2) converges is of power c in any interval.

Paley and Mary Weiss (**5**) extended this theorem for power series, i.e. they proved that if $|a_k| \to 0$ and $n_1 < n_2 < \ldots$ satisfies (1) then

$$\sum_{k=1}^{\infty} a_k z^{n_k} \tag{3}$$

converges for at least one z with $|z| = 1$; in fact the set of these z's is of power c on every arc.

Kahane (**3**) calls a sequence of integers $n_1 < n_2 < \ldots$ a *Zygmund sequence* if whenever $|a_k| \to 0$ the series (3) converges for at least one z with $|z| = 1$.

Kennedy (**4**) proved that to every $\varphi(k) \to 0$ (as $k \to \infty$) there is a sequence $n_1 < n_2 < \ldots$ for which

$$n_{k+1}/n_k > 1+\varphi(k)$$

and which is not a Zygmund sequence. Kennedy's result implies that in some sense Zygmund's theorem cannot be sharpened.

Kahane (**3**) observes that 'a slight change in the proof [of Kennedy] shows that a Zygmund sequence cannot contain arbitrarily long arithmetic progressions. Nothing more seems to be known.'

I am going to prove the following

THEOREM. *Let $n_1 < n_2 < \ldots$ be a sequence which contains two subsequences n_{k_i} and n_{l_i}, $1 \leqslant i < \infty$, satisfying*

$$k_i \to \infty, \quad k_i < l_i < k_{i+1}, \quad l_i - k_i \to \infty, \tag{4}$$

$$(n_{l_i} - n_{k_i})^{1/(l_i-k_i)} \to 1.$$

Then our sequence is not a Zygmund sequence. In other words there is a power series $\sum_{k=1}^{\infty} a_k z^{n_k}$, $|a_k| \to 0$, *which diverges for every* z *with* $|z| = 1$.

REMARK. The condition (4) can be formulated also in the following equivalent form: there exists an infinite sequence of disjoint intervals I_n such that, denoting by $|I_n|$ the length of the interval I_n and by $\mathcal{N}(I_n)$ the number of terms of the sequence $\{n_k\}$ which lie in the interval I_n, one has

$$\lim_{n\to\infty} |I_n| = \infty \tag{4a}$$

and

$$\lim_{n\to\infty} \frac{\mathcal{N}(I_n)}{\log |I_n|} = +\infty. \tag{4b}$$

It seems quite possible that if $n_1 < n_2 < \ldots$ does not satisfy (4), in other words if there is a $c > 0$ such that

$$n_u - n_v > (1+c)^r, \tag{5}$$

then it is a Zygmund sequence. I am unable to prove or disprove the above conjecture. The gap condition (5) has occurred in several previous papers (**1**) but so far it has never been decided that condition (5) is best possible.

The proof of our theorem will utilize probabilistic arguments which have been used in several previous papers (**1**).

It is easy to see that if (4) holds we may suppose that

$$n_{l_i} - n_{k_i} > i^3 \tag{6}$$

and

$$(n_{l_i} - n_{k_i})^{i/(l_i - k_i)} \to 1, \tag{7}$$

because there exists always a subsequence which satisfies these conditions.

From (6) and (7) we evidently have, for sufficiently large i,

$$2^{(l_i-k_i)/2i} > (n_{l_i} - n_{k_i})^4 > i^3 (n_{l_i} - n_{k_i})^3. \tag{8}$$

Our series will be of the form

$$\sum_{i=1}^{\infty} \frac{1}{\log i} \sum_{j=0}^{l_i-k_i} \varepsilon_j^{(i)} z^{n_{k_i+j}}, \quad \varepsilon_j^{(i)} = \pm 1, \quad 1 \leqslant i < \infty, \quad 0 \leqslant j \leqslant l_i - k_i. \tag{9}$$

We are going to show that for almost all choices of $\varepsilon_j^{(i)} = \pm 1$ the series (9) diverges for every z with $|z| = 1$. To show this we put

$$\frac{1}{\log i} \sum_{j=0}^{l_i-k_i} \varepsilon_j^{(i)} z^{n_{k_i+j}} = \sum_r Q_r^{(i)}(z), \tag{10}$$

where $Q_r^{(i)}(z)$ is the sum of i consecutive terms of (10), except the last which may have fewer than i terms, and which we shall ignore; hence $1 \leqslant r \leqslant \left[\frac{l_i - k_i}{i}\right]$. $Q_r^{(i)}(z)$ depends of course on the $\varepsilon_j^{(i)}$; this dependence we do not indicate explicitly.

Our proof will be complete if we succeed in proving that, for almost all choices of $\varepsilon_j^{(i)}$, we have that for every A and for every z with $|z| = 1$, for every $i > i_0(A)$ there is an $r = r(z, i, A)$ satisfying

$$|Q_r^{(i)}(z)| > A. \tag{11}$$

(11) clearly implies that for almost all choices of $\varepsilon_j^{(i)}$, (9) diverges for every z with $|z| = 1$. Thus we have only to prove (11).

There are clearly $2^{l_i - k_i}$ choices of the sum (10). (11) will follow immediately from the Borel–Cantelli lemma if we prove the following

LEMMA. *Let $i > i_0(A)$. Consider all the $2^{l_i - k_i}$ choices of* (10). *With the exception of $\frac{1}{i^3} 2^{l_i - k_i}$ such choices, for every z with $|z| = 1$ there is an $r = r(z)$ for which* (11) *holds.*

Put

$$z_u = e^{2\pi i u/(n_{l_i} - n_{k_i})^3}, \quad 0 \leqslant u < (n_{l_i} - n_{k_i})^3.$$

Consider a typical

$$Q_r^{(i)}(z) = \frac{1}{\log i} \sum{}' \varepsilon_j^{(i)} z^{n_{k_i +}}, \tag{12}$$

where the prime indicates that the summation is extended over i consecutive n's. There are 2^i possible choices of the sum (12). It follows from a known result (**2**) that the number of choices of the sum (12) for which

$$|Q_r^{(i)}(z)| \leqslant 2A \tag{13}$$

is less than

$$cA \log i \frac{2^i}{i^{\frac{1}{2}}} < 2^{i-1}. \tag{14}$$

For different values of r the expressions $Q_r^{(i)}(z)$ are of course independent; hence by (14) the number of sums (10) for which (13) holds for all r, $1 \leqslant r \leqslant \left[\frac{l_i - k_i}{i}\right]$, is less than

$$2^{l_i - k_i} \frac{1}{2^{[(l_i - k_i)/i]}} < 2^{(l_i - k_i)(1 - 1/2i)}. \tag{15}$$

(15) implies that, for all but $2^{(l_i-k_i)(1-1/2i)}$ choices of the sum (10),

$$|Q_r^{(i)}(z)| > 2A \tag{16}$$

holds for at least one r, $1 \leqslant r \leqslant \left[\frac{l_i-k_i}{i}\right]$. There are $(n_{l_i}-n_{k_i})^3$ choices for z_u; hence, by (8), for all but

$$(n_{l_i}-n_{k_i})^3 2^{(l_i-k_i)(1-1/2i)} < \frac{1}{i^3} 2^{l_i-k_i} \tag{17}$$

choices of (10), (16) holds for every z_u, $0 \leqslant u < (n_{l_i}-n_{k_i})^3$, for some $r = r(u)$.

To complete the proof of our lemma we have to show that if (16) holds for some u and r then, for every z on the arc (z_{u-1}, z_{u+1}),

$$|Q_r^{(i)}(z)| > A. \tag{18}$$

The proof of (18) is easy. Instead of (18) we show that

$$\left|\frac{Q_r^{(i)}(z)}{z^{n_{k_i}}}\right| > A. \tag{19}$$

$Q_r^{(i)}(z)/z^{n_{k_i}}$ is a polynomial of degree less than $n_{l_i}-n_{k_i}$, each coefficient of which is less than 1; hence its derivative on the unit circle is less than $(n_{l_i}-n_{k_i})^2$. The length of the arc (z_{u-1}, z_{u+1}) is $4\pi/(n_{l_i}-n_{k_i})^3$; hence the variation of $Q_r^{(i)}(z)/z^{n_{k_i}}$ on (z_{u-1}, z_{u+1}) is less than $4\pi/n_{l_i}-n_{k_i}$. Thus (19) follows from (16), and this completes the proof of our lemma and our theorem.

I was unable to extend this result for trigonometric series. The above method seems to give only the following weaker result: let $n_1 < n_2 < \ldots$ be an infinite sequence satisfying $n_k^{1/k} \to 1$; then there exists an everywhere-divergent trigonometric series

$$\sum_{k=1}^{\infty} (a_k \cos n_k x + b_k \sin n_k x)$$

for which $(a_k^2 + b_k^2) \to 0$.

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