

ON CHROMATIC NUMBER OF GRAPHS AND SET-SYSTEMS

By

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Professors R. PÉTER and L. KALMÁR on their 60th birthday

§ 1. Introduction¹

Let α be a cardinal number. A graph \mathcal{G} is said to have *chromatic number* α if α is the least cardinal such that, the set of vertices of \mathcal{G} is the union of α sets, where no two elements of the same set are connected by an edge in \mathcal{G} .

A graph \mathcal{G} is said to have *colouring-number* α , if α is the least cardinal such that the set of vertices of \mathcal{G} has a well-ordering $<$ satisfying the condition that for every vertex x of \mathcal{G} the number of those vertices $y < x$ of \mathcal{G} which are adjacent to x is less than α .

We consider a graph \mathcal{G} as an ordered couple $\langle g, G \rangle$ where g is a set the elements of which are called the vertices of \mathcal{G} , and G is a subset of the set of all unordered pairs of g . The elements of G are called the edges of \mathcal{G} .

As a generalization of graphs we will consider set-systems $\mathcal{H} = \langle h, H \rangle$ where h is a set and H is a set of subsets of h . It is easy to generalize the concepts of chromatic and colouring numbers for general set systems instead of graphs. We mention that several properties of set-systems have been investigated in the literature which can be expressed using the notion of chromatic-number. We do not try to give complete references but we point out one important property which can be expressed using it.

In [11] MILLER defined property **B** of a set of sets. A set H of sets is said to have property **B** if there exists a set B which meets each element of H but does not contain any of them. In [5] we have investigated property **B** in greater detail. It is obvious that a non-empty set of sets H has property **B** iff the corresponding set-system $\mathcal{H} = \langle \cup H, H \rangle$ has chromatic number 2.

Our main aim in this paper is to study the colouring- and chromatic-numbers of (infinite) graphs. The introduction of set-systems has several purposes. First we are going to generalize some easy theorems proved in this paper for general set-systems. Meantime we will state and prove some theorems for set-systems which will serve as lemmas to prove the results for graphs. Finally we will give some generalizations of known theorems concerning finite graphs, for more general finite set-systems.

§ 1. A. A brief summary of the results and the history of the problems

In § 2 we explain the notations and introduce several concepts involving graphs and set-systems.

In § 3 we give the proof of two theorems (the first of which is well known).

¹ For a detailed explanation of notations and terminology used in this paper see § 2.

3. 1 states that the colouring number is not smaller than the chromatic number provided H consists of finite sets of at least two elements. 3. 2 states under the same conditions for H that if \mathcal{H} has colouring number α , then there exists a well-ordering $<$ of h satisfying the condition appearing in the definition of colouring number such that $\text{typ } h = |h|$.

The problem considered in sections 4 and 5 has a long history. TUTTE and independently ZYKOV were the first who proved that for every integer n there exists a graph which does not contain a triangle and has chromatic number $\geq n$ (see [14], [19]). This theorem was independently proved by some other authors. See e. g. [12]. P. ERDŐS and R. RADO generalized the TUTTE—ZYKOV theorem for every infinite cardinal α (see [7]). P. ERDŐS gave a different generalization of the TUTTE—ZYKOV theorem for finite graphs. He proved that for every pair of integers n, s there exists a graph which does not contain circuits of length $\leq s$ and has chromatic number $\geq n$. This proof gives a very good estimation for the minimal number of vertices of a graph of the above property (see [4]).

The question arises whether the ERDŐS—RADO theorem has a similar generalization. That is, does there exist for every infinite cardinal α , and for every integer s a graph \mathcal{G} such that \mathcal{G} has chromatic number $\geq \alpha$ and does not contain circuits of length $\leq s$. ERDŐS' theorem in [4] trivially implies that the answer is positive if $\alpha = \omega$.

A surprising result of this paper is that for $\alpha > \omega$ the answer is no. Corollary 5. 6 implies that if \mathcal{G} does not contain a quadrilateral (or more generally an $[[i, \omega_1]]$ complete even graph for every integer i), then \mathcal{G} has colouring number at most ω .

In § 5 we prove a sequence of theorems of the type that a graph of colouring number $> \alpha$ necessarily contains certain types of subgraphs, mostly large complete even graphs. We construct some counter examples to show that the results are the best possible. The results are summarized at the end of § 5. We obtain the positive results by generalizing a construction of [11]. This is given in § 4.

On the other hand to complete the results concerning the possible generalizations of the TUTTE—ZYKOV, ERDŐS—RADO theorem in an earlier paper [6] we proved that for every α and s there exists a graph \mathcal{G} which has chromatic number $\geq \alpha$ and does not contain circuits of odd length $\leq s$. We did not know whether this result could be improved so that \mathcal{G} has only α vertices for $\alpha \geq \omega$. We give this improvement in § 7 (Theorem 7. 4).

In § 6 we prove some simple lemmas, and state generalizations of theorems of N. G. DE BRUIJN and P. ERDŐS and G. FODOR concerning set-mappings.

In § 7 we consider similar problems as in § 5, and we prove some special results concerning graphs with chromatic- (or colouring-) number $> \omega$. 7. 1 states that every graph of colouring number $> \omega$ contains an infinite path. We also prove an entirely finite graph theorem 7. 6 which states that every graph which does not contain circuits of odd length $\geq 2j+1$, has chromatic number at most $2j$.

In § 13 we come back to the problem of generalizations of the TUTTE—ZYKOV theorem or more precisely of ERDŐS' theorem in [4]. In Theorem 13. 3 we give a direct generalization of ERDŐS' theorem for finite set-systems which consist of k -element sets, after having defined in 13. 2 what we mean by the expression that such a system does not contain "short circuits". Here we use the so-called probabilistic method.

In § 8—11 we consider a problem of an entirely different type. A theorem of N. G. DE BRUIJN and P. ERDŐS [2] states that if k is an integer and every finite subgraph of a graph \mathcal{G} has chromatic number at most k , then \mathcal{G} has chromatic number at most k . (This theorem turns out to be an easy consequence of general "compactness arguments" like Tychonoff's theorem or Gödel's compactness theorem.)

R. RADO pointed out to us a possible analogue of this theorem, namely, that if every finite subgraph of a graph \mathcal{G} has colouring number at most k then \mathcal{G} has colouring number at most k . It is obvious that if this result is true it cannot be expected to be a consequence of the compactness arguments. It turns out that for $k > 2$ the result is false, but a weaker form of it is true.

We prove the following Theorem 9. 1.

If every finite subgraph of a graph \mathcal{G} has colouring number $\leq k$ then \mathcal{G} has colouring number $\leq 2k - 2$.

Theorem 9. 2 shows that this result is the best possible.

More generally we consider the problem involving four cardinals under what conditions for the cardinals $\alpha, \beta, \gamma, \delta$ is the following statement true.

Every graph \mathcal{G} of α vertices, every subgraph of fewer than γ vertices of which has colouring number $\leq \beta$, has colouring number $\leq \delta$. In § 8 we expose the general problem and give some preliminaires.

In § 9 we discuss the case $\gamma = \omega, \beta < \omega$.

In § 10 we try to generalize the negative result 9. 2 for $\gamma \geq \omega$. Here we have only partial results; we will point out Problem 10. 3 which clearly shows the range of unsolved problems.

We mention that the counterexamples given in §§ 9 and 10 are unfortunately rather involved and possibly they can be replaced by much simpler ones.

In § 11 we consider the cases $\beta \geq \omega$. As an easy consequence of the results of § 5 we obtain some positive theorems but we do not know in most cases whether they are the best possible. We point out some simple unsolved problems which seem to be difficult.

Finally in § 12 we turn back to the type of problem considered in § 5 and prove a result of this kind for set-systems $\mathcal{H} = \langle h, H \rangle$ where H consists of subsets of k elements of h , $2 < k < \omega$.

§ 2. Notations, definitions

We are going to employ the usual notations of set theory, but for the convenience of the reader we are going to collect here all the notations and conventions used in this paper which are not entirely self-explanatory.

§ 2. A. General notations, conventions

The letters x, y, z, \dots usually denote arbitrary elements or sets, the letters A, B, C, \dots denote sets. $\subseteq, \in, \circ, \cup, \cap, \cup, \cap$ denote the inclusion, the membership relation, the empty set, and the operations of forming the union or intersection of two sets and of arbitrarily many sets, respectively. Note that if A is a set of sets $\cup A$ denotes the set $\cup_{X \in A} X$. $A \sim B$ denotes the set-theoretical difference of A and B . $\mathcal{P}(A)$ denotes the set of all subsets of A .

$\{x\}$ is the set whose only element is x , $\{x, y\}$ is the unordered pair whose elements are x and y . $\langle x, y \rangle$ is the ordered pair with first term x and second term y .

If $\Phi(x)$ is an arbitrary property of the elements of the set A , $\{x \in A: \Phi(x)\}$ denotes the subset of all elements of A which possess property Φ . In some cases the set A will be omitted from the notation.

If f is a function, we will denote by $\mathcal{D}(f)$ and $\mathcal{R}(f)$ the domain and the range of f respectively. If $x \in \mathcal{D}(f)$ the value of f at x is denoted by $f(x)$ or by f_x . f is considered to be the set $\{\langle x, f(x) \rangle: x \in \mathcal{D}(f)\}$. We denote by ${}^B A$ the set of all functions with $\mathcal{D}(f) = B$, $\mathcal{R}(f) \subseteq A$. If $f \in {}^B A$ and $C \subseteq A$ we denote by $f^{-1}(C)$ the set $\{y \in B: f(y) \in C\}$. If f is a function with $\mathcal{D}(f) = A$, $B \subseteq A$, then $f \upharpoonright B$ denotes the function f restricted to B .

We assume that *ordinals* have been introduced in such a way that every ordinal coincides with the set of all smaller ordinals. The letters $\xi, \zeta, \eta, \varrho, \mu, \nu$ denote ordinals. ω is the least infinite ordinal. We call the finite ordinals integers, the letters $i, j, k, l, m, n, r, s, u, v$ denote integers. $\xi < \zeta$, $\zeta > \xi$ and $\xi \in \zeta$ are equivalent. We will denote by $+$, Σ the addition of ordinal numbers. The difference $\xi \dot{-} \zeta$ of ordinal numbers is defined to that $\xi \dot{-} \zeta = 0$ if $\xi < \zeta$ and $\xi \dot{-} \zeta = \eta$ if $\xi \cong \zeta$ and $\zeta \dot{+} \eta = \xi$.

If for a function f , $\mathcal{D}(f) = \xi$, f will be sometimes called a ξ -*termed sequence*, and its values will be written in the form f_ζ , $\zeta < \xi$. Note that if a ξ -termed sequence is defined by its elements f_ζ , $\zeta < \xi$, f does not necessarily denote the corresponding function.

If A is a set and a is a one-to-one ξ -termed sequence of range A we will briefly say that a is a *well-ordering of type ξ* of A .

By a *cardinal* we mean an initial ordinal. $\alpha, \beta, \gamma, \delta, \varepsilon, \kappa, \tau$ denote (not necessarily infinite) cardinals. Every finite ordinal is a cardinal called integer.

By $+$, Σ , \cdot , Π we denote the usual addition and multiplication of cardinals, respectively. Note that $\alpha + \beta$ is not necessarily equal to $\alpha \dot{+} \beta$, but $\alpha + \beta = \alpha \cup \beta$ if α or β is infinite.

We mention that we use the usual notations for the number-theoretical operations on integers, and that the sign \cdot of the multiplication will be sometimes omitted. In section 13 of this paper where we deal with entirely finite problems, i, j, \dots run over the set of all integers (positive or negative). There naturally we do not assume that every integer is the set of smaller integers. $\mathcal{S}_\alpha(A)$, and $\mathcal{S}_\alpha[A]$ are the sets of all subsets of A of power $< \alpha$ or of power α , respectively, i. e., $\mathcal{S}_\alpha(A) = \{x \in \mathcal{P}(A): |x| < \alpha\}$, $\mathcal{S}_\alpha[A] = \{x \in \mathcal{P}(A): |x| = \alpha\}$. We define the cardinal power α^β by $\alpha^\beta = |\alpha^\beta|$. α^+ denotes the least cardinal greater than α .

If there exists a β such that $\beta^+ = \alpha$ then α^- denotes this β , if such a β does not exist $\alpha^- = \alpha$.

The letter Θ sometimes with subscripts will denote order types.

If A is a set and R a binary relation defined on A which is a simple ordering of A of type Θ we write $\text{typ } A(R) = \Theta$. Simple ordering relations usually will be denoted by $<$ (sometimes with subscripts). $\text{typ } \xi(<)$ will be identified by ξ .

For every $x \in A$, $A \setminus \langle x = \{y \in A: y < x\}$.

When A is a simply ordered set by $<$ the subset $B \subseteq A$ is said to be *confinal* with A if for every $x \in A$ there is a $y \in B$ such that $x \leq y$. If $B, C \subseteq A$ and $x < y$ for every $x \in B, y \in C$ we write $B < C$.

Θ_1 is said to be confinal with Θ_2 if there is a set A simply ordered by $<$ and a $B \subseteq A$ confinal with it such that

$$\text{typ } A(<) = \Theta_1, \quad \text{typ } B(<) = \Theta_2.$$

By this definition the ordinal ζ is confinal with ξ iff there is a sequence $\varphi \in {}^\xi \xi$ such that $\varphi_\eta < \varphi_\rho$ for $\eta < \rho < \xi$ and $\zeta = \bigcup_{\eta < \xi} (\varphi_\eta + 1)$.

For every ζ the *confinality index* $\text{cf}(\zeta)$ is the least ordinal ξ such that ζ is confinal with ξ . For every ζ , $\text{cf}(\zeta)$ is a cardinal $\leq \zeta$ and $\text{cf}(\text{cf}(\zeta)) = \text{cf}(\zeta)$. ζ is a limit ordinal if there is no η such that $\zeta = \eta + 1$. Thus ζ is a non-limit ordinal iff $\text{cf}(\zeta) = 1$.

An infinite cardinal α is said to be a limit cardinal if $\alpha^- = \alpha$, a strong limit cardinal if $\beta < \alpha$ implies $2^\beta < \alpha$, a singular cardinal if $\text{cf}(\alpha) < \alpha$ a regular cardinal if $\text{cf}(\alpha) = \alpha$.

A regular limit cardinal is said to be inaccessible; a regular strong limit cardinal is said to be strongly inaccessible.

We denote the strictly increasing sequence of infinite cardinals by $\omega_0, \omega_1, \dots, \omega_\xi, \dots; \omega_0 = \omega$.

By the continuum hypothesis we mean the hypothesis that $2^\omega = \omega^+$, by the generalized continuum hypothesis we mean the hypothesis that $2^\alpha = \alpha^+$ for every $\alpha \geq \omega$. They will be denoted by C. H. and G. C. H., respectively.

If f is a sequence of sets with $\mathcal{D}(f) = D$ we denote the Cartesian product of the sets f_x for $x \in D$ by $\mathbf{P}_{x \in D} f_x$ i. e.

$$\mathbf{P}_{x \in D} f_x = \{g: \mathcal{D}(g) = D \text{ and } g_x \in f_x \text{ for every } x \in D\}.$$

It is convenient to use a different concept of product if the set $\mathcal{R}(f)$ is disjoint.

We denote by $\mathbf{P}_{x \in D}^* f_x$ the set

$$\{y \in \mathcal{S}(\bigcup (\mathcal{R}(f))) : |y \cap f_x| = 1 \text{ for every } x \in D\}.$$

If especially $D = k$ we use the alternative notation

$$\mathbf{P}_{x \in D}^* f_x = [f_0, \dots, f_{k-1}].$$

§ 2. B. Special notations, graphs, set-systems

By a *graph* \mathcal{G} we mean an ordered pair $\langle g, G \rangle$ where g is an arbitrary set and $G \subseteq S_2[g]$. The elements of g are the vertices of \mathcal{G} , the elements of G are the edges of \mathcal{G} . As a generalization of graphs we are going to consider *set-systems*. By a set system \mathcal{H} we mean an ordered pair $\langle h, H \rangle$ where h is an arbitrary set and $\bigcup H \subseteq h$.

It is obvious that every graph is a set-system. The well-known concepts of colouring and chromatic numbers of a graph as well as many other concepts of graph theory can be generalized for arbitrary set-systems, and many results concerning graphs can be generalized under natural conditions for arbitrary set-systems. Some of the results concerning sets of sets considered in the literature can be easily formulated and generalized using this terminology. In what follows in this section whenever a notation is defined for an arbitrary set-system $\mathcal{H} = \langle h, H \rangle$ we will use the corresponding notation for graphs by interchanging the letters \mathcal{H}, h, H by the letters \mathcal{G}, g, G , respectively.

If there is no danger of misunderstanding we will not always distinguish the set of sets H and the set-system $\mathcal{H} = \langle \cup H, H \rangle$.

DEFINITION 2. 1. We denote by $\alpha(\mathcal{H})$ the cardinal $|h|$.

DEFINITION 2. 2. We say that the set-system \mathcal{H}_1 contains the set-system \mathcal{H}_2 or \mathcal{H}_2 is a sub-set-system of \mathcal{H}_1 if $h_2 \subseteq h_1$, and $H_2 \subseteq H_1$. We briefly write then $\mathcal{H}_2 \subseteq \mathcal{H}_1$.

DEFINITION 2. 3. Let \mathcal{H} be a set-system, and $h' \subseteq h$. By the set-system spanned by h' in \mathcal{H} we mean the set-system $\mathcal{H}(h') = \langle h', H \cap \mathcal{S}(h') \rangle$. If \mathcal{G} is a graph, $g' \subseteq g$, then $\mathcal{G}(g')$ is a graph.

DEFINITION 2. 4. If H is a set of sets, we denote by $\varkappa(H)$ the least cardinal \varkappa for which $|A| \leq \varkappa$ for every $A \in H$. \mathcal{H} is said to be uniform if $|A| = |B|$ for every $A, B \in H$. A set-system \mathcal{H} is a graph iff \mathcal{H} is uniform and $\varkappa(H) = 2$.

DEFINITION 2. 5. Let H be a set of sets. H is said to have property $C(\gamma, \delta)$ if $|\cap H'| < \delta$ for every $H' \subseteq H$, $|H'| \geq \gamma$.

Let f be a sequence of sets. f is said to have property $C(\gamma, \delta)$ if $|\cap_{x \in D'} f_x| < \delta$ for every $D' \subseteq \mathcal{D}(f)$, $|D'| \geq \gamma$.

If for some δ , $|A| \geq \delta$ for every $A \in H$ and H has property $C(2, \delta)$, H is said to be almost disjoint.

DEFINITION 2. 6. Let \mathcal{H} be a set-system. A subset $h' \subseteq h$ is said to be a free set (or independent set) of \mathcal{H} if $A \not\subseteq h'$ for every $A \in H$.

DEFINITION 2. 7. Let \mathcal{H} be a set-system, $x \in h$, and $h' \subseteq h$ we denote by $V(x, h', \mathcal{H})$ the set $\{A \in H: x \in A \text{ and } A \sim \{x\} \subseteq h'\}$ and let

$$v(x, h', H) = \cup V(x, h', \mathcal{H}) \sim \{x\}$$

$$\tau(x, h', \mathcal{H}) = |v(x, h', \mathcal{H})|,$$

$\tau(x, h', \mathcal{H})$ is the valency of x in \mathcal{H} with respect to h' . If $h' = h$ we briefly write $\tau(x, h', \mathcal{H}) = \tau(x, \mathcal{H})$ and $\tau(x, \mathcal{H})$ is the valency of x in \mathcal{H} .

DEFINITION 2. 8. Let \mathcal{H} be a set-system such that $|A| \geq 2$ for every $A \in H$. \mathcal{H} is said to have chromatic number β if β is the smallest cardinal such that h is the sum of β free sets. The chromatic number of \mathcal{H} will be denoted by $\text{Chr}(\mathcal{H})$.

$$\text{Chr}(\mathcal{H}) = 1 \text{ iff } H = 0.$$

In [5] we investigated the property **B** of a set of sets H . H is said to possess property **B** if there exists a set B such that $A \cap B \neq 0$, $A \not\subseteq B$ for every $A \in H$. It is obvious that $\text{Chr}(H) = 2$ iff $H \neq 0$ and H possesses property **B**.

DEFINITION 2. 9. Let \mathcal{H} be a set system, and f a well-ordering of type ξ of the set h . Let $h_\zeta = \{f_\eta: \eta < \zeta\}$ for every $\zeta < \xi$. f is said to be a β -colouring of \mathcal{H} iff $\tau(f_\zeta, h_\zeta, \mathcal{H}) < \beta$ for every $\zeta < \xi$.

Alternatively, if $<$ is a well-ordering of h , $<$ is a β -colouring iff $\tau(x, h|_{< x}, \mathcal{H}) < \beta$ for every $x \in h$.

\mathcal{H} is said to have colouring number β if β is the smallest cardinal number for which \mathcal{H} has a β -colouring. The colouring number of \mathcal{H} is denoted by $\text{Col}(\mathcal{H})$.

\mathcal{H} has colouring number β in type ξ if $\text{Col}(\mathcal{H}) = \beta$ and there is a β -colouring of f of type ξ .

DEFINITION 2. 10. If A is a set and f is a function $f \in {}^A \mathcal{S}(A)$ and $x \notin f(x)$ for every $x \in A$ then f is said to be a *set-mapping* defined on A . A subset $B \subseteq A$ is said to be a *free set* of f if $x \notin f(y)$ and $y \notin f(x)$ for every $x, y \in B$. f is said to be of *order* δ if δ is the smallest cardinal for which $|f(x)| < \delta$ for every $x \in A$.

DEFINITION 2. 11. The graph $\mathcal{G} = \langle g, S_2[g] \rangle$ is called the *complete α -graph* if $\alpha(\mathcal{G}) = \alpha$. It will be denoted by $[[\alpha]]$.

DEFINITION 2. 12. The graph \mathcal{G} is called a *complete α, β -even graph* (or briefly an α, β -graph) if there exist g_0, g_1 such that $|g_0| = \alpha, |g_1| = \beta, g_0 \cap g_1 = 0, g_0 \cup g_1 = g$ and $G = [g_0, g_1]$. It will be briefly denoted by $[[\alpha, \beta]]$.

DEFINITION 2. 13. (i) The graph $\mathcal{G} = \langle g, G \rangle$ is said to be a *path of length i* if there are distinct elements x_0, \dots, x_{i-1} such that

$$g = \{x_0, \dots, x_{i-1}\}, \quad G = \{\{x_0, x_1\}, \dots, \{x_{i-2}, x_{i-1}\}\}.$$

(ii) If $i \geq 3$, $\mathcal{G} = \langle g, G \rangle$ is said to be a *circuit of length i* if there are distinct elements x_0, \dots, x_{i-1} such that

$$g = \{x_0, \dots, x_{i-1}\}, \quad G = \{\{x_0, x_1\}, \dots, \{x_{i-2}, x_{i-1}\}, \{x_{i-1}, x_0\}\}.$$

$\mathcal{P}(x_0, \dots, x_{i-1}), \mathcal{C}(x_0, \dots, x_{i-1})$ will denote paths and circuits of length i , respectively.

DEFINITION 2. 14. The graph $\mathcal{G} = \langle g, G \rangle$ is said to be an *infinite path* if there is a well-ordering f of type ω of the set g such that

$$\mathcal{G} = \{\{f_i, f_{i+1}\}; i < \omega\}.$$

Infinite paths will be briefly denoted by $\mathcal{P}(f)$ or by \mathcal{P}_∞ .

§ 3. Two theorems for chromatic and colouring numbers.

It is well known that $\text{Chr}(\mathcal{G}) \equiv \text{Col}(\mathcal{G})$ for every graph \mathcal{G} . As an easy generalization of this we prove

THEOREM 3. 1. Let \mathcal{H} be a set-system such that $H \subseteq \mathcal{S}_\omega(h)$ and $|A| \geq 2$ for every $A \in H$. Then $\text{Chr}(\mathcal{H}) \equiv \text{Col}(\mathcal{H})$.

PROOF. Put $\text{Col}(\mathcal{H}) = \beta$. Let f be a β -colouring of type ξ of \mathcal{H} . We define a function $\varphi \in {}^\xi \beta$ by induction on $\zeta < \xi$ as follows. Assume φ_η is defined for $\eta < \zeta, \zeta < \xi$. By the assumption and by (2. 7) and (2. 9) the set

$$A = \{\varrho < \beta; \varrho = \varphi_\eta \text{ for some } \eta < \zeta \text{ for which } f_\eta \in v(f_\zeta, h_\zeta, \mathcal{H})\}$$

has power $< \beta$. Let φ_ζ be the smallest ordinal which belongs to $\beta \sim A$.

For every $\varrho < \beta$ let $A_\varrho = \{f_\zeta \in h; \varphi_\zeta = \varrho\}$. The sets A_ϱ are obviously disjoint and their union is h . Assume $X \in H$. Then X is finite, by the assumption. Let f_ζ be

its greatest element in the well-ordering. Considering that X has at least two elements, there is an $\eta < \zeta$ such that $f_\eta \in X$. Then $f_\eta \in v(f_\zeta, h_\zeta, \mathcal{H})$ hence $\varphi_\eta \neq \varphi_\zeta$ and $X \not\subseteq A_\varrho$ for every ϱ . The sets A_ϱ are free. This proves 3. 1.

The condition $H \subseteq \mathcal{S}_\omega(h)$ of 3. 1 is necessary as is shown by the set-system $\mathcal{H} = (\omega, \mathcal{S}(\omega))$, ($\text{Col}(\mathcal{H}) = 1$, $\text{Chr}(\mathcal{H}) = \omega$). This example clearly shows that the concept of colouring number can be useful only for set-systems which consist of finite sets.

THEOREM 3. 2. *Let \mathcal{H} be a set-system, $\alpha(\mathcal{H}) = \alpha$, $\text{Col}(\mathcal{H}) = \beta$. Assume $H \subseteq \mathcal{S}_\omega(h)$. Then \mathcal{H} has a β -colouring of type α .*

3. 2 is trivial if $\alpha \equiv \beta$. Hence it is a corollary of the following

THEOREM 3. 3. *Let \mathcal{H} be a set-system, $\alpha(\mathcal{H}) = \alpha$, $\text{Col}(\mathcal{H}) = \beta$. Assume $H \subseteq \mathcal{S}_\omega(h)$, $\beta < \alpha$. Let f be an arbitrary β -colouring of type ξ of \mathcal{H} . Then there exists a β -colouring f' of type α of \mathcal{H} satisfying the following condition:*

If $f_\zeta = f'_\eta$ then $v(f_\zeta, h_\zeta, \mathcal{H}) = v(f'_\eta, h'_\eta, \mathcal{H})$ for every $\zeta < \xi$, $\eta < \alpha$.

(Here, according to the definition 2. 7, $h_\zeta = \{f_\varrho : \varrho < \zeta\}$, $h'_\eta = \{f'_\varrho : \varrho < \eta\}$ for every $\zeta < \xi$ and $\eta < \alpha$, respectively.)

PROOF. 3. 3 is trivial for $\alpha < \omega$. We assume $\alpha \equiv \omega$.

For every $\zeta < \xi$ and for every i we define $v_i(\zeta)$ and $v(\zeta)$ by induction on i as follows:

$$(1) \quad v_0(\zeta) = v(f_\zeta, h_\zeta, \mathcal{H}) \cup \{f_\zeta\}; \quad v_{i+1}(\zeta) = \bigcup_{f_\eta \in v_i(\zeta)} v(f_\eta, h_\eta, \mathcal{H}) \cup v_i(\zeta)$$

$$v(\zeta) = \bigcup_{i < \omega} v_i(\zeta).$$

It is easy to verify that

$$(2) \quad v(\zeta) = \bigcup_{f_\varrho \in v(f_\zeta, h_\zeta, \mathcal{H})} v(\varrho) \cup \{f_\zeta\} \text{ for every } \zeta < \xi.$$

Let φ be a one-to-one α -termed sequence, with $\mathcal{R}(\varphi) = \xi$. For every $\eta < \alpha$ let

$$(3) \quad A_\eta = v(\varphi_\eta) \sim \bigcup_{\varrho < \eta} v(\varphi_\varrho).$$

Then

$$h = \bigcup_{\eta < \alpha} A_\eta.$$

We define a well-ordering $<$ of h as follows.

Assume

$$(4) \quad f_\zeta, f_{\zeta'} \in h, \quad f_\zeta \in A_\varrho, \quad f_{\zeta'} \in A_\eta,$$

$f_\zeta < f_{\zeta'}$, iff either $\varrho < \eta$ or $\varrho = \eta$ and $\zeta < \zeta'$.

We are going to prove

$$(5) \quad \text{typ } h(<) = \alpha.$$

Considering $\beta < \alpha$ and that by (3) and (4)

$$\text{typ } h(<) = \sum_{\eta < \alpha} \text{typ } f^{-1}(A_\eta)(<)$$

to prove (5) it is sufficient to see that

$$(6) \quad |A_\eta| < \beta^+ \cup \omega \quad \text{for } \eta < \alpha.$$

Considering (3) this follows from

$$(7) \quad |v(\zeta)| < \beta^+ \cup \omega \quad \text{for every } \zeta < \xi.$$

Considering that $|v(f_\zeta, h_\zeta, \mathcal{H})| < \beta$ by the assumption (7) follows by transfinite induction on ζ from (2).

Hence (5), (6) and (7) are true.

By (5) there exists a one-to-one α -termed sequence f' whose range is h satisfying the condition $f'_\varrho < f'_\eta$ for $\varrho < \eta < \alpha$. Let $h'_\eta = \{f'_\varrho : \varrho < \eta\}$ for $\eta < \alpha$.

Assume that $A \in H$ and f_ζ is the element of A with the greatest subscript. Let ϱ be the ordinal for which $f_\zeta \in A_\varrho$. Then $f_\zeta \in v(\varphi_\varrho)$ by (3) and $A \subseteq v(\varphi_\varrho)$ by (1). It follows from (4) that then f_ζ is the greatest element of A in the well-ordering $<$.

Assume now $f'_\eta = f_\zeta$ for some $\eta < \alpha, \zeta < \xi$. By the above remark using $H \subseteq \mathcal{S}_\omega(h)$ it follows from the definition 2.7 that $V(f'_\eta, h'_\eta, \mathcal{H}) = V(f_\zeta, h_\zeta, \mathcal{H})$ and consequently $v(f'_\eta, h'_\eta, \mathcal{H}) = v(f_\zeta, h_\zeta, \mathcal{H})$. Hence f' satisfies the requirements of 3.3.

Note that the condition $\beta < \alpha$ can be replaced by the weaker one that $\beta \equiv \alpha$ and α is regular if $\beta = \alpha$ since the proof of (7) immediately gives that $|v(\zeta)| < \beta \cup \omega$ if β is regular. In case $\alpha = \beta$, α singular the theorem is false. We omit the simple but not entirely trivial proof of this. Condition $H \subseteq \mathcal{S}_\omega(h)$ of 3.3 is necessary as is shown by the following simple example due to E. C. MILNER.

Let $h = \omega_1 + \omega$, $H = \{X \subseteq h : X = h \sim \omega_1 \cup \{y\} \text{ for some } y \in \omega_1\}$. Then obviously $<$ is a 1-colouring of \mathcal{H} of the type $\omega_1 + \omega$, but every well-ordering of type ω_1 of h is a β -colouring of \mathcal{H} only if $\beta > \omega$.

§ 4. Lemmas. Generalization of Miller's inductive construction

First we restate a theorem which we will use later.

LEMMA 4.1 (Theorem of TARSKI). *Let $\mathcal{H} = \langle h, H \rangle$ be a set-system with $\alpha(\mathcal{H}) = \gamma \equiv \omega$. Assume that $|A| \equiv \delta$ for every $A \in H$. Then $|H| \equiv \gamma$ if one of the following conditions (i), (ii) and (iii) holds.*

- (i) *The G. C. H. is true, $\delta \equiv \omega$, H has property $\mathbf{C}(\gamma^+, \delta)$ and $\text{cf}(\gamma) \neq \text{cf}(\delta)$.*
- (ii) *The G. C. H. is true and H has property $\mathbf{C}(\gamma^+, \delta')$ for some $\delta' < \delta$.*
- (iii) *H has property $\mathbf{C}(\gamma^+, \delta')$ for some $\delta' < \omega$.*

These are really corollaries of Theorem 5 I, p. 211 and Corollary 6, p. 213 of TARSKI's paper [13]. Note that the theorems in [13] are stated under the stronger conditions that $\mathbf{C}(2, \delta)$ and $\mathbf{C}(2, \delta')$ hold, respectively, however the proofs give the somewhat stronger results stated in 4.1.

Now we need a generalization of a construction given by E. W. MILLER in [11]. This will be similar to that we have used in [5], but for the convenience of the reader we will give here all the details.

In what follows in this section $\mathcal{G} = \langle g, G \rangle$ will denote a fixed graph, with $\alpha(\mathcal{G}) = \alpha \equiv \omega$ and τ will be a fixed cardinal number > 0 . We remind that by the definition 2.7 for every $x \in g$ and $g' \subseteq g$ $V(x, g', \mathcal{G})$ is the set of edges of \mathcal{G} emanating from

x whose other endpoint is in g' , $v(x, g', \mathcal{G})$ is the set of vertices in g' connected to x in \mathcal{G} , and that $|V(x, g', \mathcal{G})| = |v(x, g', \mathcal{G})| = \tau(x, g', \mathcal{G})$ is the valency of the vertex x in \mathcal{G} with respect to g' .

DEFINITION 4.2. A subset $g' \subseteq g$ is said to be τ -closed in \mathcal{G} if $\tau(x, g', \mathcal{G}) \cong \tau$ implies that $x \in g'$, for every $x \in g$.

Considering that g itself is τ -closed for every τ and that the intersection of any number of τ -closed subsets is again τ -closed, for every $g' \subseteq g$ there exists a minimal τ -closed subset containing g' . This will be called the τ -closure of g' in \mathcal{G} and it will be denoted by $\text{Clos}(g', \mathcal{G}, \tau)$.

Let φ be a well-ordering of type α of g . We are going to define a sequence g_ξ $\xi < \alpha$ of subsets of g by transfinite induction on ξ as follows.

DEFINITION 4.3. Assume g_ζ is defined for every $\zeta < \xi$ for some $\xi < \alpha$. Put $h_\xi = \bigcup_{\zeta < \xi} g_\zeta$. If $h_\xi = g$ put $g_\xi = 0$. If $g \sim h_\xi \neq 0$ let $x_\xi = \varphi_\rho$ for the least ρ for which $\varphi_\rho \in g \sim h_\xi$ and put

$$g_\xi = \text{Clos}(h_\xi \cup \{x_\xi\}, \mathcal{G}, \tau) \sim h_\xi.$$

The following facts are immediate consequences of the definitions.

LEMMA 4.4. (i) $g = \bigcup_{\xi < \alpha} g_\xi$ and the sequence g_ξ , $\xi < \alpha$ is disjointed.

(ii) $g_\xi = h_{\xi+1} \sim h_\xi$ for every $\xi < \alpha$.

(iii) $h_{\xi+1}$ is τ -closed for every $\xi < \alpha$, and as a corollary of this $\tau(x, h_{\xi+1}, \mathcal{G}) < \tau$ for every $\xi < \alpha$, $x \in g_\xi$ provided $\xi < \zeta$.

LEMMA 4.5. Assume $x \in g_\xi$, $\xi < \alpha$. Then

(i) $\tau(x, h_\xi, \mathcal{G}) \cong \tau$ if $\tau \cong \omega$,

(ii) $\tau(x, h_\xi, \mathcal{G}) < \tau$ if $\tau < \omega$.

PROOF. $\tau(x, h_\xi, \mathcal{G}) \cong \bigcup_{\zeta < \xi} \tau(x, h_{\zeta+1}, \mathcal{G})$ by 4.3 and 4.4. $\tau(x, h_{\zeta+1}, \mathcal{G}) < \tau$ for every $\zeta < \xi$ by 4.5. The union of an increasing sequence of cardinals $< \tau$ is $\cong \tau$ if τ is infinite and is $< \tau$ if $\tau < \omega$.

LEMMA 4.6. Assume $g' \subseteq g$, $|g'| \cong \gamma$, $\omega \cup \tau^+ \cong \gamma$. Then

$$|\text{Clos}(g', \mathcal{G}, \tau)| \cong \gamma$$

provided one of the following conditions (i), (ii), (iii) holds.

(i) The G. C. H. is true, \mathcal{G} does not contain a $[[\gamma^+, \tau]]$ complete even graph and $\text{cf}(\gamma) \neq \text{cf}(\tau)$.

(ii) The G. C. H. is true and \mathcal{G} does not contain a $[[\gamma^+, \delta]]$ complete even graph for a $\delta < \tau$.

(iii) \mathcal{G} does not contain a $[[\gamma^+, \delta]]$ complete even graph for a $\delta < \omega$, $\delta < \tau$.

PROOF. Let $\varepsilon = \omega \cup \tau^+$. We define a sequence A_ξ , $\xi < \varepsilon$ of subsets of g by transfinite induction on ξ as follows.

(1) Assume A_ζ is defined for every $\zeta < \xi$ for some $\xi < \varepsilon$. Put $B_\xi = g' \cup \bigcup_{\zeta < \xi} A_\zeta$ and $A_\xi = \{x \in g - B_\xi : \tau(x, B_\xi, \mathcal{G}) \cong \tau\}$.

For every $\xi < \varepsilon$ let $H_\xi = \{v(x, B_\xi, \mathcal{G}) : x \in A_\xi\}$.

As a corollary of Definitions 2.5 and 2.12 we have

(2) If $0 < \delta \leq \tau$ and H_ξ does not possess property $C(\gamma^+, \delta)$ then \mathcal{G} contains a complete even graph $[[\gamma^+, \delta]]$.

We are going to prove by transfinite induction on ξ that

(3) $|B_\xi| \leq \gamma$ for every $\xi \leq \varepsilon$.

Assume (3) is true for every $\zeta < \xi$ for some $\xi \leq \varepsilon$. If ξ is a limit ordinal $B_\xi = g' \cup \bigcup_{\zeta < \xi} B_\zeta$ by (1) hence $|B_\xi| \leq \varepsilon \gamma = \gamma$. If $\xi = \eta + 1$ then $B_\xi = B_\eta \cup A_\eta$. It

follows from (2) that H_η possesses property $C(\gamma^+, \tau)$ if (i) holds and H_η possesses property $C(\gamma^+, \delta)$ if (ii) or (iii) hold. $\bigcup H_\eta \subseteq B_\eta$ and $|B_\eta| \leq \gamma$ by the induction hypothesis. It follows from Lemma 4.1 that $|H_\eta| \leq \gamma$ if one of the conditions (i), (ii), (iii) holds. Using again the conditions (i)...(iii) it follows that $|A_\eta| \leq \gamma$. Hence $|B_\xi| \leq \gamma$ for every $\xi \leq \beta$.

We prove

(4) B_ε is τ -closed.

If $x \in g \sim B_\varepsilon$ then $v(x, B_\varepsilon, \mathcal{G}) = \bigcup_{\xi < \varepsilon} v(x, B_\xi, \mathcal{G})$ and $\tau(x, B_\varepsilon, \mathcal{G}) < \tau$ for every $\xi < \varepsilon$. Considering that $\tau < \varepsilon$ and ε is infinite and regular it follows that there is a $\xi_0 < \varepsilon$ such that $\tau(x, B_\varepsilon, \mathcal{G}) = \tau(x, B_{\xi_0}, \mathcal{G}) < \tau$.

It follows that $\text{Clos}(g', \mathcal{G}, \tau) \subseteq B_\varepsilon$ and thus 4.6 follows from (3).

LEMMA 4.7. Assume $\omega \leq \beta$, $\tau < \beta$ and $\text{Col}(\mathcal{G}(g_\xi)) \leq \beta$ for every $\xi < \alpha$. Then $\text{Col}(\mathcal{G}) \leq \beta$.

For every $\xi < \alpha$ let $<_\xi$ be a β -colouring of the graph $\mathcal{G}(g_\xi)$.

If $x, y \in g$ let $x < y$ if and only if $x \in g_\xi, y \in g_\xi$ and either $\zeta < \xi$ or $\zeta = \xi$ and $x <_\xi y$. By 4.4 (i) $<$ is a well-ordering of g . For every $y \in g_\xi$ $v(y, g|<y, \mathcal{G}) = v(y, h_\xi, \mathcal{G}) \cup v(y, g_\xi|<_\xi y, \mathcal{G}(g_\xi))$. It follows from 4.5 that $|v(y, g|<y, \mathcal{G})| < \tau^+ \cup \beta = \beta$, and so $<$ is a β -colouring of g .

§ 5. Theorems and problems concerning graphs with $\text{Col}(\mathcal{G}) > \beta \cong \omega$

We are going to consider the following problems involving four cardinals $\alpha, \beta, \gamma, \delta$.

Let $\mathcal{G} = \langle g, G \rangle$ be a graph, $\alpha(\mathcal{G}) = \alpha$ with colouring number $> \beta$ (or chromatic number $> \beta$, respectively). Under what conditions for the cardinals $\alpha, \beta, \gamma, \delta$ does then \mathcal{G} necessarily contain a complete even graph $[[\gamma, \delta]]$?

The results of this section are relevant only if $\beta \cong \omega$.

To have a brief notation we introduce the relations

$$\text{Col}(\alpha, \beta, \gamma, \delta), \text{Chr}(\alpha, \beta, \gamma, \delta).$$

DEFINITION 5.1. The relation $\text{Col}(\alpha, \beta, \gamma, \delta)$ is said to hold if for every graph \mathcal{G} $\alpha(\mathcal{G}) = \alpha$, $\text{Col}(\mathcal{G}) > \beta$ implies that \mathcal{G} contains a complete even graph $[[\gamma, \delta]]$.

The relation $\text{Chr}(\alpha, \beta, \gamma, \delta)$ is said to hold if for every graph \mathcal{G} $\alpha(G) = \alpha$, $\text{Chr}(\mathcal{G}) > \beta$ implies that \mathcal{G} contains a complete even graph $[[\gamma, \delta]]$.

LEMMA 5.2. $\text{Col}(\alpha, \beta, \gamma, \delta)$ implies $\text{Chr}(\alpha, \beta, \gamma, \delta)$.

By 3. 1.

As a corollary of the definition we have

LEMMA 5. 3. Both relations introduced in 5. 1 are decreasing in variables α, γ, δ and are increasing in β .

DEFINITION 5. 4. For every infinite cardinal κ we denote by $\alpha(\kappa)$ the smallest cardinal $\alpha > \kappa$ for which $\text{cf}(\alpha) = \text{cf}(\kappa)$ e. g. $\alpha(\kappa) = \omega_\omega$ if $\kappa = \omega$, $\alpha(\kappa) = \omega_{\omega+\omega}$ if $\kappa = \omega_\omega$.

THEOREM 5. 5.² Assume $\beta \cong \omega$. Then $\text{Col}(\alpha, \beta, \beta^+, \delta)$ is true provided one of the following conditions (i), (ii), (iii) holds.

(i) The G. C. H. is true, $\delta^+ = \beta$ and $\alpha \leq \alpha(\delta)$.

(ii) The G. C. H. is true and $\delta^+ < \beta$.

(iii) $\delta < \omega$.

PROOF. We prove the theorem by transfinite induction on α .

If $\alpha(\mathcal{G}) \leq \beta$ then $\text{Col}(\mathcal{G}) \leq \beta$ hence $\text{Col}(\alpha', \beta, \gamma, \delta)$ holds for every $\alpha' \leq \alpha$ and for every γ, δ . Assume that $\alpha > \beta$ and $\text{Col}(\alpha', \beta, \beta^+, \delta)$ holds for every $\alpha' < \alpha$ and let \mathcal{G} be a graph with $\alpha(\mathcal{G}) = \alpha$.

(1) Let $\tau = \delta$ if (i) holds, and let $\tau = \delta^+$ if (ii) or (iii) hold.

We assume

(2) \mathcal{G} does not contain a complete $[[\beta^+, \delta]]$ and using (2) we prove $\text{Col}(\mathcal{G}) = \beta$.

We consider the sets g_ξ defined in 4. 3. We prove

(3) $|g_\xi| < \alpha$ for every $\xi < \alpha$.

To prove (3) we prove by transfinite induction on ξ the following somewhat stronger statement.

(4) $|h_\xi| \leq \beta \cup |\xi|$ for every $\xi < \alpha$.

Assume that (4) is true for every $\zeta < \xi$ for some $\xi < \alpha$. $h_\xi = \bigcup_{\zeta < \xi} g_\zeta$ hence (4)

is trivial if ξ is a limit ordinal. Assume $\xi = \zeta + 1$. Then by 4. 3 and 4. 4 $\text{Clos}(h_\zeta \cup \{x_\zeta\}, \mathcal{G}, \tau) = h_{\zeta+1}$. Then $|h_\zeta \cup \{x_\zeta\}| \leq \beta \cup |\xi|$. Put $\beta \cup |\xi| = \gamma$. Then $\gamma \cong \omega \cup \tau^+$. It follows from 4. 6 (i), (ii), (iii) and (2) that $|h_\xi| \leq \gamma = \beta \cup |\xi|$.

By (3), $\alpha(\mathcal{G}(g_\xi)) < \alpha$ for every ξ . By (2), $\mathcal{G}(g_\xi)$ does not contain a $[[\beta^+, \delta]]$. It follows from the induction hypothesis that $\text{Col}(\mathcal{G}(g_\xi)) \leq \beta$ for every $\xi < \alpha$. Considering that $\beta \cong \omega$ and $\tau < \beta$, by (1), Lemma 4. 7 implies that $\text{Col}(\mathcal{G}) \leq \beta$.

COROLLARY 5. 6. If $\text{Col}(\mathcal{G}) > \omega$ then \mathcal{G} contains an $[[i, \omega_1]]$ graph for every i .

We do not know whether in Theorem 5. 5 condition (i) the assumption $\alpha \leq \alpha(\delta)$ is necessary. The simplest unsolved problem here is

PROBLEM 5. 7. Assume the G. C. H. Is it true that $\text{Col}(\omega_{\omega+1}, \omega_1, \omega_2, \omega)$ or $\text{Chr}(\omega_{\omega+1}, \omega_1, \omega_2, \omega)$ holds, i. e., is it true that every graph with $\alpha(\mathcal{G}) = \omega_{\omega+1}$ which does not contain a $[[\omega_2, \omega]]$ has colouring number $\leq \omega_1$?

THEOREM 5. 8. Assume the G. C. H. is true, and $\beta \cong \omega$. Then $\text{Chr}(\beta^+, \beta, \beta, \beta)$ is not true.

Instead of 5. 8 we are going to prove the following slightly stronger

² It is to be remarked that we originally proved Theorem 5. 5 for the relation $\text{Chr}(\alpha, \beta, \gamma, \delta)$. R. RADO pointed out to us that our proof really gives the stronger result for colouring numbers.

THEOREM 5.9. *Assume the G. C. H. is true and $\beta \cong \omega$. Then there exists a graph \mathcal{G} for which $\alpha(\mathcal{G}) = \beta^+$, $\text{Chr}(\mathcal{G}) = \beta^+$, \mathcal{G} does not contain either a $[[\beta, \beta]]$ graph or an $[[\omega]]$ -graph.*

We do not know whether the condition $[[\omega]] \not\subseteq \mathcal{G}$ can be replaced by the stronger one $[[3]] \not\subseteq \mathcal{G}$, i. e., that \mathcal{G} does not contain a triangle. The simplest unsolved problem is

PROBLEM 5.10. *Assume the C. H. Does there exist a graph with $\alpha(\mathcal{G}) = \omega_1$ such that $\text{Chr}(\mathcal{G}) = \omega_1$, $[[\omega, \omega]] \not\subseteq \mathcal{G}$ and $[[3]] \not\subseteq \mathcal{G}$?*

An affirmative answer to 5.10 would be a consequence of the following assertion: Every graph \mathcal{G} with $\alpha(\mathcal{G}) = \alpha \cong \omega$, $\text{Chr}(\mathcal{G}) = \alpha$ contains a subgraph \mathcal{G}' with $\alpha(\mathcal{G}') = \alpha$, $\text{Chr}(\mathcal{G}') = \alpha$ such that $[[3]] \not\subseteq \mathcal{G}'$.

We do not know whether this assertion is true or false for any infinite α even if we replace $[[3]]$ by $[[k]]$ for some $3 < k < \omega$.

We have some special results on problems of this type which we preserve for later publication.

If we replace the chromatic number by colouring number we can prove the corresponding result without using the G. C. H.

THEOREM 5.11. *Assume $\beta \cong \omega$. Then there exists a graph \mathcal{G} with $\alpha(\mathcal{G}) = \beta^+$, $\text{Col}(\mathcal{G}) = \beta^+$ which does not contain either a $[[\beta, \beta]]$ graph or a circuit of odd length.*

PROOF OF THEOREM 5.9. We define the graph $\mathcal{G} = \langle g, G \rangle$ as follows. Let $g = \beta^+ \times \beta^+$. Let f be a well-ordering of type β^+ of g and let $\varphi, \psi \in \beta^+ \beta^+$ such that

$$f_\xi = \langle \varphi_\xi, \psi_\xi \rangle \text{ for } \xi < \beta^+.$$

Let further $g_\xi = \{\xi\} \times \beta^+$. $g = \bigcup_{\xi < \beta^+} g_\xi$.

(1) Let $H = \{A \subseteq g : \text{There is a set } T(A) \subseteq \beta^+, |T(A)| = \beta \text{ such that } |g_\xi \cap A| = \beta \text{ for every } \xi \in T(A) \text{ and } g_\xi \cap A = 0 \text{ for } \beta^+ \sim T(A)\}$.

Considering $H \subseteq \mathcal{S}_{\beta^+}(\beta^+)$ it follows from G. C. H. that there exists a well-ordering Φ of type β^+ of H .

We are going to define a sequence B_ξ $\xi < \beta^+$ of subsets of g by transfinite induction on ξ satisfying the following conditions.

- (2) (i) $|B_\xi| = \beta$ for $\xi < \beta^+$
 (ii) $|B_\zeta \cap B_\xi| < \beta$ for $\zeta < \xi < \beta^+$
 (iii) $\Phi_\zeta \cap B_\xi \neq 0$ for $\zeta \sim \xi < \beta^+$
 (iv) $|B_\zeta \cap g_\xi| \leq 1$ for every $\zeta < \beta^+$ for $\xi < \beta^+$.

Assume B_ζ is defined in such a way that it satisfies (2) for every $\zeta < \xi$ for some $\xi < \beta^+$. Then B_ξ can be defined using the fact that if F is a set of power $< \beta$ of Φ_ζ 's and B is a set of power $< \beta$ of B_ζ 's already defined and $\Phi_{\zeta_0} \notin F$ then by (1) and (2) there are β η 's belonging to $T(\Phi_{\zeta_0})$ such that $g_\eta \sim \bigcup B$ is non-empty.

Now we define the set of edges of G as follows.

- (3) $\{f_\xi, f_\eta\} \in G$ iff $f_\eta \in B_\xi$, $\varphi_\xi < \varphi_\eta$ and $\psi_\xi > \psi_\eta$.

It follows immediately from (3) that $[[\omega]] \not\subseteq \mathcal{G}$ because if a complete graph is contained in \mathcal{G} then either the first or the second terms of its vertices form a decreasing well-ordered sequence of ordinals.

Assume now $C_1, C_2 \subseteq g$, $C_1 \cap C_2 = 0$, $[C_1, C_2] \subseteq G$, $|C_1| = |C_2| = \beta$. Using the same idea, it follows that there are sets $D_1, D_2 \subseteq g$ such that $|D_1| = |D_2| = \beta$, $[D_1, D_2] \subseteq [C_1, C_2] \subseteq G$, and there is an $\eta < \beta^+$ for which $D_1 \subseteq g_\eta$, $D_2 \subseteq \bigcup_{\rho > \eta} g_\rho$. If $f_\zeta \neq f_\xi \in D_1$ then $D_2 \subseteq B_\zeta \cap B_\xi$ by (3), in contradiction to (2) (ii). It follows that $[[\beta, \beta]] \subseteq \mathcal{G}$.

Assume now that $\text{Chr}(\mathcal{G}) < \beta^+$. Then \mathcal{G} is the sum of fewer than β^+ free sets. Considering that β^+ is regular then there exists a free subset $C \subseteq g$ such that the set

$$T = \{\eta < \beta^+ : |C \cap g_\eta| = \beta^+\}$$

has cardinal β^+ . It follows from (1) that there exists an $\eta \in T$ such that $\Phi_\zeta \subseteq T$, $T(\Phi_\zeta) \subseteq T \sim (\eta + 1)$ for some $\zeta < \beta^+$. Then there exists $\xi > \zeta$ such that $f_\xi \in C \cap g_\eta$ and $\psi_\xi > \psi_\rho$ for every $\rho \in f^{-1}(\Phi_\zeta)$.

Then $B_\xi \cap \Phi_\zeta \neq 0$, by (2) (iii) and $B_\xi \cap \Phi_\zeta \subseteq v(f_\xi, \Phi_\xi, \mathcal{G})$ by (3). This contradicts the assumption that C is a free set, hence $\text{Chr}(\mathcal{G}) = \beta^+$.

\mathcal{G} satisfies the requirements of 5.9.

PROOF OF THEOREM 5.11. We define the graph $\mathcal{G} = \langle g, G \rangle$ as follows.

By a well-known theorem (see, e.g., [13]), there exists a set H of subsets of power β of β such that $|H| = \beta^+$ and H has property **C**(2, β). Put $g = \beta \cup H$. Define $G = \{\{x, y\} \in \mathcal{S}_2[g] : x \in g \text{ and } y \in H \text{ and } x \in y\}$.

It is obvious that $\alpha(\mathcal{G}) = \beta^+$, and that \mathcal{G} contains neither $[[\beta, \beta]]$ graphs nor circuits of odd length. We prove that $\text{Col}(\mathcal{G}) = \beta^+$. Assume that \mathcal{G} has colouring number $\leq \beta$. Then by 3.2 \mathcal{G} has a β -colouring f of type β^+ . Then there is a $\xi < \beta^+$ such that $h_\xi = \{f_\eta : \eta < \xi\}$ contains β . Then $|v(f_\xi, h_\xi, \mathcal{G})| = \beta$ in contradiction to the assumption. Hence $\text{Col}(\mathcal{G}) = \beta^+$.

Using the G. C. H. we can summarize the results of this section concerning the relations $\text{Col}(\alpha, \beta, \gamma, \delta)$ and $\text{Chr}(\alpha, \beta, \gamma, \delta)$ as follows.

If $\alpha \leq \beta$ both relations are trivially true. Hence we assume $\alpha > \beta \cong \omega$. The trivial example of the complete $[[\beta^+]]$ -graph shows that both relations are false if $\text{Max}(\gamma, \delta) > \beta^+$. We assume $\beta^+ \cong \gamma \cong \delta$.

Under these assumptions we distinguish the cases A) $\delta^+ < \beta$, B) $\delta^+ = \beta$, C) $\delta \cong \beta$.

In case A) both relations are true by 5.5 (i).

In case B) both relations are true for $\alpha \leq \alpha(\delta)$ and it is not known whether they are true or false for $\alpha > \alpha(\delta)$ if $\delta \cong \gamma \cong \beta^+ = \delta^{++}$. That means that we do not know the answer to Problem 5.7 even if we replace ω_2 by ω_1 or ω .

In case C) both relations are false by 5.9.

§ 6. Set-mappings, and ordering numbers of graphs

DEFINITION 6.1. Let g be a set and f a set-mapping on g (see Definition 2.10). f is said to have *chromatic number* β if β is the least cardinal such that g is the union of β free sets.

$\mathcal{G} = \langle g, G \rangle$ is said to be the *graph induced by the set-mapping* f defined on g if $\{x, y\} \in G$ iff $x \in f_y$ or $y \in f_x$.

As an immediate consequence of the definitions we have that if \mathcal{G} is the graph induced by the set-mapping then both have the same chromatic number.

DEFINITION 6. 2. A graph \mathcal{G} is said to have *ordering number* β if β is the least cardinal such that there exists a simple ordering $<$ of the set g such that

$$\tau(x, g | <, \mathcal{G}) < \beta \quad \text{for every } x \in g.$$

The ordering number of \mathcal{G} will be denoted by $\text{Ord}(\mathcal{G})$.

We have the following

THEOREM 6. 3. Assume f is a set-mapping of order $\cong \beta$ defined on a set g , and let $G = \langle g, G \rangle$ be the graph induced by f . Then $\text{Col}(G) \cong \beta$ if β is infinite.

6. 3 is a slight generalization of a theorem of G. FODOR [8]. His theorem states the same assertion for chromatic numbers instead of colouring numbers. 6. 3 implies this theorem by 3. 1, but it can be proved using the same ideas. We omit the proof.

THEOREM 6. 4. $\text{Col}(\mathcal{G}) \cong \text{Ord}(\mathcal{G})$ if $\text{Ord}(\mathcal{G})$ is infinite.

PROOF. Put $\text{Ord}(\mathcal{G}) = \beta$. Let $<$ be an ordering of g such that $\tau(x, g | <, \mathcal{G}) < \beta$ for every $x \in g$. Let f be the set-mapping on g defined by the stipulation

$$f(x) = v(x, g | <, \mathcal{G}).$$

Then by the definitions 2. 7 and 6. 1 \mathcal{G} is the graph induced by f and f is of order $\cong \beta$. Hence 6. 4 follows from 6. 3.

A theorem of N. G. DE BRUIJN and P. ERDŐS [2] states that if f is a set-mapping of order $\cong \beta < \omega$ then f has chromatic number $\cong 2\beta - 1$. (FODOR's theorem mentioned is a generalization of this for infinite β 's.) This theorem also has a generalization corresponding to 6. 3.

We have

THEOREM 6. 5. If f is a set-mapping of order $\cong \beta < \omega$ defined on a set g , and $\mathcal{G} = \langle g, G \rangle$ is the graph induced by it then

$$\text{Col}(\mathcal{G}) \cong 2\beta - 1.$$

If \mathcal{G} is finite the proof given in [2] applies, but the proof of the general result becomes more involved. Since we do not need this result in this paper we omit the proof but we mention that it can be carried out quite similarly to the proof of 9. 1.

Similarly to 6. 4 this theorem has the corollary that if $\text{Ord}(\mathcal{G})$ is finite then $\text{Col}(\mathcal{G}) \cong 2 \text{Ord}(\mathcal{G}) - 1$, but this corollary is not best possible because 9. 1 will imply that $\text{Col}(\mathcal{G}) \cong 2 \text{Ord}(\mathcal{G}) - 2$.

THEOREM 6. 6. Let \mathcal{G} be a graph and $\beta < \omega$. If every finite subgraph of \mathcal{G} has ordering number at most β then $\text{Chr}(\mathcal{G}) \cong \beta$.

6. 6 is an easy consequence of TYCHONOFF's compactness theorem. We omit the proof.

§ 7. Some special results and problems concerning graphs
with $\text{Col } (\mathcal{G}) > \omega$ and $\text{Chr } (\mathcal{G}) > \omega$

THEOREM 7. 1. *Every graph of colouring number $> \omega$ contains an infinite path \mathcal{P}_∞ .*

PROOF. It is obvious that if there is a $g' \subseteq g$ such that $g' \neq 0$ and $\tau(x, g', \mathcal{G}) \cong \omega$ for every $x \in g'$, then g contains an infinite path.

We assume that

(1) Every non-empty $g' \subseteq g$ contains an x such that $\tau(x, g', \mathcal{G}) < \omega$.

We define a sequence x_ξ of elements of g by transfinite induction by the stipulation that x_ξ is an element of the set $g_\xi = g \sim \{x_\zeta : \zeta < \xi\}$ satisfying $\tau(x_\xi, g_\xi, \mathcal{G}) < \omega$ whenever g_ξ is non-empty. It is obvious that there is an η such that g_η is empty and thus $g = \{x_\xi : \xi < \eta\}$. Let $x_\xi < x_\zeta$ iff $\xi < \zeta$. Then $\tau(x_\xi, g \setminus \{x_\xi, g\}, \mathcal{G}) = \tau(x_\xi, g_\xi, \mathcal{G})$ hence $\text{Ord } (\mathcal{G}) \cong \omega$ and $\text{Col } (\mathcal{G}) \cong \omega$ as a corollary of 6. 4.

In fact we proved here that if $\text{Col } (\mathcal{G}) > \omega$ then there exists a non-empty subset $g' \subseteq g$ such that $\tau(x, g', \mathcal{G}) \cong \omega$ for every $x \in g'$.³ The same idea gives the following

THEOREM 7. 2. *Assume $\beta \cong \omega$ and $\text{Col } (\mathcal{G}) > \beta$. Then there exists a non-empty subset $g' \subseteq g$ such that $\tau(x, g', \mathcal{G}) \cong \beta$ for every $x \in g'$, and as a corollary of this $|g'| \cong \beta$.*

We omit the proof, but we mention that this result is the best possible, namely we have

THEOREM 7. 3. *For every $\beta \cong \omega$ there exists a graph \mathcal{G} such that $\alpha(\mathcal{G}) = \beta$, $\text{Chr } (\mathcal{G}) = \beta$, and every non-empty subset $g' \subseteq g$ contains an element x such that $\tau(x, g', \mathcal{G}) < \beta$.*

This is shown, e. g., by the following graph. Let $g = \beta$. Let $g_\xi, \xi < \beta$ be a sequence of disjoint subsets of β each with β elements and such that $\bigcup_{\xi < \beta} g_\xi = g$.

Let $G = \{\{\eta, \zeta\} : \eta < \zeta < \beta\}$ and $\eta \in g_\xi, \zeta \in g_{\xi'}$ for $\xi > \xi'$.

It is easy to see that \mathcal{G} satisfies the requirements of 7. 3.

Now we are going to discuss a problem of different type. As a generalization of some results of [4], [7] and [14] the authors proved in [6] that for every $\beta \cong \omega$ and for every integer j there exist graphs \mathcal{G} of chromatic number $\cong \beta$ such that \mathcal{G} does not contain circuits of length $2i+1$ for $1 \leq i \leq j$. As a slight improvement of this we can prove

THEOREM 7. 4. *Assume $\beta \cong \omega$. There exists a graph \mathcal{G} with $\alpha(\mathcal{G}) = \text{Chr } (\mathcal{G}) = \beta$ which does not contain circuits of length $2i+1$ for $1 \leq i \leq j$.*

PROOF (in outline). Let $g = 2^{j^2+1}\beta$. We define the usual lexicographical ordering $<$ of g by the stipulation that if $a, b \in g, a < b$ iff $a_l < b_l$ for the least $l < 2j^2+1$ for which $a_l \neq b_l$.

Put $\{a, b\} \in G$ iff $a < b$ and $a_j < b_0 < a_{j+1} < b_1 < \dots < a_{2j^2} < b_{2j^2-j}$.

This construction is a generalization of the construction given in [7].

It is obvious that $\alpha(\mathcal{G}) = \beta$. Using the same idea as in [7] it is easy to see that $\text{Chr } (\mathcal{G}) = \beta$. To prove that \mathcal{G} does not contain circuits of odd length $\cong 2j+1$ is a matter of easy calculation. We omit the details.

³ The idea of this generalization of 7.1 was suggested to us by J. SABADUSSI.

Corollary 5.6 implies that a graph of colouring number $> \omega$ contains circuits of length $2i$ for every i , and trivial examples show that there are graphs of arbitrarily high colouring number which do not contain odd circuits at all. 7.4 shows that there are graphs of chromatic number ω_1 which do not contain "short" odd circuits. We have

THEOREM 7.5. *Every graph \mathcal{G} with $\text{Chr}(\mathcal{G}) \cong \omega$ contains circuits of length $2i+1$ for infinitely many i .*

7.5 will be a corollary of 7.7.

The following problem remains open.

PROBLEM 7.6. Is it true that for every graph \mathcal{G} of $\text{Chr}(\mathcal{G}) > \omega$ there exists an integer j such that \mathcal{G} contains odd circuits of length $2i+1$ for every $i \geq j$?

We mention that the answer is affirmative under the stronger assumption $\text{Chr}(\mathcal{G}) > \omega_2$. Since this is obviously not a final result we omit the proof. A positive answer to 7.6 would follow e. g. from the following assertion:

Every graph \mathcal{G} with $\alpha(\mathcal{G}) = \text{Chr}(\mathcal{G}) = \omega_1$ contains a subgraph \mathcal{G}' with $\alpha(\mathcal{G}') = \text{Chr}(\mathcal{G}') = \omega_1$ such that \mathcal{G}' is ω -fold connected.⁴

We do not know whether this assertion is true or false. Many similar questions can be asked even if we replace ω_1 by α and ω -fold connected by β -fold connected. Interesting problems of new character arise even if we assume that α, β both are finite but we have very little information on them.

A theorem of [2] already mentioned states that if β is finite and every finite subgraph of a graph \mathcal{G} has chromatic number at most β then \mathcal{G} has chromatic number at most β . Using this theorem 7.5 is a corollary of the following

THEOREM 7.7. *Let \mathcal{G} be a graph, $\alpha(\mathcal{G}) < \omega$ and assume that \mathcal{G} does not contain circuits of length $2i+1$ for $i \geq j$ for some $j < \omega$. Then $\text{Chr}(\mathcal{G}) \leq 2j$.*

7.7 is best possible as is shown by the example of the complete $[[2j]]$ graph.

PROOF (in outline). By a theorem of T. GALLAI [9]⁵ we can assume that the following assertion holds

(1) If $x \neq y \in g$, there exist $\mathcal{P}(x_1, \dots, x_l), \mathcal{P}'(x'_1, \dots, x'_s)$ such that $x_1 = x'_1 = x, x_l = x'_s = y$ l is even, s is odd.

We proceed by induction on $\alpha(\mathcal{G})$. Assume now that $\text{Chr}(\mathcal{G}) > 2j$. By the induction hypothesis this implies that

(2) $\tau(x, \mathcal{G}) \cong 2j$ for every $x \in g$.

Using (2) we first prove that

(3) There exists a circuit $\mathcal{C} \subseteq \mathcal{G}$ of length $\cong 4j$.⁶

Namely, let $\mathcal{P}(x_1, \dots, x_r)$ be a path of maximal length contained in \mathcal{G} . Then $v(x_1, \mathcal{G}) \subseteq \{x_2, \dots, x_r\}$. Let $N = \{i: 2 \leq i \leq r \text{ and } x_i \in v(x_1, \mathcal{G})\}$. Using the assumption that \mathcal{G} does not contain odd circuits of length $> 2j$ we have

(4) Either $i - i'$ is even or $|i - i'| < 2j$, for $i' < i \in N$. Let i_0 be the greatest element of N .

⁴ A graph $g = \langle g, G \rangle$ is said to be β -fold connected if for every $g' \subseteq g, |g \sim g'| < \beta, \mathcal{G}(g')$ is connected.

⁵ See [9], 6.3, p. 16.

⁶ By a theorem of G. DIRAC [3] (2) implies (3). But we give a direct proof of (3).

Considering that $2 \in N$ it follows from (4) that
 (5) $2i+1 \notin N$ for $i \cong j$ and $i_0 - (2i-1) \notin N$ for $i \cong j$.

If $i_0 < 4j$ then $i_0 - (2j-1) < 2j+1$ and then by (5) N contains at most $\frac{i_0}{2} < 2j$ elements, in contradiction to (2). Hence $i_0 \cong 4j$ and (3) is true.

Now let $\mathcal{C}(x_1, \dots, x_{i_0}) \subseteq \mathcal{G}$ be a circuit of length $\cong 4j$. We may assume i_0 is even. It follows easily from (1) that there exists a path $\mathcal{P}(y_1, \dots, y_l) \subseteq \mathcal{G}$ and $1 \cong i_1 < i_2 \cong i_0$ such that

$$y_1 = x_{i_1}, \quad y_l = x_{i_2}, \quad \{x_1, \dots, x_{i_0}\} \cap \{y_2, \dots, y_{l-1}\} = \emptyset$$

and $i_1 - i_2$ and $l-1$ are of different parity. But then either $\mathcal{C}(x_{i_1}, x_{i_1+1}, \dots, x_{i_2}, y_{l-1}, \dots, y_2)$ or $\mathcal{C}(y_1, \dots, y_l, x_{i_2+1}, \dots, x_{i_0}, x_1, \dots, x_{i_1-1})$ is a circuit of \mathcal{G} of odd length $> 2j$.

Finally in this section we are going to mention a problem concerning graphs with $\text{Chr}(\mathcal{G}) \cong \omega$. 7.5 implies that \mathcal{G} contains odd circuits of length i for infinitely many i , and easy examples show that there are graphs \mathcal{G} with $\text{Chr}(\mathcal{G}) = \omega$ such that they do not contain circuits of length $2i$ and $2j+1$ for infinitely many i and j . One can generally ask that what can be said of the set of those integers for which \mathcal{G} contains a circuit. We can not solve the following simple problem:

PROBLEM 7.8. Let \mathcal{G} be a graph, $\text{Chr}(\mathcal{G}) = \omega$. Let $N = \{i: \text{there is a circuit } \mathcal{C} \text{ of length } i \text{ such that } \mathcal{C} \subseteq \mathcal{G}\}$. Is it true that then

$$\sum_{i \in N} \frac{1}{i} = \infty?$$

§ 8. The problem of Rado

DEFINITION 8.1. The graph \mathcal{G} is said to possess property $\mathbf{D}(\beta, \gamma)$ if every subgraph of \mathcal{G} induced by a subset $g' \subseteq g$ of power $< \gamma$ has colouring number $\cong \beta$.

DEFINITION 8.2. The relation $R(\alpha, \beta, \gamma, \delta)$ is said to hold if every graph \mathcal{G} of $\alpha(\mathcal{G}) = \alpha$ which possesses property $\mathbf{D}(\beta, \gamma)$ has colouring number $\cong \delta$.

The problem suggested by R. RADO mentioned in the introduction was whether

$$R(\alpha, \beta, \omega, \beta) \text{ holds for every } \alpha \text{ and for } \beta < \omega.$$

We prove that the answer is negative but we also prove some positive results. First we state some preliminary results.

- THEOREM 8.3.**
- (i) $\alpha \cong \delta$ implies $R(\alpha, \beta, \gamma, \delta)$.
 - (ii) If $\gamma \cong \beta^+$ then $R(\alpha, \beta, \gamma, \delta)$ holds iff $\alpha \cong \delta$.
 - (iii) If $\beta = 0$, $\gamma > 1$ then $R(\alpha, \beta, \gamma, \delta)$ is true.
 - (iv) If $\beta = 1$ and $\gamma > 2$ then $R(\alpha, \beta, \gamma, \delta)$ is true for every $\delta \cong 1$.
 - (v) If $\alpha < \gamma$ then $R(\alpha, \beta, \gamma, \delta)$ iff $\beta \cong \delta$ or $\alpha \cong \delta$.

(i) and (ii) follow from the fact that $\text{Col}(\mathcal{G}) \cong \alpha(\mathcal{G})$ for every graph \mathcal{G} . (iii) is true because $\text{Col}(\mathcal{G}) > 0$ for every non-empty \mathcal{G} . (iv) is true because if \mathcal{G} has property $\mathcal{D}(1, 3)$ then it has no edges. (v) follows immediately from 8.1 and 8.2.

In view of 8. 3 we will usually assume $\alpha \cong \gamma$, $\alpha > \delta$, $\gamma > \beta^+$ $\beta \cong 2$ and it is obvious that under these conditions positive results can be expected only if $\beta \cong \delta$.

In what follows we will consider the case of infinite graphs, i. e., $\alpha \cong \omega$.

We mention that if we assume that $\alpha < \omega$ then interesting problems of finite graph theoretical type arise but we do not investigate them in this paper.⁷ The cases we are going to investigate in greater detail are β is finite and γ is infinite.

We need some further definitions concerning graphs and colouring numbers.

DEFINITION 8. 4. Let $\mathcal{G} = \langle g, G \rangle$ be a graph and let $t(x)$ be a function on g such that $t(x)$ is a cardinal number $\cong 1$ for every $x \in g$. Then t is said to be a colouring function of \mathcal{G} .

Let $<$ be a simple ordering of g . We say that $<$ satisfies the colouring function t of \mathcal{G} if $\tau(x, g | <, \mathcal{G}) < t(x)$ for every $x \in g$.

By 2. 9 and 8. 4 we have

THEOREM 8. 5. \mathcal{G} has colouring number $\cong \delta$ iff there is a colouring function t of \mathcal{G} and a well-ordering $<$ of g such that $<$ satisfies t and $t_x \cong \delta$ for every $x \in g$.

THEOREM 8. 6. Let \mathcal{G} be a graph ($\alpha(\mathcal{G}) = \alpha$) and let t be a colouring function of \mathcal{G} satisfying $t_x \cong \delta$, for some δ where $\delta \cong \alpha$ and $\delta < \alpha$ if α is singular. Assume g has a well-ordering $<$ satisfying t . Then g has a well-ordering $<'$ satisfying t such that

$$\text{typ } g(<') = \alpha.$$

PROOF. 8. 6 follows from 3. 3 if $\delta < \alpha$. If $\delta = \alpha$ and α is regular it follows from the remark made after the proof of 3. 7.

DEFINITION 8. 7. Let g be a set and $a \in \xi \mathcal{S}(g)$. We call a a disjoint partition of type ξ of g if $\mathcal{R}(a)$ is disjointed and $\bigcup \mathcal{R}(a) = g$.

Let $\mathcal{G} = \langle g, G \rangle$ be a graph and a a disjoint partition of g . Let $t(x)$ be a colouring function of G , and let $<$ be a simple ordering of g . We say that $<$ satisfies t with respect to a if $<$ satisfies t and $a_\eta < a_\zeta$ for every $\eta < \zeta < \xi$.

LEMMA 8. 8. Let \mathcal{G} , a and t have the same meaning as in 8. 7. Let \mathcal{G}_ζ denote the graph induced by a_ζ and let $b_\zeta = \bigcup_{\eta < \zeta} a_\eta$ for $\zeta < \xi$.

Then there is a simple ordering $<$ (well-ordering $<$) of g satisfying t with respect to a iff the following conditions (i) and (ii) hold.

(i) There exist ordering functions t_ζ of \mathcal{G}_ζ and simple ordering relations $<_\zeta$ (well-ordering relations $<_\zeta$) of a_ζ such that $<_\zeta$ satisfies t_ζ on G_ζ for every $\zeta < \xi$, respectively.

(ii) $\tau(x, b_\zeta, \mathcal{G}) + \varepsilon < t(x)$ for every $\varepsilon < t_\zeta(x)$ for every $x \in a_\zeta$ and for every $\zeta < \xi$.

(ii) is equivalent to the condition

(iii) $\tau(x, b_\zeta, \mathcal{G}) < t(x)$, and $t_\zeta(x) \cong t(x) - \tau(x, b_\zeta, \mathcal{G})$ if $t(x)$ is finite $t_\zeta(x) \cong t(x)$ if $t(x)$ infinite.

As a corollary of this the following condition is sufficient for the existence of a simple ordering $<$ (well-ordering $<$) of \mathcal{G} satisfying $t(x)$ with respect to a .

⁷ In 11. 5 we define the property $P(\alpha, \beta, \gamma, \delta)$ strongly related to $R(\alpha, \beta, \gamma, \delta)$. In [15] there are several results concerning $P(\alpha, \beta, \gamma, \delta)$ for finite α . Results of similar type might be expected for $R(\alpha, \beta, \gamma, \delta)$ too.

(iv) For every $\zeta < \xi$ there exists a colouring function t_ζ of $G(a_\zeta \cup b_\zeta)$ satisfying $t_\zeta(x) \leq t(x)$ for $x \in a_\zeta$, and a well-ordering $<_\zeta$ of $a_\zeta \cup b_\zeta$ satisfying t_ζ , and $b_\zeta <_\zeta a_\zeta$.

PROOF. By 8. 4 and 8. 7.

8. 8 is an immediate generalization of 4. 7.

§ 9. The relation $R(\alpha, \beta, \gamma, \delta)$ in case $\beta < \omega, \gamma = \omega$

THEOREM 9. 1. $R(\alpha, \beta, \omega, 2\beta - 2)$ for every $2 \leq \beta < \omega$, and for every α .

THEOREM 9. 2. $R(\omega, \beta, \omega, 2\beta - 3)$ is false if $2 \leq \beta < \omega$.

9. 1 and 9. 2 show that the answer to R. RADO's problem is affirmative iff $\beta = 2$.

We postpone the proof of the theorems to pp. 82, 85, respectively. First we need some lemmas.

LEMMA 9. 3. Let $\mathcal{G} = \langle g, G \rangle$ be a graph and t a colouring function of it, $t \in {}^g\omega$. Assume that every $g' \in \mathcal{S}_\omega(g)$ has a simple ordering which satisfies the colouring function $t \upharpoonright g'$ of $\mathcal{G}(g')$. Then g has a simple ordering $<$ satisfying t .

9. 3 is a generalization of 6. 5. The proof is an easy application of Tychonoff's compactness theorem. We omit it.

Note that the compactness theorem obviously does not imply that the simple ordering $<$ is a well-ordering. This leads to the phenomenon shown by 9. 1 and 9. 2.

LEMMA 9. 4. Let $\mathcal{G} = \langle g, G \rangle$ be a graph, $\alpha(\mathcal{G}) = \omega$, and t a colouring function of it, $t \in {}^g\omega$. Assume that every $a \in \mathcal{S}_\omega(g)$ has an ordering $<_a$ which satisfies $t \upharpoonright a$ on $\mathcal{G}(a)$. Put $s(x) = \max(2t(x) - 2, t(x))$. Then there exists a well-ordering $<$ of g satisfying $s(x)$.

PROOF. We will prove

(1) For every $a \in \mathcal{S}_\omega(g)$ there exists a set a' satisfying the following conditions:

(i) $a' \in \mathcal{S}_\omega(g)$

(ii) for every $b \in \mathcal{S}_\omega(g \sim a')$, the graph $a' \cup b$ has a simple ordering $<$ which satisfies $s \upharpoonright (a' \cup b)$ on the graph $\mathcal{G}(a' \cup b)$ with respect to the partition whose first member is a' and the second member is b .

To prove (1) we need the following

(2) Assume $a \in \mathcal{S}_\omega(g)$. Then either $a' = a$ satisfies the requirements of (1), or there exists a set $b \in \mathcal{S}_\omega(g)$ satisfying the following conditions

(i) $\tau(x, a \cup b, \mathcal{G}) \cong s(x)$ for every $x \in b$

(ii) there is an $x \in b$ such that $\tau(x, a, \mathcal{G}) \neq 0$.

To prove (2) assume that $a' = a$ does not satisfy the requirements of (1). Then $a' = a$ does not satisfy (ii) of (1). Let b be a set with minimal number of elements such that $a \cup b$ has no ordering which satisfies $s \upharpoonright (a \cup b)$ with respect to the partition with first member a , second member b . We show that this b satisfies the requirements of (2).

Assume $x \in b$, $\tau(x, a \cup b, \mathcal{G}) < s(x)$. Then by the minimality of b , $b \sim \{x\}$ satisfies (1) (ii) with $a = a'$ and so $a \cup b \sim \{x\}$ has an ordering satisfying $s(x)$ with respect to the partition $a \cup (b \sim \{x\})$ and then this ordering can be extended to an ordering

of $a \cup b$ by the stipulation that x is the last element of it and the new ordering so obtained satisfies $s \uparrow a \cup b$ on $\mathcal{G}(a \cup b)$ with respect to the partition $a \cup b$. This is a contradiction, hence (2) (i) is satisfied.

Assume now (2) (ii) is false. Then by the assumption both a and b have orderings \prec_a and \prec_b satisfying $t \uparrow a$ and $t \uparrow b$ of $\mathcal{G}(a)$ and $\mathcal{G}(b)$, respectively. Extending these simple orderings to an ordering \prec of $a \cup b$, by the stipulation $a \prec b$, the ordering \prec would satisfy $t \uparrow a \cup b$ with respect to the partition $a \cup b$. Considering that $t_x \leq s_x$ for every $x \in g$ this contradicts the definition of b , hence (2) (ii) is true and (2) is proved.

Now we prove

(3) Assume $a \in \mathcal{S}_\omega(g)$. Let $e(a)$ denote the number of edges of $\mathcal{G}(a)$. Then

$$e(a) \cong \sum_{x \in a} (t(x) - 1).$$

By the assumption there exists an ordering \prec_a of a satisfying $t \uparrow a$ of $\mathcal{G}(a)$. We have

$$e(a) = \sum_{x \in a} \tau(x, a | \prec_x, \mathcal{G}) \cong \sum_{x \in a} t(x) - 1.$$

We prove (1). Let $a \in \mathcal{S}_\omega(g)$ and assume that there is no a' satisfying the requirements of (1). We define the sequences a_i, b_i by induction on i simultaneously.

Put $a_0 = a$, assume that $a_i, i \geq 0$ and $b_{i-1}, i \geq 1$ are already defined in such a way that $a \subseteq a_i$. If $a'_i = a_i$ satisfies the requirements of (1) for a_i then $a_i = a'$ satisfies (1) as well. Thus, by (2), there exists a set b_i satisfying the requirements of (2) with $a_i = a, b_i = b$. Put $a_{i+1} = a_i \cup b_i$. Then $a \subseteq a_{i+1}$ and a_i and b_i are defined for every $i < \omega$.

We need an estimation of $e(a_{i+1})$. We prove that $e(a_{i+1}) \cong e(a_i) + \sum_{x \in b_i} (t(x) - 1) + 1$. In fact it follows from (2) (i) and (ii) that

$$e(a_{i+1}) \cong e(a_i) + 1 + \frac{1}{2} \left(\left(\sum_{x \in b_i} s(x) \right) - 1 \right)$$

and considering that $e(a_{i+1})$ is an integer this gives the result.

It follows by induction on i that

$$e(a_{i+1}) \cong \sum_{j=0}^i \sum_{x \in b_j} (t(x) - 1) + i.$$

Put

$$i_0 = \sum_{x \in a} (t(x) - 1).$$

Then

$$e(a_{i+1}) \cong \sum_{x \in a_{i+1}} (t(x) - 1) + i - i_0,$$

hence for $i > i_0$

$$e(a_{i+1}) > \sum_{x \in a_{i+1}} (t(x) - 1).$$

This contradicts (3), hence (1) is true.

Let $y \in {}^\omega g$ a well-ordering of g . For every $a \in \mathcal{S}_\omega(g)$ a' denotes a set satisfying the requirements of (1). We define the sequences a_i, b_i, c_i by induction on i , simultaneously. Assume a_j is defined for every $j < i$, and $a_j \in \mathcal{S}_\omega(g)$. Put $b_i = \bigcup_{j < i} a_j$,

$c_i = b_i \cup \{y(k_i)\}$ where k_i is the least integer k for which $x(k) \notin b_i$, and let $a_i = c_i \sim b_i$. Then by (1) $a_i \in \mathcal{S}_\omega(g)$ and a_i, b_i, c_i are defined for every i .

We have

- (4) $\{a_i\}_{i < \omega}$ is a disjoint partition of type ω of g , and $a_i = b_{i+1} \sim b_i$ for every i . Considering that $b_0 = 0$, and $b_{i+1} = c'_i$, $b'_i = b_i$ for every i . It follows from (1) (ii) that the set b_{i+1} has an ordering $<_i$ satisfying $s \uparrow b_{i+1}$ with respect to the partition $b_i \cup a_i$. As a consequence of 8.8 (iv) it follows that g has a well-ordering satisfying $s(x)$. This proves 9.4.

THEOREM 9.5. *Let $\mathcal{G} = \langle g, G \rangle$ be a graph and t a colouring function of it such that $t \in {}^g\omega$. Assume that for every $a \in \mathcal{S}_\omega(g)$ a has an ordering $<_a$ satisfying $t \uparrow a$ of $\mathcal{G}(a)$. Put $s(x) = \max(2t(x) - 2, t(x))$, for $x \in g$.*

Then there exists a well-ordering $<$ of g satisfying $s(x)$.

REMARK. Theorem 9.1 follows from 9.5, applying 9.5 for the colouring function $t(x) = \beta$, and considering that $s(x) = 2t(x) - 2$ if $t(x) \geq 2$.

PROOF. By 9.4 the theorem is true if $\alpha(\mathcal{G}) \leq \omega$. We proceed by induction on $\alpha(\mathcal{G})$. Put $\alpha(\mathcal{G}) = \alpha > \omega$ and assume that the theorem is true for every graph \mathcal{G}' with $\alpha(\mathcal{G}') < \alpha$. By the assumptions and by 9.3 there exists a simple ordering $<^*$ of g satisfying t on \mathcal{G} .

First we prove

- (1) For every $A \subseteq g$ there is a set $B \subseteq g$ satisfying the conditions

$$(i) A \subseteq B \quad (ii) |B| \leq |A| \cdot \omega \quad (iii) v(x, g | <^* x, \mathcal{G}) \subseteq B$$

for every $x \in B$.

To prove (1) we define a sequence A_i $i < \omega$ by induction on i as follows:

$$A_0 = A, \quad A_{i+1} = \bigcup_{x \in A_i} v(x, g | <^* x, \mathcal{G}) \cup A_i.$$

Considering that $<^*$ satisfies t , it follows that $|A_{i+1}| \leq |A_i| \cdot \omega$, and that $B = \bigcup_{i < \omega} A_i$ satisfies the requirements of (1).

Now let $\mathcal{B} \in {}^g\mathcal{S}(g)$ such that $\mathcal{B}(A) = B$ satisfies the requirements of (1) for A .

Let $x \in {}^g g$ be a well-ordering of g .

We define $C, D \in {}^g\mathcal{S}(g)$ by transfinite induction on $\xi < \alpha$ simultaneously.

- (2) Assume C_ζ is defined for every $\zeta < \xi$ for some $\xi < \alpha$. Put $\bigcup_{\zeta < \xi} C_\zeta = D_\xi$. If $g \sim D_\xi = 0$, put $C_\xi = 0$. If $g \sim D_\xi \neq 0$, let $y_\xi = x_\eta$ for the least η for which $x_\eta \in g \sim D_\xi$ and let $C_\xi = \mathcal{B}(D_\xi \cup \{y_\xi\}) \sim D_\xi$.

We have

- (3) C is a disjoint partition of type α of g .
 (4) If $x \in D_\xi$ then $v(x, g | <^* x, \mathcal{G}) \subseteq D_\xi$ for $\xi < \alpha$.
 (3) follows immediately from (2). If $x \in D_\xi$ then $x \in C_\zeta$ for some $\zeta < \xi$. $D_\xi \cup C_\zeta = \mathcal{B}(D_\xi \cup \{y_\zeta\})$ by (2), hence $v(x, g | <^* x, \mathcal{G}) \subseteq D_\xi \cup C_\zeta \subseteq D_\xi$ by (i) and (iii).

We prove by induction on ξ

- (5) $|C_\xi| \leq |\xi| \cdot \omega$.

Assume (5) is true for every $\zeta < \xi$.

Then $|D_\xi| \leq \sum_{\zeta < \xi} \zeta \cdot \omega = \xi \cdot \omega$. $|D_\xi \cup \{y_\xi\}| = \xi \cdot \omega$ and then by (i), (ii) and (2)

we have $|C_\xi| \leq \xi \cdot \omega$.

(6) $v(x, D_\xi, \mathcal{G}) \subseteq v(x, g | \prec^* x, \mathcal{G})$ for every $x \in C_\xi$.

Assume $y \in v(x, D_\xi, \mathcal{G})$. Then $\{x, y\} \in G$, and $y \prec^* x$. For if not then $x \prec^* y$ and $x \in v(y, g | \prec^* y, \mathcal{G})$ and $x \in D_\xi$ by (4). Hence $y \in v(x, g | \prec^*, \mathcal{G})$.

Considering that \prec^* satisfies t it follows from (6) that

(7) $t_\xi(x) = t(x) - \tau(x, D_\xi, \mathcal{G}) > 0$ for every $x \in C_\xi$, $\xi < \alpha$ hence t_ξ is a colouring function of $\mathcal{G}(C_\xi)$ for $\xi < \alpha$.

We prove

(8) If $a \in \mathcal{S}_\omega(C_\xi)$ then \prec^* satisfies $t_\xi \uparrow a$ on $\mathcal{G}(a)$. If $y \in v(x, a | \prec^* x, \mathcal{G}(a))$ then $y \in v(x, g | \prec^* x, \mathcal{G})$ and $y \notin v(x, D_\xi, \mathcal{G})$. Considering (6) it follows that

$$\tau(x, a | \prec^* x, \mathcal{G}) \subseteq \tau(x, g | \prec^* x, \mathcal{G}) - \tau(x, D_\xi, \mathcal{G}) < t_\xi(x).$$

Put $s_\xi(x) = \max(2t_\xi(x) - 2, t_\xi(x))$ for $x \in C_\xi$ and for $\xi < \alpha$. By (5) we have $\alpha(\mathcal{G}(C_\xi)) < \alpha$ for $\xi < \alpha$. By (7), (8), and by the induction hypothesis it follows that

(9) There exist well-orderings \prec_ξ of the sets C_ξ satisfying s_ξ on $\mathcal{G}(C_\xi)$, for $\xi < \alpha$.

By 8. 8, (3) and (9) there exists a well-ordering \prec of g satisfying $s_\xi(x) + \tau(x, D_\xi, \mathcal{G})$ on \mathcal{G} for every $x \in C_\xi$ and for every $\xi < \alpha$.

On the other hand $s_\xi(x) + \tau(x, D_\xi, \mathcal{G}) \subseteq s(x)$ for every x because

$$\max(2(t \sim \tau) - 2, t - \tau) + \tau \subseteq \max(2t - 2, t)$$

whenever $t - \tau > 0$, $\tau \geq 0$.

Hence the well-ordering \prec satisfies s on \mathcal{G} and 9. 5 is proved.

For the construction of a graph satisfying the conditions of 9. 2 as well as for some more complicated counter-examples we need the following

LEMMA 9. 6. Let $\mathcal{G} = \langle g, G \rangle$ be a graph, $B \in \omega(S_\omega(g) \sim \{0\})$, $A \subseteq g$ where $\{A\} \cup R(B)$ is disjoint. Let $t \in \omega$ be a colouring function of \mathcal{G} . Put $C_i = A \cup \bigcup_{j < i} B_j$. Assume that there is an i_0 such that

$$\tau(x, C_{i+1}, \mathcal{G}) \subseteq \tau(x) \text{ for every } x \in B_i \text{ for } i \geq i_0.$$

Then no well-ordering \prec of g satisfies $t(x)$ and the condition

$$A < \bigcup_{i < \omega} B_i.$$

As a corollary of this, if A is finite, there is no well-ordering \prec of g satisfying t , on \mathcal{G} .

PROOF. Let $\tau(x)$ briefly denote $\tau(x, A, \mathcal{G})$ for every $x \in \bigcup_{i < \omega} B_i$. Assume that the theorem is false, and there is a well-ordering \prec of g satisfying t on \mathcal{G} . Put $b = \bigcup_{i < \omega} B_i$.

Then $|b| = \omega$ by the assumption. It follows from 8. 8 that \prec satisfies $t(x) - \tau(x)$ on $\mathcal{G}(b)$. Then by 8. 6 there is a well-ordering \prec' of b satisfying $t(x) - \tau(x)$ on $\mathcal{G}(b)$ such that $\text{typ } b(\prec') = \omega$.

Let $b_i = \max_{\prec'}(B_i)$, $c_0 = \max_{\prec'}(C_{i_0} \sim A)$.

We prove by induction on i that $b_i \subseteq' c_0$. Assume that this is true for every $j < i$. Then $\max_{\prec'}(C_i \sim A)$ is either b_i or c_0 . If $b_i = \max_{\prec'}(C_i \sim A)$ then $\tau(b_i, C_i \sim A, \mathcal{G}) = \tau(b_i, b | \prec' b_i, \mathcal{G}) < t(b_i) - \tau(b_i)$ in contradiction to the assumption. Hence $b_i \subseteq' c_0$ for every i . This contradicts $\text{typ } b(\prec') = \omega$ and proves the first part of the theorem.

Assume that the corollary is not true. Then, by 3.1, $A \cup b$ has a well-ordering \prec of type ω satisfying $t(x)$.

Put $a' = \max_{\prec}(A)$, $A' = A \cup b | \leq a'$, $B'_i = B_i \sim A'$, $C'_i = \bigcup_{j < i} B'_j \cup A$. Then for some $i \geq i_0$, $B_i = B'_i$, $C_i = C'_i$ and a contradiction follows from the first part of the theorem which has already been proved.

DEFINITION 9.7. We define the graphs $\mathcal{G}(k, l) = \langle g(k, l), G(k, l) \rangle$ for $l \geq 3$ if $k = 2$ and for $l \geq 2$ if $k \geq 3$.

(i) $g(k, l) = \omega$.

We define $B(k, l)$ as a disjoint partition of type ω of ω .

(ii) $j \in B_i(k, l)$ if $j = l \binom{k}{2} i + s$, $0 \leq s < l \binom{k}{2}$.

We define the set of edges $G(2, l)$ for $l \geq 3$ as follows

(iii) $\{j, j+1\} \in G(2, l)$, $\{li, li+l-1\} \in G(2, l)$.

We define the set of edges $G(k, l)$ for $k \geq 3$, $l \geq 2$. First we define a partition of type $l-1$ of each $B_i(k, l)$.

(iv) Assume $j \in B_i(k, l)$, $j = l \binom{k}{2} i + s$, $0 \leq s < l \binom{k}{2}$.

$$j \in B_{i, r}(k, l) \quad \text{for } 0 \leq r < l-2 \quad \text{if } r \binom{k}{2} \leq s < (r+1) \binom{k}{2}.$$

$$j \in B_{i, l-2}(k, l) \quad \text{if } (l-2) \binom{k}{2} \leq s < l \binom{k}{2}.$$

(v) $\{jj'\} \in G(k, l)$ if $j, j' \in B_{i, r}(k, l)$ for some i and $r < l-1$ and $j < j'$, $j' - j \leq k-1$.

$$\left\{ l \binom{k}{2} i + s, l \binom{k}{2} i + \binom{k}{2} + s \right\} \in G(k, l) \quad \text{for } (l-2) \binom{k}{2} \leq s < (l-1) \binom{k}{2}.$$

If $p \neq li-1$ for some i , $p > 0$, then for every $0 \leq w < k-1$

$$\left\{ (p-1) \binom{k}{2} + v, p \binom{k}{2} + w \right\} \in G(k, l) \quad \text{for } w(k-1) - \binom{w}{2} \leq v < (w+1)(k-1) - \binom{w+1}{2}.$$

In the next lemma we are going to collect all the consequences of the above construction which we are going to use later.

LEMMA 9.8. (i) $\alpha(\mathcal{G}(k, l)) = \omega$.

(ii) $\bigcup_{i < \omega} B_i(k, l) = \omega$, and $B_i(k, l) = \bigcup_{r < l-1} B_{i, r}(k, l)$ for $k \geq 3$.

$$B_i(k, l) < B_{i'}(k, l) \quad \text{for } i < i'.$$

$$B_{i, r}(k, l) < B_{i, r'}(k, l) \quad \text{for } r < r'.$$

Assume $j = l \binom{k}{2} i + s$, $0 \leq s < l \binom{k}{2}$ i. e. $j \in B_i(k, l)$

(iii) For $l \geq 3$ we have

$$\tau(j, j, \mathcal{G}(2, l)) = \begin{cases} 0 & \text{if } j = 0 \\ 1 & \text{if } j > 0, \quad s \neq l-1 \\ 2 & \text{if } s = l-1. \end{cases}$$

(iv) For $l \geq 3$ we have

$$\tau(j, \omega \sim j, \mathcal{G}(2, l)) = \begin{cases} 1 & \text{if } s > 0 \\ 2 & \text{if } s = 0. \end{cases}$$

(v) For $l \geq 3, i > 0$

$$\tau(j, B_{i-1}(k, l) \cup B_i(k, l), \mathcal{G}(2, l)) = \begin{cases} 2 & \text{if } s > 0 \\ 3 & \text{if } s = 0 \end{cases}$$

$$\tau(j, B_i, \mathcal{G}(2, l)) = 2.$$

(vi) For $k \geq 3, l \geq 2, i > 0$

$$\tau(j, j, \mathcal{G}(k, l)) = \begin{cases} k-1 & \text{if } 0 \leq s < (l-1) \binom{k}{2} \\ k & \text{if } (l-1) \binom{k}{2} \leq s < l \binom{k}{2}. \end{cases}$$

(vii) For $k \geq 3, l \geq 2$

$$\tau(j, \omega - j, \mathcal{G}(k, l)) = \begin{cases} v & \text{if } (*) s = (li + r + 1) \binom{k}{2} - v \text{ for some } i, 0 \leq r \leq l-1, \\ & r \neq l-2, 1 \leq v \leq k-1 \\ k & \text{in the other cases.} \end{cases}$$

(viii) For $k \geq 3, l \geq 2, i > 0$

$$\tau(j, B_{i-1}(k, l) \cup B_i(k, l), \mathcal{G}(k, l)) = \begin{cases} k+v-1 & \text{if condition } (*) \text{ of (vii) holds} \\ 2k-1 & \text{in the other cases.} \end{cases}$$

(ix) If $j \in B_i(2, l)$ for $l \geq 3$ then

$$v(j, \mathcal{G}(2, l), \mathcal{G}(2, l)) \subseteq B_{i-1}(2, l) \cup B_i(2, l) \cup B_{i+1}(2, l).$$

If $j \in B_{i,r}(k, l)$ for $k \geq 3, l \geq 2$ then

$$v(j, \mathcal{G}(k, l), \mathcal{G}(k, l)) \subseteq B_{i,r-1}(k, l) \cup B_{i,r}(k, l) \cup B_{i,r+1}(k, l)$$

where

$$B_{i,r-1}(k, l) = B_{i-1,l-2}(k, l) \quad \text{if } r = 0$$

$$B_{i,r+1}(k, l) = B_{i+1,0}(k, l) \quad \text{if } r = l-2.$$

PROOF OF THEOREM 9.2. The theorem is trivial if $\beta = 2$. We assume $\beta \geq 3$. Put $\beta = k + 1$. We define a graph $\mathcal{G} = \langle g, G \rangle$.

(1) Let $A = \{a_0, \dots, a_{k-2}\}$ be a set of $k-1$ elements disjoint to ω . Put $g = A \cup \omega$.

(2) $\mathcal{G}(A)$ has no edges.

(3) $\mathcal{G}(\omega) = \mathcal{G}(2, 3)$ for $k = 2$, $\mathcal{G}(\omega) = \mathcal{G}(k, 2)$ for $k \geq 3$.

We complete the definition of \mathcal{G} by defining $v(j, A, \mathcal{G})$ for every $j \in \omega$.

(4) For $k = 2$, let $j = 3i + s$, $0 \leq s < 3$

$$v(j, A, \mathcal{G}) = 0 \quad \text{if } s = 0,$$

$$v(j, A, \mathcal{G}) = \{a_0\} \quad \text{if } s = 1 \quad \text{or } s = 2.$$

(5) For $k \geq 3$ let $j = 2 \binom{k}{2} i + s$, $0 \leq s < 2 \binom{k}{2}$.

$$v(j, A, \mathcal{G}) = 0 \quad \text{if} \quad 0 \leq s < 2 \binom{k}{2} - (k-1)$$

$$v(j, A, \mathcal{G}) = \{a_0, \dots, a_{k-v-1}\} \quad \text{if} \quad s = 2 \binom{k}{2} - v; \quad v = 1, \dots, k-1.$$

By (1) we have $\alpha(\mathcal{G}) = \omega$. Put briefly $B_i = B_i(2, 3)$ if $k=2$, $B_i = B_i(k, 2)$ if $k \geq 3$, and let $C_i = \bigcup_{j < i} B_j \cup A$. By 9.8 (ii) B_i, C_i satisfy the requirements of 9.6. It follows from 9.8 (v) and (viii) and from (4) and (5) that

$$\tau(j, C_{i+1}, \mathcal{G}) \geq 2k-1 \quad \text{for every} \quad j \in B_i, \quad i > 0.$$

It follows from 9.6 that $\text{Col}(\mathcal{G}) > 2k-1 = 2\beta-3$. Define the simple ordering \prec of g by the following stipulations $a_0 \prec \dots \prec a_{k-2}$, $A \prec \omega$; If $i < j < \omega$ then $j \prec i$.

By 9.6 (iv) and (vii) and by (4) and (5) \prec satisfies the colouring function $\beta = k+1$, and as a corollary of this every finite subgraph of \mathcal{G} has colouring number $\leq \beta$. This proves 9.2.

§ 10. The relation $R(\alpha, \beta, \gamma, \delta)$ in case $\beta < \omega, \gamma \geq \omega$

Comparing 9.1 and 9.2 we see that 9.1 is best possible for fixed α, β , $2 \leq \beta < \omega$ if $\gamma = \omega$, but the problem whether it remains best possible for $\omega < \gamma \leq \alpha$ remains open in most cases. We can prove the following results.

THEOREM 10.1. $R(\omega_n, \beta, \omega_n, 2\beta-3)$ is not true for every finite n and for $2 \leq \beta < \omega$.

THEOREM 10.2. $R(\alpha, \beta, \alpha, \beta)$ for $\beta < \alpha$, if $\text{cf}(\alpha) = \omega, \alpha > \omega$.

The simplest unsolved problem is

PROBLEM 10.3. Is $R(\omega_{\omega+1}, \beta, \omega_{\omega+1}, \delta)$ true for some $\beta \leq \delta < 2\beta-2$ for $3 \leq \beta < \omega$?

We mention that if $R(\alpha, \beta, \alpha, \beta)$ is false for some $\beta < \alpha, \alpha > \omega$ then $\alpha \in \mathbf{C}_0$. (For the definition of the class \mathbf{C}_0 see e.g. [10].) We omit the proof. We postpone the proof of 10.1 and 10.2 to pp. 90 and 91 respectively. We need some lemmas.

LEMMA 10.4. Let m, n, β be integers, $m \geq 1, \beta \geq 3$. We say that the graph $\mathcal{G} = \langle g, G \rangle$ has property $q(m, n, \beta)$ if the following conditions (i)–(v) hold.

(i) There exist sets A, B, A_t $t < m$ such that $g = A \cup B$, $A = A_0 \cup \dots \cup A_{m-1}$, where the summands are disjoint.

(ii) $|A_t| = \omega_n$ for $t < m$, $|B| = \omega_n$.

(iii) $\tau(x, g, \mathcal{G}) = 1$ for every $x \in A$ and $\mathcal{G}(A)$ has no edges.

(iv) If $D \subseteq A_t$, $|D| < \omega_n$ for some $t < m$ then \mathcal{G} has a β -colouring \prec satisfying the condition

$$A \sim (A_t \sim D) \prec B \prec A_t \sim D.$$

(v) No well-ordering \prec of g satisfying the colouring function $2\beta-3$ has the property $A \prec B$.

Assume that there is a graph \mathcal{G} which has property $q(m, n, \beta)$. If $m > 1$ then there exists a graph \mathcal{G}^* which has property $q(m-1, n+1, \beta)$. If $m=1$ there exists a graph \mathcal{G}^* such that $\alpha(\mathcal{G}^*) = \omega_{n+1}$, $\text{Col}(\mathcal{G}^*) \cong 2\beta - 2$ and $\text{Col}(\mathcal{G}(g')) \leq \beta$ for every $g' \subseteq g$, $|g'| < \omega_{n+1}$.

PROOF. Let C be a set of power ω_n . By a well-known result of A. TARSKI [13], there exists a sequence C_ξ , $\xi < \omega_{n+1}$ of subsets of C satisfying the following conditions

- (1) $\bigcup_{\xi < \omega_{n+1}} C_\xi = C$, $|C_\xi| = \omega_n$ and $|C_\zeta \cap C_\xi| < \omega_n$ for every pair $\zeta, \xi < \omega_n$.

For every $\xi < \omega_{n+1}$ let $\mathcal{G}_\xi = \langle g_\xi, G_\xi \rangle$ be a graph satisfying the conditions (i)–(v) and the following additional requirements with the sets $A_\xi, B_\xi, A_{t,\xi}$ $t < m$

- (2) $A_{m-1,\xi} = C_\xi$ for $\xi < \omega_{n+1}$.
 (3) The set containing $C, B_\xi, A_{t,\xi}$ for $t < m-1$, $\xi < \omega_{n+1}$ is disjointed.

We define a graph $\mathcal{G}^* = \langle g^*, G^* \rangle$ by

- (4) $g^* = \bigcup_{\xi < \omega_{n+1}} g_\xi$, $G^* = \bigcup_{\xi < \omega_{n+1}} G_\xi^*$.
 (5) $A_t^* = \bigcup_{\xi < \omega_{n+1}} A_{t,\xi}$ for every $t < m-1$,

$$A^* = \bigcup_{t < m-1} A_t^*, \quad B^* = C \cup \bigcup_{\xi < \omega_{n+1}} B_\xi, \quad \mathcal{G}^*(g_\xi) = \mathcal{G}_\xi \quad \text{for } \xi < \omega_{n+1}.$$

It follows immediately from (1)–(5) that

- (6) If $m > 1$ then \mathcal{G}^* satisfies the requirements (i), (ii) and (iii) of the property $q(m-1, n, \beta)$ and if $m=1$ then $\alpha(\mathcal{G}) = \omega_{n+1}$.

We prove

- (7) No well-ordering $<$ of g^* satisfying $2\beta - 3$ has the property $A^* < B^*$.

Assume that (7) is not true and let $<$ be a well-ordering of g^* satisfying $2\beta - 3$ and such that $A^* < B^*$. Then by 3. 1 and 8. 8 we can assume that

$$\text{typ } B^*(<) = \omega_{n+1}.$$

Considering that by (1), $|C| = \omega_n$ then there exists a $\xi < \omega_{n+1}$ such that $C < B_\xi$. But then by (1)–(5) $<$ is a well-ordering of g_ξ such that $A_\xi < B_\xi$ and $<$ satisfies $2\beta - 3$ on \mathcal{G}_ξ . This contradicts (1) and thus (7) is true.

(7) means that \mathcal{G}^* satisfies requirement (v) as well and that $\text{Col}(\mathcal{G}^*) > 2\beta - 3$ if $m=1$.

To prove that \mathcal{G}^* satisfies (iv) we need some preliminaries.

- (8) Put $h_\xi = \bigcup_{\zeta < \xi} g_\zeta$, $H_\xi = \bigcup_{\zeta < \xi} G_\zeta$, $\mathcal{H}_\xi = \langle h_\xi, H_\xi \rangle$.

We prove that

- (9) For every $\xi < \omega_{n+1}$ h_ξ has a well-ordering $<$ satisfying β on \mathcal{H}_ξ such that

$$A^* \cap h_\xi < h_\xi \sim A^*.$$

We can assume that $\omega_n \cong \xi$. Let $\psi \in \omega_n^\xi$ be a well-ordering of type ω_n of ξ . For every $\zeta < \omega_n$ put $D_\zeta = C_{\psi(\zeta)} \cap \bigcup_{\eta < \zeta} C_{\varphi(\eta)}$. By (1) and by the regularity of ω_n we have $|D_\zeta| < \omega_n$. By (1) $\mathcal{G}_{\psi(\zeta)}$ satisfies the requirement (iv), hence there exists a well-ordering $<_\zeta$ of $g_{\psi(\zeta)}$ satisfying β on $\mathcal{G}_{\psi(\zeta)}$ and such that

- (10) $A_{\psi(\zeta)} \sim (C_{\psi(\zeta)} \sim D_\zeta) <_\zeta B_{\psi(\zeta)} <_\zeta C_{\psi(\zeta)} \sim D$.

We choose a well-ordering \prec of h_ξ satisfying the following conditions

$$(11) \quad A^* \cap h_\xi \prec A^* \sim h_\xi,$$

\prec is an arbitrary well-ordering on $A^* \cap h_\xi$. (Note that $\mathcal{G}(A^* \cap h_\xi)$ has no edges.)

$$\text{Put } E_\zeta = \bigcup_{\eta < \zeta} (B_{\psi(\eta)} \cup C_{\psi(\eta)}), \quad F_\zeta = B_{\psi(\zeta)} \cup C_{\psi(\zeta)}.$$

Then by (1) and (3) $E_\zeta \cap F_\zeta = D_\zeta$ for $\zeta < \omega_n$.

Put $E_\zeta \prec F_\zeta \sim D_\zeta$, and for every $\zeta < \omega_n$ let \prec coincide with \prec_ζ on the set $F_\zeta - D_\zeta$.

By 8. 8 and (10) every well-ordering \prec of h_ξ satisfying (11) satisfies the requirements of (9).

(6), (7) and (9) prove the theorem in case $m = 1$.

Assume $m > 1$ and $D \subseteq A_t^*$, $|D| \leq \omega_n$ for some $t < m - 1$. Let $\xi_0 < \omega_{n+1}$ such that $D \cap A_{t, \xi} = 0$ for every $\xi \geq \xi_0$. By (9) there exists a well-ordering \prec_{ξ_0} of h_{ξ_0} such that

$$(12) \quad A^* \cap h_{\xi_0} \prec_{\xi_0} h_{\xi_0} \sim A^*.$$

For every $\xi \geq \xi_0$, by (1), there exists a well-ordering \prec'_ξ of g_ξ satisfying β and such that

$$(13) \quad A_\xi \sim A_{t, \xi} \prec'_\xi B_\xi \prec'_\xi A_{t, \xi}.$$

We choose a well-ordering \prec of g^* satisfying the following conditions

$$(14) \quad A^* \sim (A_t^* \sim D) \prec B^*,$$

\prec is an arbitrary well-ordering on $A^* \sim (A_t^* \sim D)$.

On B^* we choose \prec so that

$$h_{\xi_0} \prec C \sim h_{\xi_0} \prec \bigcup_{\xi \geq \xi_0} B_\xi.$$

Let \prec coincide with \prec_{ξ_0} on h_{ξ_0} , and with \prec'_ξ on B_ξ for $\xi \geq \xi_0$.

$B^* \prec A_t^* \sim D$ and \prec is arbitrary on $A_t^* \sim D$. Using (1), (12) and (13) it is easy to verify that every well-ordering \prec of g^* satisfying the requirements of (14) satisfies β and is such that $A^* \sim (A_t^* \sim D) \prec B^* \prec A_t^* \sim D$. Hence \mathcal{G}^* satisfies the requirement (iv) of $\varphi(m - 1, n, \beta)$ for $m > 1$. In view of (6) and (7) this concludes the proof of 10. 4.

LEMMA 10. 5. For every $m \geq 1$, $3 \leq \beta < \omega$ there exists a graph $\mathcal{G}(g, G)$ which has property $q(m, 0, \beta)$ of 10. 4.

PROOF. Put $\beta = k + 1$.

(1) Let A be a set $A \cap \omega = 0$, $|A| = \omega$. Put $B = \omega$. It is obvious that there exists a graph $\mathcal{G} = \langle g, G \rangle$ satisfying the following stipulations

(2) $\mathcal{G}(A)$ has no edges.

(3) $\mathcal{G}(\omega) = \mathcal{G}(k, m + 2)$.

(Note that $\mathcal{G}(k, m + 1)$ would be good as well except in case $k = 2$, $m = 1$, where $\mathcal{G}(2, 2)$ is not defined.)

Assume $j \in B_i(k, m + 2)$ i. e. $j = (m + 2) \binom{k}{2} i + s$ where $0 \leq s < (m + 2) \binom{k}{2}$.

We conclude the definition of \mathcal{G} by defining $\tau(j, A, \mathcal{G})$ for $j \in \omega$, and by

(4) $\tau(x, B, \mathcal{G}) = 1$ for $x \in A$.

(5) Assume $k = 2$. Put

$$\tau(j, A, \mathcal{G}) = \begin{cases} 1 & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases}$$

(6) Assume $k \geq 3$. Put

$$\tau(j, A, \mathcal{G}) = \begin{cases} k-v & \text{whenever } v \text{ satisfies condition } (*) \text{ of Lemma 9.8 (vii)} \\ 0 & \text{in the other cases.} \end{cases}$$

We define the sets A_0, \dots, A_{m-1} .

First we define the subsets $B^{(0)}, \dots, B^{(m)}$ of B as follows.

(7) If $k=2$, $j = (m+2)i+s$, $0 \leq s \leq m+2$

$$j \in B^{(t)} \quad \text{if } s=t \quad \text{for } 0 \leq t \leq m-1$$

$$j \in B^{(m)} \quad \text{if } s=m \text{ or } s=m+1.$$

(8) If $k \geq 3$ put

$$B^{(t)} = \bigcup_{i < \omega} B_{i,t}(k, m+2) \quad \text{for } 0 \leq t < m-1$$

$$B^{(m-1)} = \bigcup_{i < \omega} (B_{i,m-1}(k, m+2) \cup B_{i,m}(k, m+2)), \quad B^{(m)} = 0.$$

(9) Put

$$A_t = \{x \in A : x \in v(j, A, \mathcal{G}) \text{ for some } j \in B^{(t+1)}\}$$

for $0 \leq t < m$ if $k=2$ and

$$A_t = \{x \in A : x \in v(j, A, \mathcal{G}) \text{ for some } j \in B^{(t)}\}$$

for $0 \leq t < m$ if $k \geq 3$.

It is an immediate consequence of the definitions that \mathcal{G}, A, B, A_t , $0 \leq t < m$ satisfy the requirements (i), (ii), (iii) of $\varphi(m, 0, \beta)$ stated in 10.4.

Put briefly $B_i = B_i(k, m+2)$ for every i and $C_i = \bigcup_{j < i} B_j \cup A$.

It follows from 9.8 (v) and (viii) and from (5) and (6) that for every $i > 0$ for every $j \in B_i$ we have

$$\tau(j, C_{i+1}, \mathcal{G}) = 2k-1 = 2\beta-3.$$

It follows from 9.6 that no well-ordering $<$ of g satisfying $2\beta-3$ has the property $A < B$. Hence \mathcal{G} satisfies requirement (v) of 10.4 as well.

Assume $D \subseteq A_t$, $|D| < \omega$ for some $t < m$. Choose a well-ordering $<$ of g satisfying the following conditions:

(10) $A \sim (A_t \sim D) < B < A_t \sim D$,

$<$ is an arbitrary well-ordering on the sets $A \sim (A_t \sim D)$ and $A_t \sim D$.

It remains to choose $<$ on $B = \omega$. There exists an i_0 such that $v(j, D, \mathcal{G}) = 0$ for every $j \in B_i(k, m+2)$ for $i > i_0$.

Let $\bigcup_{j \equiv i_0} B_j < \omega \sim \bigcup_{j \equiv i_0} B_j$ and let $<$ coincide with $>$ on the set $\bigcup_{j \equiv i_0} B_j$. To choose $<$ on the set $\omega \sim \bigcup_{j \equiv i_0} B_j$ we distinguish the cases (I) $k=2$ (II) $k \geq 3$.

(I) Choose $<$ so that

$$B_i < B_{i'} \quad \text{for every } i_0 < i < i' < \omega$$

and choose $<$ as an ordering of the circuit $\mathcal{G}(B_i)$ satisfying the condition

$$v(j, B_i, \mathcal{G}(B_i)) \leq 1 \quad \text{for every } j \notin B^{(t+1)}$$

$$v(j, B_i, \mathcal{G}(B_i)) \leq 2 \quad \text{for every } j \in B^{(t+1)}.$$

(II) Put briefly $B_{i,r} = B_{i,r}(k, m+2)$ for $0 \leq r \leq m$

$$D_0 = \bigcup_{0 \leq r < t} B_{i_0+1,r}, \quad E_0 = B_{i_0+1,t}$$

$$D_{i+1} = \bigcup_{t < r \leq m} B_{i_0+1+i} \cup \bigcup_{0 \leq r < t} B_{i_0+2+i,r}$$

for every $i < \omega$.

Choose $<$ so that

$$D_0 < D_1 < E_0 < D_2 < E_1 < D_3 < \dots$$

$<$ coincides with $>$ on D_i , $<$ coincides with $<$ on E_i .

To see that \mathcal{G} satisfies the requirement (iv) of $\varphi(m, 0, \beta)$ of 10. 4 it is obviously sufficient to see that $\tau(x, g | < x, \mathcal{G}) \leq k$ for every $x \in g$ whenever $<$ satisfies the requirements of (10). If $x \notin \omega$ this is trivial from (2) and (4). If $x \in \bigcup_{j \leq i_0} B_j$ this follows from (5) and (6) using 9. 8 (iv) and (vii). In case (I) for $x \in \omega - \bigcup_{j \leq i_0} B_j$ the statement follows from (7), (9) and 9. 8 (v) and (ix). In case (II) using (8), (9) and 9. 8 (vii) and (ix) the statement follows easily for every $x \in D_i$. For $x \in E_i$ we have to use (8), (9), 9. 8 (vi) and (ix) and the following fact which easily follows from the definition 9. 7 of the graph $\mathcal{G}(k, m+2)$:

If $j \in B_{i,r}$ for some $0 \leq r < m$ then $\tau(j, \omega \sim (j \cup B_{i,r}), \mathcal{G}(k, m+2)) = 1$. This proves that \mathcal{G} satisfies the requirements of 10. 5.

LEMMA 10. 6. For every $m, n, \beta, m \geq 1, 3 \leq \beta < \omega$ there is a graph \mathcal{G} which has the property $\varphi(m, n, \beta)$ of 10. 4.

PROOF. By induction on n . If $n=0$ 10. 6 is true for every β, m satisfying the requirements, by 10. 5. Assume that 10. 6 is true for some n for every $m \geq 1, 3 \leq \beta < \omega$. Then by 10. 4 it is true for $n+1$, for every $m \geq 1$ and $3 \leq \beta < \omega$.

PROOF OF THEOREM 10. 1. 10. 1 is trivial if $\beta=2$. Assume $3 \leq \beta < \omega$. If $n=0$ 10. 1 is true by 9. 2. Assume $n>0$. By 10. 6 there exists a graph \mathcal{G} which has property $\varphi(1, n-1, \beta)$ of 10. 4. By 10. 4 then there exists a graph \mathcal{G}^* such that $\alpha(\mathcal{G}^*) = \omega_n$, $\text{Col}(\mathcal{G}^*) \cong 2\beta-2$ and $\text{Col}(\mathcal{G}^*(g')) \leq \beta$ for every $g' \subseteq g^*, |g'| < \omega$. This proves 10. 1.

We need a further lemma for the proof of Theorem 10. 2.

LEMMA 10. 7. Let $\mathcal{G} = \langle g, G \rangle$ be a graph which possesses property $\mathbf{D}(\beta, \gamma)$. Let $A \subseteq g, |A| = \varepsilon$ such that $\beta \leq \varepsilon, \omega \leq \varepsilon^+ < \gamma$. Then there exists a subset B satisfying the following conditions:

(i) $A \subseteq B, |B| = \varepsilon$.

(ii) For every $C \subseteq g \sim B, |C| < \gamma$ there exists a β -colouring $<$ of $\mathcal{G}(B \cup C)$ such that $B < C$.

PROOF. First we prove that there exists a set B satisfying the condition (i) and the following condition

(iii) For every $C \subseteq g \sim B, |C| \leq \varepsilon$, there exists a β -colouring $<$ of $\mathcal{G}(B \cup C)$ such that $B < C$.

Assume that there is no B satisfying (i) and (iii). That means that for every $A \subseteq B, |B| = \varepsilon$ there exists a $\varphi(B)$ satisfying the following conditions

(1) $\varphi(B) \subseteq g \sim B; |\varphi(B)| \leq \varepsilon$, and $\mathcal{G}(B \cup \varphi(B))$ has no β -colouring $<$ satisfying $B < \varphi(B)$.

We define a sequence $A_\xi, \xi < \varepsilon^+$ of subsets of g by transfinite induction ξ as follows.

(2) $A_0 = A$. Assume that A_ζ is defined for every $\zeta < \xi$ for some $0 < \xi < \varepsilon^+$ in such a way that $|A_\zeta| \leq \varepsilon$. Then $|\bigcup_{\zeta < \xi} A_\zeta| = \varepsilon$. Put $A_\xi = \varphi(\bigcup_{\zeta < \xi} A_\zeta)$. Then $|A_\xi| \leq \varepsilon$ and A_ξ is defined for every $\xi < \varepsilon^+$.

Put further $D = \bigcup_{\xi < \varepsilon^+} A_\xi$. Then $|D| = \varepsilon^+$. By the assumption and by 3.2 $\mathcal{G}(D)$ has a β -colouring \prec such that

(3) $\text{typ } D(\prec) = \varepsilon^+$.

Using (2), (3) and the regularity of ε^+ it is easy to see that

(4) There is a $\xi_0 < \varepsilon^+$ and an $x_0 \in D$ such that

$$\bigcup_{\zeta < \xi_0} A_\zeta = D \prec x_0.$$

It follows from (4) that $\bigcup_{\zeta < \xi_0} A_\zeta < A_{\xi_0}$. Considering that \prec is a β -colouring of $\mathcal{G}(D)$ this contradicts (1) and (2).

Hence there is a set B satisfying the conditions (i) and (iii). We prove that the same set B satisfies (ii).

Let $C \subseteq g \sim B, |C| < \gamma$. By the assumptions there exists a β -colouring \prec' of $\mathcal{G}(B \cup C)$.

We define a sequence B_i of subsets of $B \cup C$ by induction on i as follows.

(5) $B_0 = B; B_{i+1} = \bigcup_{x \in B_i} v(x, B \cup C \prec' x, \mathcal{G}(B \cup C))$.

Put $D = \bigcup_{i < \omega} B_i$.

Considering that $|B| = \varepsilon$ and $\tau(x, B \cup C \prec' x, \mathcal{G}(B \cup C)) < \beta$ because of \prec' is a β -colouring it follows that

(6) $B \subseteq D \subseteq B \cup C; |D| = \varepsilon$, for every $x \in D$

$$v(x, B \cup C \prec' x, \mathcal{G}(B \cup C)) \subseteq D.$$

Considering that B satisfies (iii) we can choose a β -colouring \prec of $\mathcal{G}(D)$ such that $B \prec D \sim B$. We define the well-ordering \prec^* of $B \cup C$ as follows.

(7) \prec^* and \prec coincide on D \prec^* coincides with \prec' on $(B \cup C) \sim D = C \sim D$ $D \prec^* C \sim D$.

\prec^* obviously satisfies $B \prec^* C$. To prove that it is a β -colouring by (7) it is sufficient to see that $\tau(x, B \cup C \prec^* x, \mathcal{G}(B \cup C)) < \beta$ for every $x \in C \sim D$.

Assume $y \in v(x, B \cup C \prec^* x, \mathcal{G}(B \cup C))$ for some $x \in C \sim D$.

We prove that then $y \in v(x, B \cup C \prec' x, \mathcal{G}(B \cup C))$. If $y \in C \sim D$ this follows from (7). If $y \in D$ and $\{yx\} \in G$ then $y \prec' x$ for if not then $x \in v(y, B \cup C \prec' y, \mathcal{G}(B \cup C))$ and $x \in D$, by (6).

Considering that \prec' is a β -colouring of $\mathcal{G}(B \cup C)$ this implies the statement and B satisfies (ii).

PROOF OF THEOREM 10. 2. Let $\mathcal{G} = \langle g, G \rangle$ be a graph, $\alpha(\mathcal{G}) = \alpha$, which has property $D(\beta, \alpha)$. By the assumptions there exists a $\varphi \in {}^\omega \alpha$ such that

(1) $\bigcup_{i < \omega} \varphi_i = \alpha$, $\varphi_i < \varphi_{i'}$, φ_i is a cardinal, $\varphi_i \cong \beta \cup \omega$ for every $i < i' < \omega$.

Let $h \in {}^\omega g$ be a disjoint partition of g satisfying

(2) $|h_i| = \varphi_i$ for $i < \omega$.

If $A \subseteq g$, $|A| = \varepsilon$, $\beta \cup \omega \leq \varepsilon < \alpha$ then by the assumption and by 10. 7 there exists a set B satisfying the conditions (i), (ii) of 10. 7.

(3) Let $\mathcal{B}(A)$ denote such a set for every A satisfying the above conditions.

We define a disjoint partition a of type ω of g as follows. Assume that a_j is defined for every $j < i$ in such a way that $|a_j| < \alpha$.

Put $a_i = \mathcal{B}(\bigcup_{j < i} a_j \cup h_i) \sim \bigcup_{j < i} a_j$ for $i < \omega$.

Then by (2) and (3) we have $|a_i| < \alpha$ and by (2) a is a disjoint partition of g .

By the assumption and by (3) the set $\bigcup_{j \leq i} a_j$ has a β -colouring $<_i$ such that

$\bigcup_{j < i} a_j < a_i$. Lemma 8. 8 implies that then $\text{Col}(\mathcal{G}) \leq \beta$.

§ 11. The relation $R(\alpha, \beta, \gamma, \delta)$ in cases $\beta \cong \omega$. Problems

In case $\beta \cong \omega$, our results are rather incomplete. Using the results of § 5 we obtain some positive results but in most cases we cannot prove that they are the best possible. We are going to state some typical unsolved problems.

As an immediate consequence of Definition 2. 12 we have

LEMMA 11. 1. Assume $0 < \beta < \beta'$. Then $[[\beta, \beta']]$ has colouring number $> \beta$.

LEMMA 11. 2. Assume that for some $0 < \beta < \beta' < \gamma$, $\text{Col}(\alpha, \delta, \beta', \beta)$. Then $R(\alpha, \beta, \gamma, \delta)$.

PROOF. Let \mathcal{G} be a graph $\alpha(\mathcal{G}) = \alpha$ which has property $\mathbf{D}(\beta, \gamma)$. Then \mathcal{G} does not contain a $[[\beta, \beta']]$, by 11. 1. Hence by 5. 1, $\text{Col}(\mathcal{G}) \leq \delta$.

THEOREM 11. 3. Assume that G. C. H. holds and $\beta \cong \omega$. Then

(i) $R(\alpha, \beta, \beta^{++}, \beta^+)$ holds for every $\alpha \leq \alpha(\beta)$

(ii) $R(\alpha, \beta, \beta^{+++}, \beta^{++})$ holds for every α .

PROOF. By 5. 5 (i) and (ii) $\text{Col}(\alpha, \beta^+, \beta^{++}, \beta)$ and $\text{Col}(\alpha, \beta^{++}, \beta^{+++}, \beta)$ hold in cases (i) and (ii), respectively. Hence 11. 3 follows from 11. 2 in both cases.

Comparing this result with 8. 3 we see that the following problems remain open. Assume $\beta \cong \omega$.

Is $R(\alpha, \beta, \beta^{++}, \beta)$ true for some (or every) $\alpha > \beta$?

Is $R(\alpha, \beta, \beta^{++}, \beta^+)$ true for some (or every) $\alpha > \alpha(\beta)$?

Are $R(\alpha, \beta, \gamma, \beta)$ or $R(\alpha, \beta, \gamma, \beta^+)$ true for some $\beta^{++} \leq \gamma \leq \alpha$?

9. 2 gives a positive answer to the last problem in case of $(\alpha) = \omega$, $\alpha > \omega$.

We state the simplest unsolved problems.

PROBLEM 11. 4. (i) Is $R(\omega_2, \omega, \omega_2, \omega)$ true?

(ii) Is $R(\omega_{\omega+1}, \omega, \omega_2, \omega_1)$ true?

DEFINITION 11. 5. The relation $P(\alpha, \beta, \gamma, \delta)$ is said to hold if every graph with $\alpha(\mathcal{G}) = \alpha$ has chromatic number $\leq \delta$ provided every subgraph of power $< \gamma$ of it has chromatic number $\leq \beta$.

The problem involved in the relation P is well known and is stated, e. g., in [5]. The DE BRUIJN—ERDŐS theorem [2] mentioned in the introduction states

$P(\alpha, \beta, \omega, \beta)$ for every finite β and for every α . No non-trivial result is known in cases $\beta \cong \omega$. The simplest unsolved problem (assuming the G. C. H.) is $P(\omega_2, \omega, \omega_2, \omega)$. This should be compared with 11. 4 (i). We mention that while as a corollary of 11. 3 we have $R(\omega_2, \omega, \omega_2, \omega_1)$ the problem whether $P(\omega_2, \omega, \omega_2, \omega_1)$ holds is unsolved.

We mention the following

THEOREM 11.5. *Assume $\beta \cong \omega$. If the β^+ -product space of an α -termed sequence of two-point discrete topological spaces is γ -compact then both*

$$P(\alpha, \beta, \gamma, \beta) \text{ and } R(\alpha, \beta, \gamma, \beta) \text{ hold.}$$

For the concepts appearing in 11. 5 see, e. g., [10]. The condition $\beta \cong \omega$ can be omitted in case of P but not in case of R as is shown by 9.2.⁸

We omit the proof which is well known in case of P and is easy in case of R .

§ 12. Some remarks and problems on general set systems

As a generalization of the problems investigated in 5. 1 one can consider problems of the following type. What kind of special set-systems are necessarily contained in a set-system $\mathcal{H} = \langle h, H \rangle$ with $\alpha(\mathcal{H}) = \alpha$, $\text{Col}(\mathcal{H}) > \beta$ or $\text{Chr}(\mathcal{H}) > \beta$, respectively. It is obvious that it is possible to obtain immediate generalizations of Theorem 5. 5 for uniform set-systems (see Definition 2. 4) with $3 \cong \kappa(\mathcal{H}) < \omega$ but we do not know whether these results are best possible and so we do not investigate this problem in this paper. However in case $\kappa(\mathcal{H}) = 3$ some simpler problems arise which are of a different type from the ones considered in case $\kappa(\mathcal{H}) = 2$. We just formulate a result and a problem concerning one of them.

THEOREM 12. 1. *Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set-system with $\alpha(\mathcal{H}) = \alpha$, $\kappa(\mathcal{H}) = k$, $2 \cong k < \omega$ and let β be an infinite cardinal number. Then one of the following conditions (i), (ii) holds*

- (i) *There are disjoint subsets $h_0, h_1 \subseteq h$ such that $|h_0| = k - 1$, $|h_1| = \beta^+$ and $h_0 \cup \{x\} \in H$ for every $x \in h_1$.*
- (ii) *$\text{Col}(\mathcal{H}) \cong \beta$.*

In case $k = 2$, 12. 1 is a trivial special case of 5. 5, in case $k > 2$ it can be proved easily using the idea of 5. 5. We omit the proof.

From 12. 1 we have

COROLLARY 12. 2. *Under the assumptions of 12. 1 one of the following conditions (i) and (ii) is true*

⁸ In [10] it is proved that the α -product space of an α termed sequence of two point discrete topological spaces is not α compact for a wide class of cardinals α . As a generalization of this result using G. C. M. in [16] it is proved the ω_1 -product space of an α -termed sequence of two point discrete topological spaces is not α compact for many cardinals α . It is not known whether these results hold for every α . 11.5 is stated here merely to show that there are some connections between the problems treated in the relations P, R and in [10] respectively. Since these problems are not the topic of this paper we omit references to further relevant results. As to the definition of the concepts used in 11. 5 and in this remark we refer to [10].

(i) For every $l > k$, there exists a subset $h' \subseteq h$ such that $|h'| = l$ and $|\{A \in H: A \subseteq h'\}| \geq l - k + 1$.

(ii) $\text{Col}(\mathcal{H}) \leq \omega$.

We do not know whether $l - k + 1$ is best possible in 12. 1. We mention a very special problem we cannot solve.

PROBLEM 12. 3. Let \mathcal{H} be a uniform set-system $\alpha(\mathcal{H}) = \omega_1, \kappa(\mathcal{H}) = 3$. Is it true that one of the following conditions (i) or (ii) holds?

(i) There is $h' \subseteq h, |h'| = 5$ such that $|\{A \in H: A \subset h'\}| \geq 4$.

(ii) $\text{Col}(\mathcal{H}) \leq \omega$ (or at least $\text{Chr}(\mathcal{H}) \leq \omega$).

Note that using the method of [17] theorem 28 we know that if C. H. holds and 4 is replaced by 5, then the answer is negative and as a corollary of 12. 2 we know that 12. 3 is true if 4 is replaced by 3.

§ 13. On chromatic number of finite set-systems

First we prove a very simple and special theorem

THEOREM 13. 1. Let $\mathcal{H} = \langle h, H \rangle$ be a finite uniform set-system, such that $\alpha(\mathcal{H}) = n, \kappa(\mathcal{H}) = 3$. Assume that \mathcal{H} has property C(2, 2) i. e., for $X \neq Y \in H, |X \cap Y| \leq 1$. Then h contains an independent subset h' such that $|h'| \geq \lfloor \sqrt{2n} \rfloor$.⁹

PROOF. Let h' be a maximal independent subset of h . Put $|h'| = r$. By the maximality of h' for every $x \in h \sim h'$ there is an element A_x of H such that $A_x \sim \{x\} \subseteq h'$. It follows from the assumption that $|A_x \cap A_y| \leq 1$ for $x \neq y \in h \sim h'$. We obtain

$$|h \sim h'| = n - r \leq \binom{r}{2} \quad \text{or} \quad r \geq \lfloor \sqrt{2n} \rfloor.$$

We do not investigate here some possible generalizations of this theorem for uniform set-systems with $\kappa(\mathcal{H}) > 3$, but we are going to discuss one possible improvement of this theorem which will turn out to be false. Namely one would guess that the result $r \geq \lfloor \sqrt{2n} \rfloor$ trivially obtained above is very far from being the best possible, and that it can be replaced by $r > cn$ with some real number c .

As a consequence of the main result of this section it will turn out that it is not true, and even $r > n^{1-\varepsilon}$ is not true for some fixed $\varepsilon > 0$.

Before stating the main theorem we mention that a graph $\mathcal{G} = \langle g, G \rangle$ contains a circuit of length $\leq s$ iff there is a subset G' of G such that $0 < |G'| = t \leq s$ and $|\cup G'| \leq t$.

Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set-system, $\kappa(H) = k, 2 \leq k < \omega$. The above remark makes it possible that without defining the concept of a circuit for $k \geq 3$, we define the concept of s -circuitless uniform set-systems for $s < \omega$.

DEFINITION 13. 2. Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set-system, $\kappa(H) = k, 2 \leq k < \omega$. \mathcal{H} is said to be s -circuitless if for every $1 \leq t \leq s$, and for every $H' \subseteq H, |H'| = t$

$$|\cup H'| \geq 1 + (k - 1)t.$$

⁹ In this section $[x]$ denotes the integer part of the real number x . In what follows we use many other usual notations of number theory not introduced in § 2. $\delta, \varepsilon, \eta$ denote real numbers.

Using this concept our main theorem is an immediate generalization of a theorem of P. ERDŐS [4] already mentioned in the introduction.

THEOREM 13.3. *For every $k \geq 3$, and for every s there is a real number $\varepsilon_{k,s} > 0$ and an integer $n_{k,s}$ such that for every $n > n_{k,s}$ there exist a uniform set-system $\mathcal{H} = \langle h, H \rangle$ such that $\alpha(\mathcal{H}) = n$, $\varkappa(\mathcal{H}) = k$, \mathcal{H} is s -circuitless and h contains no independent subset of more than $n^{1-\varepsilon_{k,s}}$ elements.*

From 13.3 we get

COROLLARY 13.4. *For every $k \geq 2$, r , s there exist uniform set-systems \mathcal{H} with $\varkappa(\mathcal{H}) = k$ such that \mathcal{H} is s -circuitless and $\text{Chr}(\mathcal{H}) \cong r$ or as an equivalent formulation to this, for every $k \geq 2$ and for every s there exists a uniform set-system \mathcal{H} , with $\alpha(\mathcal{H}) = \omega$, $\varkappa(\mathcal{H}) = k$, which is s -circuitless, and has chromatic number ω .*

We mention that at present we find hopeless the exact determination of $\varepsilon_{k,s}$ appearing in 13.3.

We postpone the proof of 13.3. First we state and prove a simpler theorem which shows that 13.3 is in some respect best possible.

THEOREM 13.5. *Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set-system such that $\varkappa(\mathcal{H}) = k$, $2 \leq k$. Assume that for some $t \geq 1$, and for every $H' \subset H$, $|H'| = t$, $|UH'| \geq 2 + (k-1)t$. Then the colouring number of \mathcal{H} is at most t .*

As a corollary of this if $\alpha(\mathcal{H}) = n$ then \mathcal{H} contains an independent subset of $\cong n/t$ elements.

PROOF. By the assumption $|V(x, h, \mathcal{H})| < t$ for every $x \in h$, and by 3.1.

The estimation $\text{Col}(\mathcal{H}) \cong t$ is obviously not best possible, but we do not investigate this.

PROOF OF THEOREM 13.3. We will use the probabilistic method described e. g. in [4]. First we briefly outline the proof.

We will consider a set h of n elements. Then we will choose an $H \subseteq S_k[h]$ of $[n^{1+\eta}]$ elements at random (where η will be determined later). The idea of our proof is that for the most choices of H , the condition that \mathcal{H} is s -circuitless is rarely violated, and on the other hand every subset of at least $n^{1-\varepsilon_{k,s}}$ elements of h will contain many elements of H . So we will find an H such that omitting few elements of it we obtain an H' so that the resulting set-system $\langle h, H' \rangle$ is s -circuitless and does not contain an independent set of $n^{1-\varepsilon_{k,s}}$ elements.

Put $N = [n^{1+\eta}]$.

Let $A_N = \{H : H \subseteq S_k[h] \text{ and } |H| = N\}$.

Clearly

$$(1) |A_N| = \binom{\binom{n}{k}}{N}.$$

Let l, m be integers. Denote by $A_N(l, m)$ the set

$$(2) \{H \in A_N : \text{there exist an } h' \subseteq h, |h'| = m \text{ such that } |H \cap S_k[h']| \cong l\}.$$

We want an upper estimation for the cardinality of $A_N(l, m)$.

For every fixed $h' \subseteq h$, $|h'| = m$ the number of $H \in A_N$ satisfying (2) is exactly

$$(3) \quad \sum_{i=0}^l \binom{m}{k} \binom{m}{i} \binom{\binom{n}{k} - \binom{m}{k}}{N-i}.$$

Hence by (2) and (3) we have

$$(4) \quad |A_N(l, m)| \cong \binom{n}{m} \left(\sum_{i=0}^l \binom{m}{k} \binom{\binom{n}{k} - \binom{m}{k}}{N-i} \right).$$

We prove the following lemma

(5) Assume $0 < \eta < \frac{1}{s}$, $l = n$, $m = [n^{1-\varepsilon_{k,s}}]$ where $0 < \varepsilon_{k,s} < \frac{h}{2k}$. Then

$$|A_N(l, m)| = o \left(\binom{n}{k} \right).$$

To prove (5) observe that

$$(6) \quad \binom{\binom{n}{k} - \binom{m}{k}}{N-i} \cdot \binom{\binom{n}{k}}{N}^{-1} < \left(1 - \frac{\binom{m}{k}}{\binom{n}{k}} \right)^{N-i} < \left(1 - \frac{\binom{m}{k}}{\binom{n}{k}} \right)^{\frac{N}{2}}$$

provided n is big enough and $i < \frac{N}{2}$. Hence under the conditions of (5) by (4) and (6) we have

$$|A_N(l, m)| \binom{\binom{n}{k}}{N}^{-1} < \binom{n}{m} \left(1 - \frac{\binom{m}{k}}{\binom{n}{k}} \right)^{\frac{N}{2}} \sum_{i=0}^n \binom{m}{k} < 2^n n^{kn} e^{-n^{1+\frac{n}{2}}} = o(1).$$

This proves (5).

For an arbitrary $H \in A_N$ we denote by $z(H)$ the number of elements of the set:

$$Z(H) = \{h' \subseteq h : |h'| = (k-1)t, t \leq s \text{ and } |\mathcal{S}_k[h'] \cap H| \cong t\}.$$

We denote by z the expected number of $z(H)$ if $H \in A_N$ is chosen at random.

Considering that the number of those h' which have $(k-1)t$ elements is $\binom{n}{(k-1)t}$,

and that for every h' , ($|h'| = (k-1)t$) there are at most $\binom{(k-1)t}{t} \binom{\binom{n}{k}}{N-t}$ H 's satisfy-

ing the condition $|\mathcal{S}_k[h'] \cap H| \cong t$ we have

$$(7) \quad z \cong \left(\binom{n}{k} \right)^{-1} \left(\sum_{t=2}^s \binom{n}{(k-1)t} \binom{(k-1)t}{t} \binom{n}{N-t} \right).$$

An easy computation shows that then there exists a real number c such that

$$(8) \quad z < c \left(\frac{N}{n} \right)^s.$$

It follows from (8) that for all but $o \left(\binom{n}{k} \right)$ systems $H \in A_N$ we have $z(H) \cong \left(\frac{N}{n} \right)^s \log n$.

If $h' \in z(H)$ then $|h' \cap H| \cong \mathcal{S}_k[h'] \cong \binom{(k-1)s}{k}$.

By (5) and (8) if n is sufficiently large then there exists an $H \in A_N$ such that it satisfies (9) and (10).

(9) For every $h' \subseteq h$, $|h'| \cong [n^{1-\varepsilon_k s}]$ we have $|H \cap \mathcal{S}_k[h']| \cong n$.

(10) The set $\{x: x \in H \text{ and } x \subseteq h' \text{ for some } h' \in Z(h)\}$ has at most $\binom{(k-1)s}{k} \left(\frac{N}{n} \right)^s \log n$ elements.

Considering that this number is smaller than n , for sufficiently large n , omitting the elements of the set defined in (10) from H we obtain a set H' , such that the set-system $\mathcal{H} = \langle h, H' \rangle$ has no independent set of $[n^{1-\varepsilon_k s}]$ elements, by (9), and is s -circuitless by (10). This proves 13.3.

The question arises how large can s be as a function of n so that our set-system is s -circuitless, and the chromatic number is still unbounded. Our proof gives that if $s = o(\log n)$ then the chromatic number can be unbounded; we omit the details.

To show that this best possible we outline the proof of the following

THEOREM 13.6. *Let $\mathcal{H} = \langle h, H \rangle$ be a uniform set system $\alpha(\mathcal{H}) = n$, $\alpha(\mathcal{H}) = k$. Assume that there is a real number $c > 0$ such that \mathcal{H} is s -circuitless for some $s \cong c \log n$. Then there exists an integer $m_0(c)$ such that $\text{Chr}(\mathcal{H}) \cong m_0(c)$ for every n . As a corollary of this \mathcal{H} contains a free set of $\cong \left\lfloor \frac{n}{m_0(c)} \right\rfloor$ elements.*

To prove 13.6 it is convenient to make the following

DEFINITION 13.7. Let $\mathcal{H} = \langle h, H \rangle$ an arbitrary set-system. We say that \mathcal{H} has *quasi-colouring number* β if β is the least cardinal for which there exists a well-ordering $<$ of h satisfying the following condition.

Whenever $V \subseteq V(x, h) < x$, \mathcal{H} and $A \cap B = \{x\}$ for every $A \neq B \in V$ then $|V| < \beta$. The quasi-colouring number of \mathcal{H} will be denoted by $\text{Col}^*(\mathcal{H})$.

As an immediate consequence of the definitions we have

$$\text{Col}^*(\mathcal{H}) \cong \text{Col}(\mathcal{H})$$

and as an easy generalization of 3.1 one can prove

THEOREM 13.8. *Under the conditions of 3.1*

$$\text{Chr}(\mathcal{H}) \cong \text{Col}^*(\mathcal{H}).$$

13.8 is to be seen quite similarly to 3.1. We omit the details.

If \mathcal{H} is a graph then obviously $\text{Col}(\mathcal{H}) = \text{Col}^*(\mathcal{H})$. For general set-systems in some sense, the quasi-colouring number seems to be a more appropriate generalization of the colouring numbers of graphs than the ordinary colouring number defined in 2.9. Throughout this paper we preferred the ordinary colouring number because the quasi colouring number fails to possess some simple and important properties of the former one. For example theorems 3.2 and 3.3 are no longer true for quasi colouring numbers. We omit the simple but not entirely trivial example we have for this fact. Though a detailed examination of the quasi colouring number might be useful, in this paper we are going to use it only in the proof of 13.6.

PROOF OF 13.6 (in outline). Considering that by 13.8, $\text{Chr}(\mathcal{H}) \cong \text{Col}^*(\mathcal{H})$ it is sufficient to see that $\text{Col}^*(\mathcal{H})$ is bounded.

Assume $\text{Col}^*(\mathcal{H}) \cong m$. It is easy to see by induction on $\alpha(\mathcal{H})$ that then for every $x \in h$ there exists a $V_x \subseteq V(x, \mathcal{H})$ such that $|V_x| \cong m$ and $A \neq B \in V_x$ implies $A \cap B = \{x\}$.

Let x_0 be an arbitrary element of h .

Define $V(i)$ by induction on i as follows

$$V(0) = \{x_0\} \cup \bigcup V_{x_0}$$

$$V(i+1) = V_i \cup \bigcup_{x \in V_i} (\bigcup V_x).$$

Considering that \mathcal{H} is $[c \log n]$ circuitless it is easy to see that then

$$n \cong |V([c \log n])| \cong ((m-1)(k-1))^{[c \log n]-1}.$$

Hence $m \cong m_0(c)$.

In connection with the problems considered so far the following problem arises:

Let n be large, and let $\mathcal{H} = \langle h, H \rangle$ be a uniform set-system with $\alpha(\mathcal{H}) = k$, $\alpha(\mathcal{H}) = n$ and such that if $x \neq y \in H$ then $|x \cap y| \leq 1$ i. e., H has property **C(2, 2)**. How large does k have to be in order that the system should have property **B** (i. e., chromatic number 2)?¹⁰

The same question can be asked if we only assume that H has property **C(2, r)** for $2 \leq r < k$.

The following theorem shows that the right order of magnitude for k is $c \log n$.

¹⁰ As far as we know T. GALLAI raised the problem if there exist k , and n such that \mathcal{H} satisfies the requirement of this problem. An affirmative answer is given in [18]. 13.9 seems to be a stronger result in this respect.

THEOREM 13. 9. For sufficiently large n there exists a uniform set-system $\mathcal{H} = \langle h, H \rangle$ such that $\alpha(\mathcal{H}) = 2^{10n}$, $\chi(H) = n$, $|H| = 2^{11n}$, H has property $C(2, 2)$ and for every independent subset $h' \subseteq h$, $|h'| \leq \frac{2^{10n}}{2}$.

As a corollary of this H does not possess property B .

The proof can be carried out using the probabilistic method described above. We omit the details.

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