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ON CLIQUES IN GRAPHS

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ABSTRACT

A clique is a maximal complete subgraph of a graph. Moon and Moser obtained bounds for the maximum possible number of cliques of different sizes in a graph of n vertices. These bounds are improved in this note.

Let $G(n)$ be a graph of n vertices. A non empty set S of vertices of G forms a complete graph if each vertex of S is joined to every other vertex of S. A complete subgraph of G is called a clique if it is maximal i.e., if it is not contained in any other complete subgraph of G.

Denote by $g(n)$ the maximum number of different sizes of cliques that can occur in a graph of n vertices. In a recent paper [1] Moon and Moser obtained surprisingly sharp estimates for $g(n)$. In fact they proved (throughout this paper $\log n$ will denote logarithm to the base 2) that for $n \geqq 26$

$$n - [\log n] - 2[\log\log n] - 4 \leqq g(n) \leqq n - [\log n] \tag{1}$$

In the present note we shall improve the lower bound on $g(n)$. Denote by $\log_k n$ the k-times iterated logarithm and let $H(n)$ be the smallest integer for which $\log_{H(n)} n < 2$. Let $n_1 = [n - \log n - H(n)]$ and for $i > 1$ define n_i as the least integer satisfying

$$2^{n_i} + n_i - 1 \geqq n_{i-1}. \tag{2}$$

Now we prove the following

THEOREM. $g(n) \geqq n - \log n - H(n) - 0(1)$.

$H(n)$ increases much slower then the k-fold iterated logarithm thus our theorem is an improvement on (1). It seems likely that our theorem is very close to being best possible but I could not prove this. In fact I could not even prove that

$$\lim_{n=\infty} (g(n) - (n - \log n)) = \infty.$$

The proof of our theorem will use the method of Moon and Moser [1]. We construct our graph $G(n)$ as follows: The vertices of our $G(n)$ are $x_1, \cdots, x_{n_1}$; $y_1, \cdots y_{n_2}$; $z_1 \cdots z_m$, where $n_1 = [n - \log n - H(n)]$, n_2 is defined by (2) and $m = n - n_1 - n_2$. Clearly $m = H(n) + 0(1)$. Any two x's and any two y's are joined. Further for $1 \leqq j < n_2$ y_j is joined to every x_i except to the x_i satisfying

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$$2^{j-1}+j-2 < i \leqq 2^j + j - 1$$

and y_{n_2} is joined to every x_i except to those satisfying

$$2^{n_2-1} + n_2 - 2 < i \leqq n_1 \ (n_1 \leqq 2^{n_2} + n_2 - 1).$$

Now we use the vertices z_k, $1 \leqq k \leqq m$, z_k is joined to y_j for $1 \leqq j \leqq n_{k+2}$ and to the x_i for $1 \leqq i \leqq n_{k+1}$. No two z's are joined. This completes the definition of our $G(n)$.

Now we show that our $G(n)$ contains a clique for every

$$n_{m+2} < t \leqq n_1 \tag{3}$$

and since by $m = H(n) + 0(1)$ and (2) n_{m+2} is less than an absolute constant independent of m, (3) implies our Theorem.

Assume first $n_2 \leqq t \leqq n_1$. For $t = n_2$ the set of all y's and for $t = n_1$ the set of all x's gives the required cliques. For $n_1 < t < n_2$ we construct our clique of t vertices as follows: We distinguish two cases. If $n_1 - t < 2^{n_2-1}$ we consider the unique binary expansion

$$n_1 - t = 2^{j_1} + \cdots + 2^{j_r}, \ 0 \leqq j_1 < \cdots < j_r < n_2 - 1.$$

If $2^{n_2-1} \leqq n_1 - t < 2^{n_2}$ (this last inequality always holds by the definition of n_1 and n_2) we consider the unique binary expansion

$$n_1 - t - (n_1 - 2^{n_2-1} - n_2 + 2) = 2^{n_2-1} + n_2 - 2 - t = 2^{j_1} + \cdots + 2^{j_r},$$
$$0 \leqq j_1 < \cdots j_r < n_2 - 1.$$

In the first case consider the clique determined by $y_{j_1}, \cdots, y_{j_r}$ and all the x's which are joined to all the y_{j_u}, $u = 1, \cdots, r$, in the second case we consider the clique determined by $y_{j_1}, \cdots, y_{j_r}$, y_{n_2} and all the x's joined to y_{n_2} and to all the y_{j_u}, $u = 1, \cdots, r$. A simple argument shows that this construction gives a clique having t vertices. (To see this observe that y_j, $1 \leqq j < n_2$ is joined to $n_1 - 2^{j-1} - 1$ x's and y_{n_2} is joined to $2^{n_2-1} + n_2 - 2$ x's and no x is joined to every vertex of our clique since no z is joined to y_{n_2} or to an x which is not joined to y_{n_2}).

Let now $n_{s+2} + 1 \leqq t \leqq n_{s+1} + 1$, $0 < s \leqq m$. If $t = n_{s+2} + 1$ then the complete graph having the vertices z_s, $y_1, \cdots, y_{n_{s+2}}$ is a clique of size t (no x is joined to all these vertices), if $t = n_{s+1} + 1$ then the complete graph having the vertices z_s, $x_1, \cdots x_{n_{s+1}}$ is a clique of size t (no y is joined to all these vertices). If $n_{s+2} + 1 < t \leqq n_{s+1}$ we consider the graph spanned by the vertices $z_s, y_1, \cdots, y_{n_{s+2}}$, $x_1, \cdots, x_{n_{s+1}}$ (z_s is joined to all these x's and y's) and argue as in the case $s = 0$. This completes the proof of (3) and of our Theorem.

It would be easy to replace 0(1) by an explicit inequality, but I made no attempt to do so since it is uncertain to what extent our Theorem is best possible.

Reference

1. J. W. Moon and L. Moser, *On cliques in graphs*, Israel J. of Math. 3 (1965), 23–28.

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