Reprinted from JOURNAL OF COMBINATORIAL THEORY
Vol. 1, No. 1, June 1966
Printed in Italy

NOTES

On the Construction of Certain Graphs

Denote by $G(n)$ a graph of n vertices and by $G(n;\, m)$ a graph of n vertices and m edges. $I(G)$ denotes the cardinal number of the largest independent set of vertices (i.e., the largest set $x_{i1}, \ldots, x_{r1}$, $r = I(G)$ of vertices of G no two of which are joined by an edge). $v(x)$, the valency of the vertex x, denotes the number of edges incident to x, $c_1 \ldots$ will denote positive absolute constants.

1. Turán [8] proved that every $G(n;\, [n^2/4] + 1)$ contains a triangle and that the only graph $G(n;\, [n^2/4])$ which does not contain a triangle is defined as follows: Its vertices are $x_1, \ldots, x_{[n/2]};\ y_1, \ldots, y_{[(n+1)/2]}$, and its edges are (x_i, y_j), $1 \le i \le [n/2]$, $1 \le j \le [(n+1)/2]$; in other words if $G(n;\, [n^2/4])$ does not contain a triangle then

$$I\left(G\left(n;\left[\frac{n^2}{4}\right]\right)\right) = \left[\frac{n+1}{2}\right].$$

Andrásfai [1] has investigated the following question: Let $u < [(n+1)/2]$. Determine the largest integer $f(n, u)$ for which there is a $G(n; f(n, u))$ which contains no triangle and for which $I(G) \le u$. Andrásfai determines $f(n, u)$ for $u \ge [2n/5]$. It is clear that $f(n, u) \le un/2$ since the $v(x)$ vertices joined to x must be independent (for otherwise our graph would contain a triangle); hence $v(x) \le u$ for all vertices of G thus G has at most $un/2$ edges. Andrásfai [1] in fact determines all graphs for which

$$f(n, u) = un/2 \tag{1}$$

for $u \ge [2n/5]$ and gives some examples of graphs satisfying (1) for $u > n/3$.

In the present note, I will construct graphs for which (1) holds and

$$u = I(G) = n^{1-c+o(1)}\ , \qquad c = \frac{5 \log 2 - 3 \log 3}{2 \log 2}. \tag{2}$$

Denote by $g(n)$ the largest integer so that every graph of n vertices which contains no triangle satisfies $I(G\ (n)) \geq g(n)$. A very special case of the well-known theorem of Ramsay [7] implies $g(n) \to \infty$ as $n \to \infty$. Szekeres and I [2] proved that $g(n) \geq \sqrt{2n} + O(1)$ and I showed first by a direct construction that $g(n) < n^{1-c_1}$ [3] and later by a "probabilistic" method that $g(n) < x_2 n^{1/2} \log n$. I cannot at present decide whether $g(n) < c_3 n^{1/2}$ is true, in fact perhaps $g(n) = \sqrt{2n} + O(1)$. It would be of interest to construct all graphs satisfying (1) — this may be difficult or impossible — or at least to decide if (1) is possible if $u = n_{1/2+\varepsilon}$ I cannot even show that $f(n, u) = (1 + o(1))\ un/2$ can hold if $u = n_{1/2+\varepsilon}$. The construction given here does not seem to help to settle this problem. The construction given in [3] only yields $f(n, u) = (1 + o(1))\, un/2$ and not (1) for $u > n^{1-c_1}$.

I conjectured and Kleitman [6] proved the following result: Denote by $\{A_i\}\ 1 \leq i \leq 2^n$ the 2^n sequences of 0's and 1's of length n. Put $A_i = (\varepsilon_1^{(i)}, \ldots, \varepsilon_n^{(i)})$, $(\varepsilon_n^{(i)} = 0$ or $1)$. Define

$$d(A_i, A_j) = \sum_{r=1}^{n} \mid \varepsilon_r^{(i)} - \varepsilon_r^{(j)} \mid .$$

Let $A_{i_1}, \ldots, A_{i_s}$ be a family of sequences satisfying

$$d(A_{i_u},\ A_{i_v}) \leq 2k,\ \ k < n/2,\ \ 1 \leq u < v \leq s.$$

Then

$$\max s = \sum_{l=0}^{k} \binom{n}{l} \tag{3}$$

We have equality in (3) if the A's are the sequences having at most k l's.

Using Kleitman's theorem we now construct our graphs as follows: Put $n = 3k + 1$. The vertices of our graph will be the sequences $\{A_i\}$, $1 \leq i \leq 2^n$; A_i and A_j are joined if and only if

$$d(A_i,\ A_j) \geq 2k + 1.$$

Our graph has 2^{3k+1} vertices and $2^{3k}\sum_{i=0}^{k}\binom{3k+1}{i}$ edges. It is easy to see that our graph contains no triangle. To see this, observe that if it would contain a triangle we could assume without loss of generality that one of its vertices has all its coordinates 0, i.e., is (0, ..., 0). The other two vertices must be sequences containing at least $2k+1$ ones and hence they must coincide in at least $k+1$ places, or their distance is $\leq 2k$; thus they are not joined. In other words our graph contains no triangle. The valency of each vertex of our graph clearly equals

$$\sum_{i=0}^{k}\binom{3k+1}{i}.$$

On the other hand if $A_{i_1}, \ldots, A_{i_s}$ is an independent set of vertices we must evidently have $d(A_{i_u}, A_{i_v}) \leq 2k$ (for if not then by definition A_{i_u} and A_{i_v} are joined and the set was not independent). But then by the theorem of Kleitman

$$\max s = \sum_{i=0}^{k}\binom{3k+1}{i} = V(X_i), \qquad 1 \leq i \leq 2^{3k+1}.$$

In other words, $I(G) = V(x_i)$, $1 \leq i \leq 2^{3k+1}$, and thus (1) holds for our graph. A simple computation using Stirling's formula shows that (2) is also satisfied.

This construction could be generalized if the following generalization of Kleitman's result would hold: Let $t_r \geq 1$, $1 \leq r \leq n$, and denote by $\{B_i\}$, $1 \leq i \leq \prod_{r=1}^{n}(t_r+1)$, the sequences of the form $(\delta_1, \ldots, \delta_n)$, $0 \leq \delta_r \leq t_r$. Let $B_i = (\delta_1^{(i)}, \ldots, \delta_n^{(i)})$, $B_j = (\delta_1^{(j)}, \ldots, \delta_n^{(j)})$, define $d(B_i, B_j) = \sum_{r=1}^{n} |\delta_1^{(r)} - \delta_j^{(r)}|$. Let $k < \frac{1}{2}\sum_{r=1}^{n} t_r$ and let $B_{i_1}, \ldots, B_{i_s}$ be a family of sequences satisfying

$$d(B_{i_u}, B_{i_v}) \leq 2k, \quad 1 \leq u < v \leq s.$$

Then s is maximal if the B_{i_u} are the sequences satisfying $\sum_{r=1}^{n}\delta_r \leq k$. But even if this would be true we could not improve (2) by this method.[1]

[1] Kleitman showed that this generalization is false, but perhaps it holds if all the t_r's are equal.

2. A graph is called k-chromatic if its vertices can be split into k classes so that no two vertices of the same class are joined, but such a splitting is not possible into fewer than k classes. Tutte and Zykov were the first to show that for every integer k there is a k-chromatic graph which contains no triangle. Rado and I [5] showed that for every infinite cardinal m there is a graph of m vertices which contains no triangle and which has chromatic number m.

A very simple and intuitive proof of this result could be given if the following conjecture of Czipszer and myself would hold: Is it true that the unit sphere of an m-dimensional Hilbert space is not the union of fewer than m subsets of diameter less than $2 - \varepsilon$. The unit sphere of the m-dimensional Hilbert space is the set of all transfinite sequences of real numbers $\{x_\alpha\}$ where α runs through an index set of power m and $\Sigma_\alpha x_\alpha{}^2 \leq 1$ (all but denumerably many of the x_α's are 0). As far as I know this conjecture has not even been settled for $m = \aleph_1$.

If the answer to our conjecture is affirmative our graph can be constructed as follows: The vertices of our graph are the sequences $\{x_\alpha\}$, $\Sigma_\alpha x_\alpha{}^2 \leq 1$, where all the x_α are rational and only a finite number of them are different from 0. Clearly our graph has m vertices and the points of the m-dimensional unit sphere defined by these vertices are dense in the unit sphere. Two vertices are joined if their distance (in the m-dimensional Hilbert space) is greater than $\sqrt{3}$. Clearly this graph contains no triangle and the diameter of any independent set is $\leq \sqrt{3}$. Thus if the answer to our conjecture is affirmative, the vertices of our graph cannot be split into the union of fewer than m independent sets, i.e., our graph is m-chromatic.

References

1. B. Andrasfai, Graphentheoretische Extremalprobleme, *Acta Math. Acad. Sci. Hungar.* **15** (1964), 413–438.
2. P. Erdös and G. Szekeres, A Combinatorial Problem in Geometry, *Compositio Math.* **2** (1935), 463–470.
3. P. Erdös, Remarks on a Theorem of Ramsay, *Bull. Res. Council Israel Sect. F* **7** (1957), 21–24.
4. P. Erdös, Graph Theory and Probability, II. *Canad. J. Math.* **13** (1961), 346–352.
5. P. Erdös and R. Rado, A Construction of Graphs without Triangles Having Pre-assigned Order and Chromatic Number, *J. London Math. Soc.* **35** (1960), 445–448.

6. D. J. KLEITMAN, Families of Non-disjoint Subsets, *Journal of Combinatorial Theory* **1**, (1966) 153-155.

7. F. P. RAMSAY, Collected papers, pp. 82–111; see also reference 2.

8. P. TURAN, On the Theory of Graphs, *Colloq. Math.* **3** (1954), 19–30.

P. ERDÖS,
Panjab University
Chandigarh, India

Printed in Italy by I.T.E.C. - Milan