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ON THE SOLVABILITY OF SOME EQUATIONS IN DENSE SEQUENCES OF INTEGERS

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In a previous paper [1], making use of a simple combinatorial result of Kleitman [4], we showed that if $a_1 < a_2 < \cdots$ is an infinite sequence of integers for which there are infinitely many x satisfying the inequality $Ax = \sum_{a_i \le x} 1/a_i > c_1 (\log x)/(\log\log x)^{1/2}$, then the equations $(a_i, a_j) = a_r$, $r < i < j$, $[a_{i_1}, a_{j_1}] = a_{r_1}$, $i_1 < j_1 < r_1$, have infinitely many solutions. We also showed that this theorem cannot be improved in a specific sense, namely that the constant c_1 cannot be replaced by an arbitrarily small constant. More precisely, we constructed a sequence satisfying the hypothesis

$$\sum_{a_i \leqslant x} 1 > c_2 x/(\log\log x)^{1/2}, \tag{1}$$

but nevertheless the equation $[a_{i_1}, a_{j_1}] = a_{r_1}$, $i_1 < j_1 < r_1$, is not solvable.

In the present paper, $c, c_1, c_2, \cdots$ will denote absolute constants; p denotes primes; $P(n)$ is the greatest and $p(n)$ the smallest prime factor of n. Denote the sequence $a_1 < a_2 < \cdots$ by A.

We shall say that the sequence $u_1 < u_2 < \cdots$ *possesses property* I if the equation $u_i q = u_j$, $p(q) > P(u_i)$ has no solutions.

In this paper we shall show that the behavior of the equation $(a_i, a_j) = a_r$ is completely different from that of the equation $[a_i, a_j] = a_r$.

We shall prove the following theorem.

Theorem. *Let $a_1 < \cdots$ be a sequence of integers for which the equation*

$$(a_i, a_j) = a_r, \quad r < i < j, \tag{2}$$

has no solutions. Then

$$\sum \frac{1}{a_i \log a_i} < c. \tag{3}$$

We shall make a few preliminary comments. By means of partial summation, we easily find from the theorem in our paper [2] that if equation (2) has no solutions, then for every k we have the equality

$$\liminf_{x\to\infty} \sum_{a_i \leqslant x} \left(\frac{x}{\prod_{r=2}^{k} \log_r x} \right)^{-1} = 0$$

($\log_r x$ denotes the rth iteration of the logarithm).

Therefore relations similar to (1) cannot exist in this case.

The sequence $b_1 < \cdots$ is called *primitive* if there exists no number dividing all the remaining terms of the sequence. It is well known [3] that for every primitive sequence we have the inequality

$$\sum \frac{1}{b_i \log b_i} < c_3, \tag{4}$$

* Editor's note. The present translation incorporates suggestions made by the authors.

and also (see [2]) the equation

$$\lim_{x=\infty} \sum_{b_i \leqslant x} \frac{1}{b_i} \left(\frac{\log x}{(\log \log x)^{1/2}} \right)^{-1} = 0, \tag{5}$$

and this relation cannot be refined.

We prove that if $a_1 < a_2 < \cdots$ is an infinite sequence for which equation (2) is not solvable, then

$$\lim_{x \to \infty} \sum_{a_i \leqslant x} \frac{1}{a_i} \left(\frac{\log x}{(\log \log x)^{1/2}} \right)^{-1} = 0. \tag{6}$$

The proof of equation (6) is rather complex, and we shall come back to it later. The relations (3), (4), (5), and (6) prompt the following question. Let $b_1 < b_2 < \cdots$ be an infinite primitive sequence. Do there exist a constant $c > 0$ and a sequence $a_1 < \cdots$ for which equation (2) is not solvable and $a_n \ll b_n^2$? We are unable to answer this question.

Now let us consider the proof of the theorem. We shall make use of the following lemma due to Alexander.

Lemma 1. *Let $a_1 < a_2 < \cdots$ be a sequence with Property* I. *Then*

$$\sum_i \frac{1}{u_i \log u_i} < c_4. \tag{7}$$

If $u_i \nmid u_j$ (i.e. if the sequence $u_1 < u_2 < \cdots$ is primitive), then the inequality (7) is proved in [3]. The proof of Lemma 1 resembles the proof given in [3], but for the sake of completeness we shall sketch it here. We easily see that condition I means (see [3]) that $u_i q = u_j q'$, $p(q) > P(u_i)$, $p(q') > P(u_j)$.

Making use of the sieve of Eratosthenes, we conclude that the number of integers $u_i q \leq x$, $p(q) \geq P(u_1)$, is greater than

$$\prod_{p \leqslant P(u_i)} \left(1 - \frac{1}{p}\right) - 2^{u_i}. \tag{8}$$

From (8) we easily obtain the inequality

$$\sum_i \prod_{p \leqslant P(u_i)} \left(1 - \frac{1}{p}\right) \Big/ u_i \leqslant 1, \tag{9}$$

whence, with the use of Mertens' theorem,

$$\prod_{p<y} \left(1 - \frac{1}{p}\right) < c/\log y,$$

follows the proof of our lemma.

We now define a subsequence $A(a_i)$ of the sequence A in the following manner: a_j belongs to $A(a_i)$ if a_i is the largest a for which the equation $a_j = a_i q$, $p(q) > P(a_i)$, is solvable. Let A' be a subsequence of the sequence A which is not included in any subsequence $A(a_i)$. Clearly $A = A' \cup_{i=1} A(a_i)$. Therefore

$$\sum_k \frac{1}{a_k \log a_k} = \sum_{a_k \text{ in } A'} \frac{1}{a_k \log a_k} + \sum_{i=1}^{\infty} \sum_{a_k \text{ in } A(a_i)} \frac{1}{a_k \log a_k}. \tag{10}$$

Evidently the subsequence A' possesses Property I. Thus, by virtue of Lemma 1, we have the inequality

$$\sum_{a_k \text{ in } A'} \frac{1}{a_k \log a_k} < c_4. \tag{11}$$

We now prove Lemma 2.

Lemma 2.

$$\sum_{a_k \text{ in } A(a_i)} \frac{1}{a_k \log a_k} < \frac{c_5}{a_i P(a_i)^{1/2}}.$$

It is easily seen ($q_1 < q_2 < \cdots$ ranges over the set of all primes) that

$$\sum \frac{1}{n\,(P(n))^{1/2}} = \sum_{m=1}^{\infty} \frac{1}{q_m^{1/2}} \prod_{i=1}^{m} \left(1 + \frac{1}{q_i}\right) < \sum_{m=1}^{\infty} \frac{c \log q_m}{q_m^{3/2}} < \infty.$$

Our Theorem 1, therefore, follows immediately from (10), (11), and Lemma 2. To complete the proof it remains only to prove Lemma 2. Let $a_i q_r^i$, $r = 1, \cdots, p(q_r^{(i)}) > P(a_i)$, be integers of the subsequence $A(a_i)$. Clearly, the subsequence $q_r^{(i)}$ possesses property I. If it did not, and if $q_{r_2}^{(i)}/q_{r_1}^{(i)}$ is an integer satisfying the inequality $p(q_{r_2}^{(i)}/q_{r_2}^{(i)}) > P(q_{r_1}^{(i)})$, then $a_i q_{r_2}^{(i)}$ (which belongs to the subsequence $A(a_i)$) can be written in the form $a_l q$, $P(q) > P(a_l)$, $a_l = a_i q_r^{(i)}$, $q_{r_2}^{(i)}/q_{r_1}^{(i)} = q$, in contradiction with the maximality of a_i.

We now show that there exist no two coprimes $q_r^{(i)}$. In order to see this, we first of all make use of the fact that equation (2) has no solutions. Namely, assuming that $(q_{r_1}^{(i)}, q_{r_2}^{(i)}) = 1$, we find $(a_i q_{r_1}^{(i)}, a_i q_{r_2}^{(i)}) = a_i$. In other words, equation (2) has a solution, which contradicts our assumption.

Lemma 3. *Let the sequence* $q_1 < \cdots$ *possess Property* I, $(q_i, q_j) \neq 1$, $p(q_i) > t$. *Then*

$$\sum_i \frac{1}{q_i \log q_i} \leqslant c_5 / t^{1/2}.$$

The correctness of Lemma 2 follows immediately from Lemma 3, Since

$$\sum_{a_k \text{ in } A(a_i)} \frac{1}{a_k \log a_k} = \sum_r \frac{1}{a_i q_r^{(i)} \log a_i q_r^{(i)}} \leqslant \frac{1}{a_i} \sum_r \frac{1}{q_r^{(i)} \log q_r^{(i)}} < c_5 / a_i p(a_i)^{1/2}.$$

Thus there remains only to show the correctness of Lemma 3. It is highly probable that Lemma 3 is not the strongest one possible and that the expression $c_5/t^{1/2}$ may be replaced by c_5/t.

For the proof of Lemma 3 let us first assume that there exists an i for which

$$\sum_{p \mid q_i} \frac{1}{p} \leqslant \frac{1}{t^{1/2}}. \tag{12}$$

Since there exist no two coprimes q, then every q^r must be divisible by at least some p, where $p \mid q_i$. Hence

$$\sum_r \frac{1}{q_r \log q_r} \leqslant \sum_{p \mid q_i} \frac{1}{p} \sum' \frac{1}{q_{r/p} \log q_r}, \tag{13}$$

where the stroke indicates the summation ranges over all q such that $p \mid q$. The sequence q_r/p clearly possesses Property I (except for the fact that one of the numbers q_r/p may be unity). Hence, by virtue of Lemma 1,

$$\sum' \frac{1}{q_r \log q_r} < 1 + c_3. \tag{14}$$

From inequalities (12), (13), and (14), we find

$$\sum_r \frac{1}{q_r \log q_r} < (1+c_3) \sum_{p|q_i} \frac{1}{p} \leqslant \frac{1+c_3}{t^{1/2}}.$$

which proves the lemma.

To complete our proof let us now assume that inequality (12) does not hold for q_r. Let l be an integer and $x > x_0(l)$ large. Consider the integers which do not exceed x by $q_r(t)$, where all the prime factors of t are larger than q_r. Since the sequence q_r possesses property I, we find, just as in Lemma 1, that the integers

$$q_r m, \quad r = 1, 2, \ldots, l, \quad m < x / q_r, \tag{15}$$

are distinct. Denote the numbers of the form (15) by $u_1, u_2, \cdots, u_s$. We find, by virtue of Mertens' Theorem and the sieve of Eratosthenes, that

$$s = (1 + O(1)) \sum_{r=1}^{l} \frac{x}{q_r} \prod_{p=P(q_r)} \left(1 - \frac{1}{p}\right) > Cx\left(\sum_r \frac{1}{q_r \log q_r}\right) + O(x). \tag{16}$$

Clearly, all the prime factors of u are greater than t, and since inequality (12) does not hold, we have

$$\sum_{p|u_i} \frac{1}{p} > \frac{1}{t^{1/2}}.$$

Hence on the one hand

$$\sum_{i=1}^{s} \sum_{p|u_i} \frac{1}{p} > \frac{s}{t^{1/2}}, \tag{17}$$

and on the other

$$\sum_{i=1}^{s} \sum_{p|u_i} \frac{1}{p} < \sum_{u=1}^{x} \sum_{\substack{p|n \\ p>t}} \frac{1}{p} < \sum_{p>t} \frac{x}{p^2} < \frac{x}{t}. \tag{18}$$

Thus from inequalities (17) and (18) we find the inequality

$$s < x / t^{1/2}. \tag{19}$$

Therefore, inequalities (16) and (19) lead to the inequality

$$\sum_{r=1}^{l} \frac{1}{q_r \log q_r} < c_5 t^{1/2}, \tag{20}$$

and since the last inequality holds for every l the proof of Lemma 3, and therefore of the theorem, is complete.

Our proof does not make use of the combinatorial result of Kleitman [4]. We do not know how to deal with the equation $[a_i, a_j] = a_r$ without making use of Kleitman's result.

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