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ON DECOMPOSITION OF GRAPHS

By

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§ 1. Introduction. Notations

We are going to use the notations introduced in our paper [1]*, § 2. A graph $\mathcal{G}$ is an ordered pair $\langle g, G\rangle$ where the elements of the sets g and G are the vertices and the edges of $\mathcal{G}$ respectively. We assume that the reader is familiar to the usual terminilogy of graph-theory. The aim of this paper is to consider two kinds of decompositions of graphs called vertex- and edge-decompositions respectively.

DEFINITION 1. 1. Let $\mathcal{G}=\langle g, G\rangle$ be a graph and let $\mathcal{G}_\xi=\langle g_\xi, G_\xi\rangle$, $\xi<\zeta$ be a sequence of type ζ of graphs.

The sequence $\mathcal{G}_\xi$, $\xi<\zeta$ is said to be a *vertex decomposition* of $\mathcal{G}$ if the g_ξ are disjoint, $\bigcup_{\xi<\zeta} g_\xi=g$ and $\mathcal{G}_\xi$ is the subgraph $\mathcal{G}(g_\xi)$ of $\mathcal{G}$ spanned by g_ξ in $\mathcal{G}$.

The sequence $\mathcal{G}_\xi$, $\xi<\zeta$ is said to be an *edge-decomposition* of $\mathcal{G}$ if $g_\xi=g$ for every $\xi<\zeta$, the G_ξ are disjoint and $\bigcup_{\xi<\zeta} G_\xi=G$.

The cardinal number $|\zeta|$ will be said the *type of the decomposition*, and the graphs $\mathcal{G}_\xi$ will be called the *members of the decomposition* in both cases.

We mention that the expression decomposition is usually used for the edge decompositions, in some cases vertex decompositions are called colourings.

Our problems will be of the following type. Let $\mathcal{G}$ be a graph and Φ a property of the graph usually expressing that $\mathcal{G}$ is "small" in a certain sense. Let further Φ' be a stronger property usually expressing that the graph having property Φ' is even "smaller". We investigate the problem if then $\mathcal{G}$ necessarily has vertex- or -edge-decompositions of relatively small types where the members of the decomposition all have property Φ'.

We will investigate these problems in details, when Φ and Φ' are properties expressing that $\mathcal{G}$ does not contain complete α-graphs. We also have results when the properties in question are that $\mathcal{G}$ does not contain rectangles or is a tree. We are going to discuss the different problems in different sections and we give a short summary of the results there. We have very little information on edge-decomposition problems. We discuss some obviously not final results for them since we think that some of the open problems are fundamental. Though we have been motivated mostly by infinite graphs when starting this paper almost all the problems are relevant for finite graphs, and there remain some interesting unsolved problems for finite graphs too.

* These are mostly the usual notations of set theory. We mention that ordinals are introduced so that every ordinal is the set of smaller ordinals. In some parts of the paper we consider finite problems and we use negative integers as well. In these parts we naturally do not assume that an integer is the set of smaller integers.

§ 2. The decomposition problems for graphs characterized by complete subgraphs. Further notations and definitions. Preliminaries

DEFINITION 2. 1. Let $\beta(\mathcal{G})$ denote the least cardinal number for which the graph $\mathcal{G}$ does not contain a complete β-graph $[\beta]$.

We have obviously $\beta(\mathcal{G})\geqq 2$ if $g\neq 0$ and $\beta([\beta])=\beta^+$. $\beta(\mathcal{G})=2$ iff $g\neq 0$ and $\mathcal{G}$ has no edges. These graphs will be called *independent graphs*.

We remind the reader that by [1] 2. 1 $\alpha(\mathcal{G})$ denotes the cardinal $|g|$.

We are going to consider the following problems involving four cardinals $\alpha, \beta, \gamma, \delta$. Is it true that every graph $\mathcal{G}$ with $\alpha(\mathcal{G})=\alpha$, $\beta(\mathcal{G})\leqq\beta$ has a vertex-decomposition or an edge-decomposition $\mathcal{G}_\xi$, $\xi<\gamma$ of type γ such that $\beta(\mathcal{G}_\xi)\leqq\delta$ holds for all members $\mathcal{G}_\xi$, $\xi<\gamma$ respectively?

To have a brief notation we introduce the symbols

$$[\alpha, \beta]\to[\gamma, \delta],\ (\alpha, \beta)\to(\gamma, \delta).$$

DEFINITION 2. 2. $[\alpha, \beta]\to[\gamma, \delta]$, $(\alpha, \beta)\to(\gamma, \delta)$ denote that the answer to the above problem is yes in case of vertex-decomposition or edge-decomposition respectively. As usual $[\alpha, \beta]\not\to[\gamma, \delta]$, $(\alpha, \beta)\not\to(\gamma, \delta)$ denote the negations of the respective statements.

Both symbols are obviously decreasing in the cardinals standing on the left and increasing in the cardinals standing on the right.

We always assume $\beta, \delta\geqq 2$.

The following statements are trivial.

2. 3 a) For every α, β

$$[\alpha, \beta]\to[1, \delta] \quad\text{and}\quad (\alpha, \beta)\to(1, \delta)$$

iff $\delta\geqq\beta$.

b) For every α, β $[\alpha, \beta]\to[\alpha, 2]$ and if $\alpha\geqq\omega$, then $(\alpha, \beta)\to(\alpha, 3)$.

Hence the relavant cases are only $\alpha>\gamma\geqq 2$, $\delta<\beta$. Note that $[\alpha, \beta]\to[\gamma, 2]$ means that each graph $\mathcal{G}$ with $\alpha(\mathcal{G})=\alpha$, $\beta(\mathcal{G})\leqq\beta$ has chromatic number at most γ, hence in case of vertex-decompositions the case $\delta=2$ is very important. On the other hand, we trivially have $(\alpha, \beta)\not\to(\gamma, 2)$ if $\alpha>1$, $\beta\geqq 2$ and γ is arbitrary. Hence for edge-decompositions the simplest relevant case is $\delta\geqq 3$.

There is an obvious connection between the decomposition problems and the Ramsey problems treated in the partition symbol $\alpha\to(\beta_\xi)^r_\gamma$ defined in [2] and rather completely discussed in [3].

Using our present terminology we redefine the special cases $r=1, 2$ of the symbol. The rather trivial case $r=1$ is connected with vertex-decompositions, the case $r=2$ is connected with edge-decompositions.

DEFINITION 2. 4. The symbols $\alpha\to(\beta_\xi)^r_\gamma$ $(r=1, 2)$ denote that the following statements are true.

The complete α-graph $[\alpha]$ has no vertex-decomposition or edge-decomposition $\mathcal{G}_\xi$, $\xi<\gamma$ of type γ satisfying $\beta(\mathcal{G}_\xi)\leqq\beta_\xi$ for $\xi<\gamma$ for $r=1$ or $r=2$ respectively.

$\alpha\not\to(\beta_\xi)^r_\gamma$ denotes the negation of the respective statements.

If all the β_ξ are equal to β we use the notation $\alpha \to (\beta)^r_\gamma$. If $\sum_{j<i} \gamma_j = \gamma$ and γ_j β_ξ equal to β_j we use the notation

$$\alpha \to \big((\beta_0)_{\gamma_0}, \ldots, (\beta_{i-1})_{\gamma_{i-1}}\big)^r_\gamma$$

or

$$\alpha \to \big((\beta_0)_{\gamma_0}, \ldots, (\beta_{i-1})_{\gamma_{i-1}}\big)^r.$$

In case some γ_j is 1 we omit it. As a corollary of the results of [2] (see Theorem 39) corresponding to every sequence $(\beta_\xi)_{\xi<\gamma}$ for every r $(r=1, 2)$ and for every γ there is an α for which $\alpha \to (\beta_\xi)^r_\gamma$ holds.

Let $\alpha(\beta_\xi, \gamma, r)$ denote the least α of this kind. This function is said to be the generalized Ramsey function. We will use for it the same obvious abbreviations as for the corresponding symbols.

As an immediate consequence of the definitions we have

2. 5. Assume $\alpha < \beta$, $\gamma, \delta \geqq 2$. Then

$$[\alpha, \beta] \to [\gamma, \delta] \quad \text{and} \quad (\alpha, \beta) \to (\gamma, \delta)$$

hold iff $\alpha < \alpha(\delta, \gamma, r)$ for $r=1$ and $r=2$ respectively.

The if part holds for every β in both cases.

2. 5 shows that we get new problems only in case $\alpha \geqq \beta$.

As an easy consequence of theorems of [2] and [3] we have

2. 6. A) $\alpha \to (\beta)^1_\gamma$ holds iff $\beta < \alpha$ and $\gamma < \alpha$ or $\beta = \alpha$ and $\gamma < \operatorname{cf}(\alpha)$ provided $\beta > 1$, $\alpha \geqq \omega$.

B) $2^\alpha \nrightarrow (3)^2_\alpha$ for every α.

As a corollary of 2. 5 and 2. 6 we obtain

2. 7.
$$[\alpha, \beta] \to [\operatorname{cf}(\alpha), \alpha]$$
$$(2^\gamma, \beta) \to (\gamma, 3)$$

hold for every β and for every infinite α.

Hence in case of edge-decompositions we have a best possible positive result if $\alpha \leqq 2^\gamma$.

The following lemma establishes a connection between the two types of decompositions.

Lemma 1. *Let $\mathscr{G}$ be a graph which has a vertex-decomposition $\mathscr{G}_\xi$, $\xi < \gamma$ of type γ satisfying $\beta(\mathscr{G}_\xi) \leqq \delta$ for every $\xi < \gamma$. Assume further that $\gamma \nrightarrow (\delta)^2_{\gamma'}$ holds for some γ'. Then $\mathscr{G}$ has an edge-decomposition $\mathscr{G}'_\eta$, $\eta < \gamma'+1$ of type $\gamma'+1$, such that $\beta(\mathscr{G}'_\eta) \leqq \delta$ for every $\eta < \gamma'+1$.*

Proof. By the assumption the complete γ-graph $[\gamma]$ with the set of vertices γ has an edge decomposition $\mathscr{G}''_\eta$, $\eta < \gamma'$ satisfying $\beta(\mathscr{G}''_\eta) \leqq \delta$. For an arbitrary $x \in g$ let $\xi(x)$ be the unique ordinal ξ for which $x \in g_\xi$.

We define the edge-decomposition $\mathscr{G}'_\eta$, $\eta < \gamma'+1$ of $\mathscr{G}$ as follows. Let $\{x, y\} \in G$ be arbitrary

if $\xi(x) \neq \xi(y)$, $\{x, y\} \in G'_\eta$ iff $\{\xi(x), \xi(y)\} \in G''_\eta$ for some $\eta < \gamma'$,

if $\xi(x) = \xi(y)$, $\{x, y\} \in G'_{\gamma'}$.

$\mathscr{G}''_\eta$, $\eta < \gamma'+1$ is obviously an edge-decomposition of type $\gamma'+1$ of $\mathscr{G}$. It is obvious that $\beta(\mathscr{G}''_{\gamma'}) \leqq \delta$ because of $\beta(\mathscr{G}_\xi) \leqq \delta$ for $\xi < \gamma$. On the other hand $\beta(\mathscr{G}'_\eta) \leqq \delta$ since $\beta(\mathscr{G}''_\eta) \leqq \delta$ for $\eta < \gamma'$.

Corollary 1. Assume $[\alpha, \beta] \to [\gamma, \delta]$, $\gamma \leqq 2^{\gamma'}$, $\delta \geqq 3$. Then $(\alpha, \beta) \to (\gamma'+1, \delta)$.

PROOF. By Lemma 1 and by 2. 6. B).

2. 8. Assume that $[\alpha, \beta] \nrightarrow [\gamma, \delta]$ holds for every $\gamma < \alpha$, $\alpha \geqq \omega$. Then there is a graph $\mathcal{G}$, with $\alpha,(\mathcal{G}) = \alpha$, $\beta(\mathcal{G}) \leqq \beta$ such that no vertex-decomposition $\mathcal{G}_\xi$, $\xi < \gamma$ of type $\gamma < \alpha$ of $\mathcal{G}$ satisfies $\beta(\mathcal{G}_\xi) \leqq \delta$ for every $\xi < \gamma$.

PROOF. For every $\gamma < \alpha$ let $\mathcal{G}_\gamma$ be a graph satisfying the requirements of $[\alpha, \beta] \nrightarrow [\gamma, \delta]$. We may assume that the sequence g_γ, $\gamma < \alpha$ is disjointed. Put $\mathcal{G} = \langle g, G \rangle$ $g = \bigcup_{\gamma < \alpha} g_\gamma$, $G = \bigcup_{\gamma < \alpha} G_\gamma$. $\mathcal{G}$ obviously satisfies the requirements of 2. 8.

§ 3. The vertex decomposition problem for graphscharacterized by complete subgraphs

As we have already mentioned $[\alpha, \beta] \rightarrow [\gamma, 2]$ means that if $\alpha(\mathcal{G}) = \alpha$ and $\beta(\mathcal{G}) \leqq \beta$ then $\mathcal{G}$ has chromatic number $\leqq \gamma$. It was proved by P. ERDŐS and R. RADO that for every infinite α

$$[\alpha, 3] \nrightarrow [\gamma, 2] \quad \text{holds for every} \quad \gamma < \alpha$$

in other words by 2. 8 this means that for every infinite α there exists a graph of power α not containing triangles and having chromatic number α (see [4]). Using a well-known compactness argument this also implies that for every finite γ there exists an $\alpha_\gamma < \omega$ for which $[\alpha_\gamma, 3] \nrightarrow [\gamma, 2]$ holds.

This result was obtained by several other people (for references see [1]) and a very good estimation for α_γ is given by ERDŐS [5].

We are going to prove the following generalizations of this theorem.

THEOREM 1. *For every infinite cardinal α and for every integer $\delta \geqq 2$*

$$[\alpha, \delta + 1] \nrightarrow [\gamma, \delta] \quad \textit{holds for every} \quad \gamma < \alpha.$$

THEOREM 2. *For every infinite cardinals α, δ $[\alpha^\delta, \delta^+] \nrightarrow [\gamma, \delta]$ holds for every $\gamma < \alpha$.*

As a corollary we obtain

THEOREM 3. *Assume G. C. H. (generalized continuum hypothesis). Let α be infinite, $\alpha \geqq \beta$, $\alpha > \gamma$, $\beta > \delta \geqq 2$ then $[\alpha, \beta] \nrightarrow [\gamma, \delta]$.*

In view of the preliminaries collected in § 2 this is a best possible negative result which settles all the problems concerning the vertex-decomposition symbol. From Theorem 3 and 2. 8 we obtain

COROLLARY 2. Assume G. C. H., $\alpha \geqq \omega$, $\alpha \geqq \beta$, $\beta > \delta \geqq 2$. Then there exists a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \alpha$, $\beta(\mathcal{G}) = \beta$ such that for every $\gamma < \alpha$ and for every vertex-decomposition $\mathcal{G}_\xi$, $\xi < \gamma$ of type γ of $\mathcal{G}$ we have $\beta(\mathcal{G}_\xi) > \delta$ for some $\xi < \delta$.

In case α is a limit cardinal Corollary 2 is a slightly stronger statement than Theorem 3.

We postpone the proofs of Theorems 1 and 2. First we prove Theorem 3 using these theorems.

PROOF OF THEOREM 3. Considering that $\beta > \delta$ by the monotonicity of our symbol it is sufficient to prove that

$$[\alpha, \delta^+] \nrightarrow [\gamma, \delta]$$

holds for $\alpha \geqq \delta^+$, $\alpha > \gamma$.

If α is regular, then $\alpha^\delta = \alpha$ by G. C. H. and the statement follows from Theorems 1 and 2 for finite and infinite α respectively. If α is singular then $\alpha > \delta^+$, hence there is a regular α' satisfying $\max(\gamma, \delta^+) \leqq \alpha' < \alpha$. $[\alpha', \delta^+] \nrightarrow [\gamma, \delta]$ holds for this α', hence by the monotonicity we have $[\alpha, \delta^+] \nrightarrow [\gamma, \delta]$. Note that the special case $\alpha = \omega$ of Theorem 3 implies

COROLLARY 3. For every finite γ and δ, $\gamma \geqq 2$, $\delta \geqq 2$ there is an $\alpha_{\gamma,\delta} < \omega$ such that

$$[\alpha_{\gamma,\delta}, \delta^+] \nrightarrow [\gamma, \delta].$$

This has been proved previously by P. ERDŐS and A. ROGERS in [6], where a good estimation for $\alpha_{\gamma,\delta}$ is given. We return to the discussion of this result in § 4, where we consider further refinements of the vertex-decomposition problem.

For the proof of Theorems 1, 2 we need some definitions and lemmas.

DEFINITION 3. 1. Let $\alpha, \delta > 0$. We define the usual lexicographical ordering of the set ${}^\delta\alpha$ as follows. Let $f \neq h \in {}^\delta\alpha$, $f \prec_{\alpha,\delta} h$ iff $f_\xi < h_\xi$ for the least ξ for which $f_\xi \neq h_\xi$.

We denote $\text{typ}\, {}^\delta\alpha(\prec_{\alpha,\delta})$ briefly by $\text{typ}\, {}^\delta\alpha$.* In case $\delta = 0$ we put $\text{typ}\, {}^\delta\alpha = 1$.

LEMMA 2. *Assume δ is finite, $\delta \geqq 1$, $\alpha \geqq \omega$, α is regular. Then $\prec_{\alpha,\delta}$ is a well ordering of ${}^\delta\alpha$.*

(A) *Let $\prec_{\alpha,\delta}$ be briefly denoted by $\prec$. Assume $g \subseteqq {}^\delta\alpha$, $\text{typ}\, g(\prec) = \text{typ}\, {}^\delta\alpha$. Let further $g = \bigcup_{\xi<\gamma} g_\xi$ for some cardinal $\gamma < \alpha$. Then there is a $\xi < \gamma$ such that $\text{typ}\, g_\xi(\prec) = \text{typ}\, {}^\delta\alpha$.*

(B) *Let $g \subseteqq {}^\delta\alpha$, $\text{typ}\, g(\prec) = \text{typ}^\delta\, \alpha$. For an arbitrary $\xi < \alpha$ let $a_\xi = \{f \in g \cap {}^\delta\alpha\colon f_0 = \xi\}$. Then the set*

$$\{\xi < \alpha\colon \text{typ}\, a_\xi(\prec) = \text{typ}\, {}^{\delta-1}\alpha\}$$

has power α.

Lemma 2 is well-known and we omit the proof.

We need

LEMMA 3. *Let $1 \leqq \delta < \omega$, $1 \leqq l$, $\alpha \geqq \omega$, α regular. Assume $g \subseteqq {}^\delta\alpha$, $\text{typ}\, g(\prec_{\alpha,\delta}) = \text{typ}\, {}^\delta\alpha$. Let $a_k = \langle i(k), j(k)\rangle$, $k < l\cdot\delta$ be an enumeration of all the pairs $\langle i, j\rangle$, $i < l$, $j < \delta$ such that*

(1) $\quad a_k = \langle i, j\rangle,\ \ a_{k'} = \langle i', j'\rangle,\ \ j < j'$ *implies* $k < k'$ *for every* $i, i' < l$.

Then there exists an increasing sequence ξ_k, $k < l\cdot\delta$ of ordinals $< \alpha$ and a sequence f^i, $i < l$ of elements of g such that

$$f^i_j = \xi_k \quad \textit{if} \quad \langle i, j\rangle = a_k.$$

PROOF. For every $k < l\cdot\delta$ consider the sequence $\langle i(k), j\rangle = a_{k(j,k)}$, $j \leqq j(k)$. Then $k(j, k)$, $j \leqq j(k)$ is an increasing sequence of integers $\leqq k$, $k(j(k), k) = k$.

We define the sequence ξ_k, $k < l\cdot\delta$ of ordinals less than α by induction on k

* Note that in case δ is finite ${}^\delta\alpha$ equals to the ordinary ordinal power α^δ, a notation we can not use since we denoted by α^δ the ordinal power and if $\alpha \geqq \omega$ this equals to α and not to $\text{typ}\, {}^\delta\alpha$.

as follows. Assume that for some $k < l \cdot \delta$ $\xi_{k'}$ is defined for every $k' < k$ in such a way that for the set

$$A_{k'} = \{f \in {}^{\delta}\alpha \cap g : f_j = \xi_{k(j,k')} \quad \text{for} \quad j \leqq j(k')\}$$

$$\text{typ } A_{k'}(\prec_{\alpha,\delta}) = \text{typ}^{\,\delta - j(k') - 1}\alpha \tag{2}$$

holds. Put $k\big(j(k)-1, k\big) = k'$ if $j(k) > 0$. In case $j(k) = 0$ put $j(k') = -1$, $A_{k'} = g$. Then $k(j, k) = k(j, k')$ for $j \leqq j(k) - 1 = j(k')$.

The set $A_{k'}$ has a natural isomorphism on a subset $B_{k'} \subseteq {}^{\delta - j(k') - 1}\alpha$, with respect to the orderings $\prec_{\alpha,\delta}$ and $\prec_{\alpha,\delta - j(k') - 1}$.

It follows from Lemma 2/B that there are α ξ for which

$$\text{typ } \{f \in B_{k'} : f_0 = \xi\} \; (\prec_{\alpha,\delta - j(k') - 1}) = \text{typ}^{\,\delta - j(k') - 2}\alpha.$$

Put $\xi = \xi_k$ for the least ξ of this type satisfying $\xi > \xi_{k-1}$. Then

$$A_k = \{f \in {}^{\delta}\alpha \cap g : f_j = \xi_{k(j,k)} \quad \text{for} \quad j \leqq j(k)\} =$$

$$= \{f \in {}^{\delta}\alpha \cap g : f_j = \xi_{k(j,k)} = \xi_{k(j,k')} \quad \text{for} \quad j \leqq j(k) - 1 \quad \text{and} \quad f_{k(j,k)} = f_k = \xi_k\}.$$

Hence $\text{typ } A_k(\prec_{\alpha,\delta}) = \text{typ}^{\,\delta - j(k') - 2}\alpha = \text{typ}^{\,\delta - j(k) - 1}\alpha$. Thus the increasing sequence ξ_k, $k < l, \delta$ of ordinals is defined and satisfies (2) for every $k < l \cdot \delta$. For every $i < l$ there is a $k < l \cdot \delta$ such that $a_k = \langle i, \delta - 1 \rangle$.

By (1) $\text{typ } A_k(\prec_{\alpha,\delta}) = \text{typ}^{\,0}\alpha = 1$.

Let f^i be the unique element of A_k. Then by (1) $f_j^i = \xi_{k(j,k)}$ for $j \leqq j(k) = \delta - 1$. This means that ξ_k and f^i satisfy the requirements of Lemma 3.

PROOF OF THEOREM 1. By monotonicity we may assume that $\alpha = \gamma^+$ and so α is regular. We are going to define a graph $\mathcal{G} = \langle g, G \rangle$. g will be a subset of ${}^{\delta+1}\alpha$.

(1) Put $g = \{f \in {}^{\delta+1}\alpha : f_0 < \ldots < f_\delta\}$.

Let $f, h \in g, f \neq h$.

(2) Put $\{f, h\} \in G$ if there exists a j, $2 \leqq j \leqq \delta$ such that $f_{j-1} < h_0 < f_j < h_1$.

We prove that the graph $\mathcal{G}$ defined by (1) and (2) satisfies the requirements. It is obvious that $\alpha(\mathcal{G}) = \alpha$.

First we prove

$$[\delta + 1] \not\subseteq \mathcal{G}. \tag{3}$$

Let $f^0, \ldots, f^\delta$ be $\delta + 1$ different elements of $\mathcal{G}$. We have to prove that there are two f^i not connected in $\mathcal{G}$. We may choose the notation so that $f_0^0 \leqq f_0^i$ for $i \leqq \delta$. Assume that f^0 is adjacent to each f^i for $1 \leqq i \leqq \delta$. Then by (1) and (2) for every i, $1 \leqq i \leqq \delta$ there is a j_i, $1 \leqq j_i \leqq \delta$ such that

$$f_{j_i - 1}^0 < f_0^i < f_{j_i}^0 < f_1^i. \tag{4}$$

It follows that there are $i \neq i'$, $1 \leqq i, i' \leqq \delta$ satisfying (4) with $j_i = j_{i'}$. It follows that $f_0^i < f_1^{i'}$ and $f_0^{i'} < f_1^i$, hence by (1) $f_0^i < f_j^i$ for $1 \leqq j \leqq \delta$ and $f_0^{i''} < f_j^i$ for $1 \leqq j \leqq \delta$. Hence by (2) $\{f^i, f^{i'}\} \notin G$. This proves (3).

Now we prove

(5) Let $\mathcal{G}_\xi$, $\xi < \gamma$ be a vertex-decomposition of type γ of $\mathcal{G}$. Then $\beta(\mathcal{G}_\xi) = \delta + 1$ for some $\xi < \gamma$ i.e. $[\delta] \subseteq \mathcal{G}_\xi$ for some $\xi < \gamma$.

Let $\prec_{\alpha,\delta+1}$ be briefly denoted by $\prec$.

It is well known that for the set g defined in (1) we have $\operatorname{typ} g(\prec)=\operatorname{typ}{}^{\delta+1}\alpha$. It follows from Lemma 2/A that there is a $\xi<\gamma$ that

$$\operatorname{typ} g_\xi(\prec)=\operatorname{typ}{}^{\delta+1}\alpha.$$

Hence by the definition of the vertex-decomposition it is sufficient to prove

(6) If $g'\subseteq g$, $\operatorname{typ} g'(\prec)=\operatorname{typ}{}^{\delta+1}\alpha$ then $\mathscr{G}(g')\supseteq[\delta]$.

To prove (6) we have to define a sequence $f^0, \dots, f^{\delta-1}$ of elements of g' so that every pair of them is connected in $\mathscr{G}$. Considering (1) and (2) it is sufficient to define the f^i-s for $i<\delta$ so that $f^i\in g'$ and the ordinal numbers f_j^i, $i<\delta, j<\delta+1$ satisfy the following conditions

(7) $f_0^i<\dots<f_\delta^i$ for $i<\delta$.

(8) For every $i<i'<\delta$ $f^i_{i'-i}<f_0^{i'}<f^i_{i'-i+1}<f_1^{i'}$.

Let $a_k=\langle i(k), j(k)\rangle$, $k<\delta\cdot(\delta+1)$ be the following enumeration of the pairs $\langle i,j\rangle$, $i<\delta, j<\delta+1$. If $a_k=\langle i,j\rangle$, $a_{k'}=\langle i',j'\rangle$ then $k<k'$ iff either $i+j<i'+j'$ or $i+j=i'+j'$ and $i<i'$. It is easy to see that

(9) any of the following conditions a)—d) imply $k<k'$

a) $i=i'$ and $j<j'$

b) $i<i'$ and $j=i'-i, j'=0$

c) $i<i'$ and $j=i'-i+1, j'=1$

d) $i>i'$ and $j=0, j'=i-i'+1$.

It follows from Lemma 3 that there is an increasing sequence ξ_k, $k<\delta\cdot(\delta+1)$ of ordinals and a sequence f^i, $i<\delta$ of elements of g' satisfying $f_j^i=\xi_k$ for $a_k=\langle i,j\rangle$, $k<\delta\cdot\delta+1$. Considering that the enumeration a_k satisfies (9) the ordinal numbers f_j^i satisfy (7) and (8). This proves (6). By (3) and (5) $\mathscr{G}$ satisfies the requirements of Theorem 1.*

For the proof of Theorem 2 we need further preliminaries.

LEMMA 4. *Let $\delta\geqq\omega$. Let $\succ_{\alpha,\delta}$ denote the converse of the lexicografical ordering $\prec_{\alpha,\delta}$ defined in 3. 1. Let $g\subseteq{}^\delta\alpha$ and assume that g is well-ordered by $\succ_{\alpha,\delta}$. Then $|g|\leqq\delta$.*

PROOF in outline: If this is not true, then there exists a $g\subseteq{}^\delta\alpha$ such that $\operatorname{typ} g(\succ_{\alpha,\delta})=\delta^+$.

It is easy to see by induction on $\xi<\delta$, that for every $\xi<\delta$ there is an $f^\xi\in{}^\xi\alpha$ such that the set

$$A_\xi=\{f\in g: f\restriction\xi\neq f^\xi\}$$

has power $\leqq\delta$. Then $f^\eta=f^\xi\restriction\eta$ for every $\eta<\xi<\delta$. The set $g-\bigcup_{\xi<\delta}A_\xi$ has power δ^+, because $|g|=\delta^+$ but for each element f of it $f\restriction\xi=f^\xi$ for every $\xi<\delta$. This is a contradiction.

DEFINITION 3. 2. Let A be a set ordered by the relation $\prec$. We say that $\prec$ is a δ-well ordering of A if every subset $B\subseteq A$ well-ordered by the converse ordering $\succ$ has power $<\delta$. ω-well-ordered sets are the ordinary well-ordered sets. Our previous lemma states that ${}^\delta\alpha$ is δ^+-well-ordered by $\prec_{\alpha,\delta}$ for every infinite δ.

The next lemma contains the essential idea of the proof of Theorem 2.

* The proof of Theorem 1 makes use of an idea of E. SPECKER [7] used for the proof of $\operatorname{typ}^3\omega\to(\operatorname{typ}^3\omega, 3)^2$. The same idea is used in [4].

LEMMA 5. *Let $\alpha, \delta \geqq \omega$. Then ${}^{\delta}\alpha$ is not the union of less than α sets δ-well-ordered by $\prec_{\alpha,\delta}$.*

PROOF. Let g_ξ, $\xi < \gamma$ be a disjointed sequence of subsets of ${}^{\delta}\alpha$, δ-well-ordered by $\prec_{\alpha,\delta}$. We have to prove that

$$ {}^{\delta}\alpha \subsetneqq \bigcup_{\xi<\gamma} g_\xi. $$

First we prove

(1) Let $g \subseteqq {}^{\delta}\alpha$ be δ-well-ordered by $\prec_{\alpha,\delta}$. Then for every $f \in g$ there is an $\eta < \delta$ such that for every $h \in g$ $h \restriction \eta = g \restriction \eta$ implies that $h(\eta) \leqq f(\eta)$.

In fact if such an η does not exist then for every $\eta < \delta$ there is an $h^\eta \in g$ such that $h^\eta \restriction \eta = f \restriction \eta$ and $h^\eta(\eta) > f(\eta)$. But then $h^\eta \succ_{\alpha,\delta} h^{\eta'}$ for every $\eta < \eta' < \delta$ and $\{h^\eta\}_{\eta<\delta}$ is a subset of g of power δ well-ordered by $\succ_{\alpha,\delta}$, a contradiction.

It follows immediately from (1) that the following assertion holds.

(2) Let $g \subseteqq {}^{\delta}\alpha$ be δ-well-ordered by $\prec_{\alpha,\delta}$. For every $f \in g$ let $\eta(f)$ be an ordinal $< \delta$ satisfying (1). Let $f, h \in g$ be such that $\eta(f) = \eta(h) = \eta$ and $f \restriction \eta = h \restriction \eta$. Then $f(\eta) = h(\eta)$.

(3) Put $g = \bigcup_{\xi<\gamma} g_\xi$. For every $f \in g$ put $\xi(f) = \xi$ for the unique ξ for which $f \in g_\xi$ and let $\eta(f)$ be an ordinal $< \delta$ satisfying (1) with $g_{\xi(f)}$ instead of g.

(4) Put $a_\eta = \{f \in g : \eta(f) = \eta\}$ for $\eta < \delta$. We have

$$ g = \bigcup_{\eta<\delta} a_\eta. \tag{5} $$

We define a function $f \in {}^{\delta}\alpha - g$ by defining f_η, $\eta < \delta$ by transfinite induction on η as follows. Suppose that for some $\eta < \delta$ $f_{\eta'}$ is defined for every $\eta' < \eta$, hence $f \restriction \eta$ is defined. It follows from (2) and (4) that for each $\xi < \gamma$ there is an ordinal number $\varrho(\xi, \eta)$ such that for every $h \in g_\xi \cap a_\eta$ $h \restriction \eta = f \restriction \eta$ implies $h(\eta) = \varrho(\xi, \eta)$. Considering that $\gamma < \alpha$ there is an ordinal number $f_\eta < \alpha$ such that $f_\eta \neq \varrho(\xi, \eta)$ for every $\xi < \gamma$. This defines the function f and it is obvious that $f \in {}^{\delta}\alpha$. Assume $f \in g$. Then by (3) $f \in g_\xi$ for $\xi = \xi(f)$. Let $\eta(f) = \eta$. Considering that $f \restriction \eta = f \restriction \eta, f_\eta = \varrho(\xi, \eta)$ a contradiction, since $f \in g_\xi \cap a_\eta$. This proves Lemma 5.

PROOF OF THEOREM 2. We define a graph $\mathcal{G} = \langle g, G \rangle$. Let $g = {}^{\delta}\alpha$. Then $\alpha(\mathcal{G}) = \alpha^\delta$. Let f be a one-to-one mapping of α^δ onto g. Let $\{f_\varrho, f_\sigma\}$ $\varrho < \sigma < \alpha^\delta$ be arbitrary. Put $\{f_\varrho, f_\sigma\} \in G$ iff $f_\varrho \succ_{\alpha,\delta} f_\sigma$.*

First we prove a lemma.

(1) *Let $a \subseteqq {}^{\delta}\alpha$ and let $\beta \geqq \omega$ be arbitrary. Then $[\beta] \subseteqq \mathcal{G}(a)$ iff a is not β-well-ordered by $\prec_{\alpha,\delta}$.*

The only if part is trivial from the definition of $\mathcal{G}$. Assume a is not β-well-ordered by $\prec_{\alpha,\delta}$. Then by 3. 2 it contains a subset $b \subseteqq a$ such that $|b| = \beta$ and typ $b\,(\succ_{\alpha,\delta}) = \beta$, i.e. b is well-ordered by $\succ_{\alpha,\delta}$. It is well known that then b contains a subset $c \subseteqq b$, $|c| = \beta$ such that for each $f_\varrho, f_\sigma \in c$ $f_\varrho \succ_{\alpha,\delta} f_\sigma$ iff $\varrho > \sigma$.

For the convenience of the reader we mention that this can be easily proved using a theorem of [3] which in terms of the partition symbol defined in 2. 4 states

* The idea of graph definitions of this type goes back to SIERPIŃSKI. We call it a Sierpińskisation of the complete graph.

that $\beta \to (\beta, \omega)^2$ holds for every $\beta \geqq \omega$. The proof is simply a Sierpinskisation of the complete graph with set of vertices b, using the two different well-orderings.

It is obvious that the complete graph with vertices c is a subgraph of $\mathcal{G}$. This proves (1).

By lemma 4 and (1) we have $[\delta^+] \not\subseteq \mathcal{G}$, hence $\beta(\mathcal{G}) = \delta^+$. On the other hand if $\gamma < \alpha$ and $\mathcal{G}_\xi$, $\xi < \gamma$ is a vertex-decomposition of type γ of $\mathcal{G}$ then by Lemma 5, there is a $\xi < \gamma$ such that g_ξ is not δ-well-ordered. Then, by (1) $[\delta] \subseteq \mathcal{G}(g_\xi)$, hence $\beta(\mathcal{G}_\xi) \geqq \delta^+$. This proves the theorem.

We wish to discuss two possible strengthenings of the theorems of this section. It is possible that under the conditions of Theorem 3 there always exists a graph $\mathcal{G}$, with $\alpha(\mathcal{G}) = \alpha$, $\beta(\mathcal{G}) = \beta$ such that for every vertex-decomposition $\mathcal{G}_\xi$, $\xi < \gamma$ of type γ of $\mathcal{G}$ we have $\beta(\mathcal{G}_\xi) = \beta$ for some $\xi < \gamma$. As a corollary of Theorem 3 this is trivial if β is not a limit cardinal, and it is easy to see that if β is a limit cardinal then a further condition $\beta < \alpha$ or $\gamma < \mathrm{cf}(\alpha)$ is necessary. In case $\beta = \omega$ we can prove the following

THEOREM 4. *Let $\alpha \geqq \omega$ be regular. Then there exists a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \alpha$, $\beta(\mathcal{G}) = \omega$ such that, fo. every $\gamma < \alpha$ and for every vertex-decomposition $\mathcal{G}_\xi$, $\xi < \gamma$ of it, $\beta(\mathcal{G}_\xi) = \omega$ for some $\xi < \gamma$.*

We postpone the proof.

We do not know whether this result can be generalized for limit cardinals $\beta > \omega$. The simplest unsolved problem is

PROBLEM 1. Assume G.C.H. Does there exist a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \omega_{\omega+1}$, $\beta(\mathcal{G}) = \omega_\omega$ such that for every vertex-decomposition $\mathcal{G}_\xi$, $\xi < \omega_\omega$ of type ω_ω of $\mathcal{G}$, $\beta(\mathcal{G}_\xi) = \omega_\omega$ holds for some $\xi < \omega_\omega$?

The second generalization of Theorem 3 would be that under the conditions of Theorem 3 and the additional condition $\alpha > \beta$ or $\mathrm{cf}(\alpha) > \gamma$ the following assertion is true. There exists a graph $\mathcal{G}$ with $\alpha(\mathcal{G}) = \alpha$, $\beta(\mathcal{G}) = \beta$ such that for every vertex-decomposition $\mathcal{G}_\xi$, $\xi < \gamma$ of type γ, there is a $\xi < \gamma$ for which $\mathcal{G}_\xi$ contains a subgraph isomorphic to $\mathcal{G}$.

We did not investigate this problem very closely, but we mention that e.g. the graph constructed for the proof of Theorem 1 has this stronger property in case α is regular.

Now we give the

PROOF OF THEOREM 4. Let $g = {}^2\alpha$. Assume $f, h \in g$ and $f(0) \leqq h(0)$. Put $\{f, h\} \in G$ iff $f(0) < g(0)$ and $f(1) > g(1)$. Obviously $\alpha(\mathcal{G}) = \alpha$.

We prove

(1) $$\beta(\mathcal{G}) = \omega.$$

It is obvious that $\beta(\mathcal{G}) \geqq i$ for every $i < \omega$. If $\beta(\mathcal{G}) > \omega$ then $[\omega] \subseteq \mathcal{G}$, i.e. $\mathcal{S}_2[g'] \subseteq \mathcal{G}$ for some $g' \subseteq g$, $|g'| = \omega$. We may choose the notations so that $g' = \{f^i\}_{i<\omega}$ where $f^i \neq f^j$ and $f^i(0) \leqq f^j(0)$ for every $i < j < \omega$. Then by the definition of $\mathcal{G}$ $f^i(0) < f^j(0)$ and $f^i(1) > f^j(1)$ for every $i < j < \omega$. This is a contradiction, hence (1) is proved.

Let now $\mathcal{G}_\xi$, $\xi < \gamma$ be an arbitrary vertex-decomposition of type $\gamma < \alpha$ of $\mathcal{G}$. Then by Lemma 2/A there is a $\xi < \gamma$ such that

$$\mathrm{typ}\, g_\xi (\prec_{\alpha,2}) = \mathrm{typ}\, {}^2\alpha.$$

It follows from Lemma 3 that for every $i<\omega$ there exists a sequence $\{f^j\}_{j<i}$ of elements of g_ξ satisfying the conditions

$$f^0(0)<\ldots<f^{i-1}(0)<f^{i-1}(1)<\ldots<f^0(1).$$

Hence by the definition of $\mathscr{G}$ $[i]\subseteqq\mathscr{G}_\xi$ for every $i<\omega$, hence $\beta(\mathscr{G}_\xi)=\omega$. This proves Theorem 4.

§ 4. Further refinements of the vertex-decomposition problem

As a corollary of a theorem of [6] which also follows from our Theorem 2 we know that for every integer $\beta, \gamma\geqq 2$ there exists a finite graph $\mathscr{G}$ with $\beta(\mathscr{G})=\beta+1$ such that in every vertex-decomposition $\mathscr{G}_\xi$, $\xi<\gamma$ of type γ of $\mathscr{G}$ there is a member $\mathscr{G}_\xi$ with $\beta(\mathscr{G}_\xi)=\beta+1$. L. LOVÁSZ pointed out to us that possibly this theorem can be improved so that one can find such a graph $\mathscr{G}$ which satisfies even further conditions expressing that $\mathscr{G}$ does not contain "greater" graphs, other than the complete β-graph $[\beta]$. He gave in the special case $\beta=3$, $\gamma=2$ a tricky construction of a graph $\mathscr{G}$ which does not contain a quadrilateral with a diagonal (hence does not contain a [4]-graph) but has no vertex-decomposition of type two where the members do not contain triangles.

It turned out that using the methods of our paper [1] one can easily prove a very general result in this direction. This will be given in Theorem 5. An interesting feature of this theorem is that unlike the previous results it does not generalize for infinite graphs. In Theorem 6 we prove that if a graph $\mathscr{G}$ with $\beta(\mathscr{G})=\beta+1$, $2\leqq\beta<\omega$ does not contain a subgraph of $\beta+1$-vertices and $\binom{\beta+1}{2}-1$ edges (i.e. a complete $\beta+1$-graph minus one edge) then it has a vertex-decomposition $\mathscr{G}_\xi$, $\xi<\omega$ of type ω such that $\beta(\mathscr{G}_\xi)\leqq\beta$ for every $\xi<\omega$.

Before stating the theorems, for the convenience of the reader we recall some definitions given in [1].

DEFINITION 4. 1. A pair $\mathscr{H}=\langle h, H\rangle$ is said to be a *set-system* if $\cup H\subseteqq h$. $\mathscr{H}$ is said to be *uniform* if $|A|=|B|$ for every pair $A, B\in H$. If $\mathscr{H}$ is uniform then the cardinal of the elements of H will be denoted by $\varkappa(H)$. A graph $\mathscr{G}$ is a uniform set-system with $\varkappa(H)=2$. The *chromatic number of a set-system* $\mathscr{H}$ denoted by Chr $(\mathscr{H})$ is the least cardinal γ for which there is a decomposition h_ξ, $\xi<\gamma$ of type γ of h such that no h_ξ contains an element of H as a subset.

Let s be an integer, and let $\mathscr{H}$ be a uniform set-system with $\varkappa(\mathscr{H})=k$, $2\leqq k<\omega$. $\mathscr{H}$ is said to be s-circuitless if for every $1\leqq t\leqq s$, $H'\subseteqq H$, $|H'|=t$ we have $|\cup H'|\geqq 1+(k-1)t$. A graph $\mathscr{G}$ as a set system is s-circuitless iff it does not contain circuits of length $\leqq s$.

We need the following

DEFINITION 4. 2. Let $\mathscr{G}$ be a graph with $\beta(\mathscr{G})=\beta+1$, $2\leqq\beta<\omega$. Let $\mathscr{G}_{[\beta]}$ denote the set system $\langle g, \mathscr{G}_{[\beta]}\rangle$ where

$$G_{[\beta]}=\{g'\subseteqq g\colon |g'|=\beta \quad\text{and}\quad \mathscr{S}_2[g']\subseteqq G\}$$

i.e. the system of complete β-subgraphs of $\mathscr{G}$. Let s be an integer. The graph $\mathscr{G}$ is said to be β, s circuitless if $\mathscr{G}_{[\beta]}$ is s-circuitless.

We need the following

LEMMA 6. *Let $2 \leqq \beta < \omega$. Let $\mathscr{H} = \langle h, H \rangle$ be a uniform s-circuitless set system with $\varkappa(H) = \beta$, $s > \beta$. Let $\mathscr{G}_{\mathscr{H}}$ be the graph $\langle g_{\mathscr{H}}, G_{\mathscr{H}} \rangle$ defined by the following stipulations:*

$$g_{\mathscr{H}} = h, \ G_{\mathscr{H}} = \bigcup_{A \in H} \mathscr{S}_2[A].$$

Then $\mathscr{G}_{\mathscr{H}[\beta]} = \mathscr{H}$, $\beta(\mathscr{G}_{\mathscr{H}}) = \beta + 1$ and $\mathscr{G}_{\mathscr{H}}$ is β, s-circuitless.

PROOF. Let $g' \subseteqq h$, $|g'|$, $2 \leqq t \leqq \beta + 1$. We prove by induction on t that $\mathscr{S}_2[g'] \subseteqq \subseteqq G_{\mathscr{H}}$ implies that there is an $A \in \mathscr{H}$ for which $g' \subseteqq A$. For $t = 2$ this is trivial by the assumption. Assume $2 < t \leqq \beta + 1$ and that the statement is true for $t - 1$. Let $g' \subseteqq h$, $|g'| = t$, $\mathscr{S}_2[g'] \subseteqq G_{\mathscr{H}}$. Assume that $g' \not\subseteqq A$ for any $A \in \mathscr{H}$. The by the induction hypothesis there is an $H' = \{A_0, \ldots, A_{t-1}\} \subseteqq H$, $|H'| = t$ so that the A_i-s contain the different subsets of $t - 1$ elements of the set g'. Then

$$|\bigcup H'| \leqq t + t(\beta - t + 1) = t(\beta - t + 2) \leqq t \cdot (\beta - 1)$$

since $t \geqq 3$.

Considering that $\mathscr{H}$ is s-circuitless for $s \geqq \beta + 1$ this is a contradiction, hence $g' \subseteqq A$ for some $A \in H$. In case $\beta + 1 = t$ this is impossible, hence $[\beta + 1] \notin \mathscr{G}_{\mathscr{H}}$. In case $t = \beta$ the statement just proved implies that $\mathscr{G}_{\mathscr{H}[\beta]} = \mathscr{H}$. Hence by the assumption and by definition 4. 2 $\mathscr{G}_{\mathscr{H}}$ is β, s-circuitless.

THEOREM 5. *Let β, γ, s be integers, $\beta, \gamma \geqq 2$. There is a finite graph $\mathscr{G}$ with $\beta(\mathscr{G}) = \beta + 1$ such that $\mathscr{G}$ is β, s-circuitless and every vertex-decomposition $\mathscr{G}_\xi$, $\xi < \gamma$ of type γ of $\mathscr{G}$ contains a member $\mathscr{G}_\xi$ with $\beta(\mathscr{G}_\xi) = \beta + 1$.*

PROOF. By Corollary 13. 4 of [1] there exists a uniform set-system $\mathscr{H} = \langle h, H \rangle$ with $\varkappa(H) = \beta$, Chr $(\mathscr{H}) \geqq \gamma + 1$ which is s' circuitless for $s' = \max(s, \beta + 1)$. Then by Lemma 6 the graph $\mathscr{G}_{\mathscr{H}}$ satisfies $\beta(\mathscr{G}_{\mathscr{H}}) = \beta + 1$ and is β, s-circuitless.

Let $\mathscr{G}_\xi$, $\xi < \gamma$ be an arbitrary vertex-decomposition of type γ of $\mathscr{G}_{\mathscr{H}}$. Then g_ξ, $\xi < \gamma$ is a decomposition of h. Considering Chr $(\mathscr{H}) \geqq \gamma + 1$ there is an $A \in H$ and a $\xi < \gamma$ such that $A \subseteqq g_\xi$ but then by the definition of $\mathscr{G}_{\mathscr{H}}$ $\mathscr{S}_2[A] \subseteqq \mathscr{G}_{\mathscr{H}}(g_\xi)$, hence $[\beta] \in \mathscr{G}_\xi$, $\beta(\mathscr{G}_\xi) = \beta + 1$.

Note that the proof of Theorem 13. 4 of [1] makes use of the so called probabilistic method.*

THEOREM 6. *Let $\mathscr{G}$ be a graph with $\beta(\mathscr{G}) = \beta + 1$, $2 \leqq \beta < \omega$. Assume that $\mathscr{G}$ does not contain a subgraph $\mathscr{G}' = \langle g', G' \rangle$ such that $|g'| = \beta + 1$, $G' = \mathscr{S}_2[g'] - \{x\}$ for some $x \in \mathscr{S}_2[g']$. Then there is a vertex-decomposition $\mathscr{G}_\xi$, $\xi < \omega$ of type ω of $\mathscr{G}$ such that*

$$\beta(\mathscr{G}_\xi) \leqq \beta \quad \textit{for every} \quad \xi < \omega.$$

PROOF. Let $\mathscr{G}_{[\beta]} = \langle g, G_{[\beta]} \rangle$ be the uniform set-system defined in 4.2. Then it satisfies the requirements of Theorem 12. 1 of [1] with $\beta = \omega$ and therefore has

* After having prepared this manuscript the authors learned that L. LOVÁSZ obtained a constructive proof of Theorem 13. 4 of [1] and so a proof of Theorem 5. We will point out that this gives a constructive proof of the theorem of P. ERDŐS [5] concerning the existence of graphs containing no short circuits and having large chromatic number. His proof is to appear in the next volume of *Acta Math. Acad. Sci. Hung.*

chromatic number $\leqq\omega$. This means that there is a decomposition g_ξ, $\xi<\omega$ of type ω of g such that $A \nsubseteq g_\xi$ for any $A\in G_{[\beta]}$. Considering the definition 4. 2 of $G_{[\beta]}$ this means that $\mathscr{G}_\xi=\mathscr{G}(g_\xi)$ satisfies the requirements of the theorem.

We mention that Theorem 6 is trivial for $\beta=2$ even with 2 instead of ω. This is no longer true for $\beta>2$. The reason for this is that a β, 2-circuitless graph satisfies the conditions of Theorem 6 for $\beta\geqq 3$ but not for $\beta=2$ since each graph is 2, 2-circuitless.

It is known that there are 2, 3-circuitless graphs of arbitrarily high chromatic numbers but by 5. 6 of [1] a 2, 4-circuitless graph has chromatic number $\leqq\omega$.

§ 5. The edge-decomposition symbol

In view of the preliminaries collected in § 2 a best possible negative result similar to Theorem 3 would be that under the conditions

$$(1)\qquad \alpha>2^\gamma,\quad \alpha\geqq\omega\cdot\beta,\quad \beta>\delta\geqq 3,\quad \gamma\geqq 2,\quad (\alpha,\beta)\nrightarrow(\gamma,\delta)$$

hold. In fact the condition $\alpha>2^\gamma$ is necessary by 2. 7, the case $\alpha<\beta$ is covered by 2. 5, by the discussion of the Ramsey function while the other conditions except $\alpha\geqq\omega$ exclude trivial and irrelevant cases. The case $\alpha<\omega$ will be discussed separately.

We do not know any theorem which would disprove (1), but we have only partial results. The only genuine result relevant to the problem we have is the following

THEOREM 7. *Let* $\alpha=(2^{(2^\gamma)^+})^+$, $\gamma\geqq\omega$. *Then* $(\alpha,\omega)\nrightarrow(\gamma,\delta)$ *holds for every* $\delta<\omega$.

COROLLARY 4. Assume G.C.H. Then $(\omega_{\varrho+4},\omega)\nrightarrow(\omega_\varrho,\delta)$ holds for every ϱ and for every $\delta<\omega$.

The simplest unsolved problem is

PROBLEM 2. Assume G.C.H. Is $(\omega_i,\omega)\rightarrow(\omega,\delta)$ true for $i=2$ or $i=3$ and for some $\delta<\omega$?

The only other information we have is a further corollary of the results concerning the partition symbol 2. 4.

THEOREM 8. A) *Let* $\alpha=(2^\gamma)^+$, $\gamma\geqq\omega$. *Then*

$$(\alpha,\alpha)\nrightarrow(\gamma,\gamma^+).$$

B) *Assume G.C.H,* $\alpha\geqq\omega$ *then* $(\alpha^+,\alpha^+)\nrightarrow(\gamma,\alpha)$ *for* $\gamma<\operatorname{cf}(\alpha)$.

COROLLARY 5. Assume G.C.H. Then

$$(\omega_{\xi+1},\omega_{\xi+1})\nrightarrow(\gamma,\omega_\xi)\quad\text{for}\quad \gamma<\operatorname{cf}(\omega_\xi).$$

All other instances of (1) remain unsolved. We will point out only one special case which seems intrigueing.

PROBLEM 3. Does $(\alpha,\omega_1)\rightarrow(\gamma,\omega)$ hold for any pair $\alpha>2^\gamma$, $\gamma\geqq\omega$?

PROOF OF THEOREM 7. The graph we construct will be a subgraph of the graph constructed for the proof of Theorem 4. Put $\beta=(2^\gamma)^+$ (then $\alpha=(2^\beta)^+$) and $g=\{f\in{}^2\alpha: f_0<\beta\}$. Assume $f,h\in g, f_0\leqq g_0$.

(1) Put $\{f, h\} \in G$ iff $f_0 < g_0$ and $f_1 > g_1$. Then as in the proof of Theorem 4 we have

$$\alpha(\mathcal{G}) = \alpha, \quad \beta(\mathcal{G}) = \omega.$$

Let $\mathcal{G}_\xi$, $\xi < \gamma$ be an arbitrary edge-decomposition of type γ of $\mathcal{G}$. We prove

(2) There is a $\xi < \gamma$ such that $\beta(\mathcal{G}_\xi) = \omega$.

This is a slightly stronger statement than that of Theorem 7. (To prove the theorem it would be sufficient to show that for every $\delta < \omega$ there is a ξ depending on δ such that $\beta(\mathcal{G}_\xi) > \delta$.) Let $\nu < \mu < \alpha$. We define an edge-decomposition $\mathcal{G}_\xi(\nu, \mu)$, $\xi < \gamma$ of type γ of the complete graph with set of vertices β as follows.

(3) Let $\eta < \zeta < \beta$ be arbitrary. Let f, h be the elements of g satisfying

$$f_0 = \eta, \quad f_1 = \mu, \quad h_0 = \zeta, \quad h_1 = \nu.$$

Then by (1) $\{f, h\} \in G$, hence $\{f, h\} \in G_\xi$ for some $\xi < \gamma$.

Put $\{\eta, \zeta\} \in G_\xi(\nu, \mu)$ for this ξ. By Theorem 4 of [2] we have $(2^\gamma)^+ \to (\gamma^+)^2_\gamma$ for every $\gamma \geqq \omega$. This means that by 2. 4 corresponding to every $\nu < \mu < \alpha$ there exists a $\xi(\nu, \mu) < \gamma$ and a set $B(\nu, \mu) \subseteqq \beta$ such that

$$\mathcal{S}_2[B(\nu, \mu)] \subseteqq G_{\xi(\nu,\mu)}(\nu, \mu) \tag{4}$$

and

$$|B(\nu, \mu)| = \gamma.^* \tag{5}$$

(6) Let A be the set of those functions a with $\mathcal{D}(a) = 2$, for which $a_0 \in \beta$, $a_1 \in \mathcal{S}_\gamma[\beta]$. Considering that $((2^\gamma)^+)^\gamma = (2^\gamma)^+$, we have

$$|A| = \beta. \tag{7}$$

We define an edge-decomposition $\mathcal{G}^0_a$, $a \in A$ of type β of the complete graph with set of vertices α as follows.

(8) Let $\nu < \mu < \alpha$. Then by (4) and (6) $a_0 = \xi(\nu, \mu)$, $a_1 = B(\nu, \mu)$ for some $a \in A$. Put then $\{\nu, \mu\} \in \mathcal{G}^0_a$.

By (7) this edge-decomposition has type β. Considering again Theorem 4 of [2] it follows that there exists an $a \in A$ and a subset $C \subseteqq \alpha$, such that

(9) $|C| = \beta^+$ and $\mathcal{S}_2[C] \subseteqq G^0_a$.

Put $a_0 = \xi$, $a_1 = B$, $D = \{f \in g : f_0 \in B$ and $f_1 \in C\}$. We have by (8) and (9) $\xi(\nu, \mu) = \xi$, $B(\nu, \mu) = B$ for every $\nu < \mu$, $\nu, \mu \in C$. If $f, h \in D, f_0 \leqq h_0$ and $\{f, h\} \in G$ then by (1) $f_0 < h_0, f_1 > h_1, f_0, h_0 \in B, f_1, h_1 \in C$. Hence $B(h_1, f_1) = B$, $\xi(h_1, f_1) = \xi$, hence by (4) $\{f_0, h_0\} \in G_\xi(h_1, f_1)$ and that means by (3)

$$\{f, h\} \in G_\xi.$$

It follows that

$$\mathcal{G}(D) \subseteqq \mathcal{G}_\xi. \tag{10}$$

Using (5) and (9) it follows that for every $i < \omega$ there are sequences of ordinals

$$f_0^0 < \dots < f_0^{i-1} < \beta, \qquad f_0^j \in B \quad \text{for} \quad j < i$$
$$\alpha > f_1^0 > \dots > f_1^{i-1}, \qquad f_1^j \in B \quad \text{for} \quad j < i.$$

* The theorem gives a $B(\nu, \mu)$ with $|B(\nu, \mu)| = \gamma^+$ but we choose $B(\nu, \mu)$ as a set of power γ for the sake of argument given for the proof of (7).

Let f^j, $j<i$ be the corresponding sequence of elements of g and let $g'=\{f^j\}_{j<i}$. Then $|g'|=i$, by (1) $\mathscr{S}_2[g']\subseteq G$, by the definition of D we have $g'\subseteq D$, hence by (10) $\mathscr{S}_2[g']\subseteq G_\xi$. That means $[i]\subseteq\mathscr{G}_\xi$ for every $i<\omega$ and thus $\beta(\mathscr{G}_\xi)=\omega$. This proves (2).

REMARKS. The set D obtained in the proof is such that $|B|=\gamma$, $|C|=\beta^+$. Using the G.C.H. we obtain e.g. in the simplest case $\gamma=\omega$ that $|B|=\omega$, $|C|=\omega_3$. If the hypothesis is assumed this can be improved so that $|B|=\omega_1$ since in the proof of (7) we can use that $((2^\omega)^+)^{\omega_1}=\omega_2=(2^\omega)^+$.

We mention that even assuming the G.C.H. we cannot decide the following

PROBLEM 4. Let $\mathscr{G}'$ be the subgraph of $\mathscr{G}$ defined in the proof of Theorem 7 spanned by the set $g'=\{f\in{}^2\alpha: f_0<(2^\omega)^+, f_1<(2^\omega)^+\}$.

Does then $\mathscr{G}'$ have an edge-decomposition $\mathscr{G}_\xi$, $\xi<\omega$ of type ω with members $\beta(\mathscr{G}_\xi)<\omega$ for $\xi<\omega$?

To prove Theorem 8 we need

LEMMA 7. *Let α, β, β', γ, δ be cardinals such that*

a) $$\alpha\nrightarrow(\beta,\beta')^2$$

and

b) $$\alpha\rightarrow(\beta',(\delta)_\gamma)^2_{\gamma+1}$$

hold. Then $(\alpha,\beta)\nrightarrow(\gamma,\delta)$.

PROOF. By a) the complete graph with set of vertices α has an edge decomposition $\mathscr{G}^0$, $\mathscr{G}^1$ of type 2 such that $\beta(\mathscr{G}^0)\leqq\beta$, $\beta(\mathscr{G}^1)\leqq\beta'$. The graph $\mathscr{G}^0$ satisfies the requirements of our theorem. In fact if $\mathscr{G}_\xi$, $\xi<\gamma$ is an arbitrary edge-decomposition of type γ of $\mathscr{G}^0$, then $\mathscr{G}^1$, $\mathscr{G}_\xi$, $\xi<\gamma$ is an edge-decomposition of type $\gamma+1$ of the complete graph with the set of vertices α. Then by b) considering that $\beta(\mathscr{G}^1)\leqq\beta'$ there is a $\xi<\gamma$ such that $\beta(\mathscr{G}_\xi)>\delta$.

PROOF OF THEOREM 8. By Theorem 7 of [2] and by Theorem 1 of [3] we have

$$(2^\gamma)^+\rightarrow((2^\gamma)^+,(\gamma^+)_\gamma)^2_{\gamma+1}\quad\text{for}\quad\gamma\geqq\omega,\qquad\alpha^+\nrightarrow(\alpha^+,\alpha^+)^2\quad\text{for}\quad\alpha\geqq\omega$$

and $\alpha^+\rightarrow(\alpha)^2_\gamma$ if the G.C.H. holds and $\gamma<\mathrm{cf}(\alpha)$, $\alpha\geqq\omega$. Hence the theorem follows from Lemma 7 in both cases.

NOTE. If we assume G.C.H., then by the results of [3] $\alpha^+\nrightarrow(\omega_1,\beta')^2$ holds iff $\beta'=\alpha^+$ and $\mathrm{cf}(\alpha)=\omega$. On the other hand, $\alpha^+\rightarrow(\alpha^+,(3)_\omega)^2$ holds if $\mathrm{cf}(\alpha)=\omega$. Hence no information concerning Problem 3 can be obtained using the above method.

§ 6. The edge-decomposition symbol for finite graphs

Let now α, β, γ, δ be finite, and assume $\alpha\geqq\beta>\delta\geqq 3$. It is obvious from the definition 2.4 of the Ramsey function that if $\beta>\alpha((\delta)_\gamma,\gamma,2)$ then $(\alpha,\beta)\nrightarrow(\gamma,\delta)$ holds.

It is not known whether $(\alpha,\beta)\rightarrow(\gamma,\delta)$ holds for any $\beta\leqq\alpha((\delta)_\gamma,\gamma,2)$. Probably for every integer $\gamma\geqq 2$ and $\delta\geqq 3$ there is an $\alpha_{\gamma,\delta}$ such that $(\alpha_{\gamma,\delta},\delta+1)\rightarrow(\gamma,\delta)$ holds.

The special case $\gamma=2$, $\delta=3$ was suggested by the authors as a problem to several people. It is known that $\alpha((3)_2, 2, 2)=6$ and in fact it was prowed by several people* that there is an α such that $(\alpha, 6)\nrightarrow(3, 2)$ holds.

L. PÓSA proved the existence of an α for which

$$(\alpha, 5)\nrightarrow(3, 2)$$

holds, but the problem whether $(\alpha, 4)\rightarrow(3, 2)$ holds for every α, is still unsolved.

We outline PÓSA's proof.

By Corollary 3 there exists a graph $\mathcal{G}'=\langle g', G'\rangle$ such that $\beta(\mathcal{G}')=4$ and ϱ' has no vertex-decomposition $\mathcal{G}_0', \mathcal{G}_1'$ such that $\beta(\mathcal{G}_i')\leqq 3$ for $i<2$. Let g consist of g' and of one new vertex x, and let $G=G'\bigcup_{y\in g'}\{x, y\}$, i.e. the new vertex is connected to each of the old ones.

Then $\beta(\mathcal{G})=5$ and $\mathcal{G}$ has no edge-decomposition $\mathcal{G}_0, \mathcal{G}_1$ of type two with $\beta(\mathcal{G}_i)\leqq 3$ for $i<2$ for if not, then put $g_i'=\{y\in g'\colon \{x, y\}\in G_i\}$ for $i<2$.

Then $\mathcal{G}'(g_0')\leqq\mathcal{G}_1$, $\mathcal{G}'(g_1')\subseteqq\mathcal{G}_0$, hence $\mathcal{G}'(g_0')$, $\mathcal{G}'(g_1')$ is a vertex-decomposition of $\mathcal{G}'$ with $\beta(\mathcal{G}'(g_i'))\leqq 3$ for $i<2$, a contradiction.

§ 7. A special edge decomposition problem

We say that a graph $\mathcal{G}$ is a tree if it does not contain circuits. The problem arises what graphs have an edge-decomposition of type γ where all the members are trees. In Theorem 9 we give a necessary and sufficient condition for $\gamma\geqq\omega$. NASH WILLIAMS gave in [8] a necessary and sufficient condition in case $\gamma<\omega$. The condition is that every finite subgraph of i elements has at most $(i-1)\cdot\gamma$ edges. The necessity of this condition is obvious. Our Theorem 11 gives a more difficult necessary condition. It states that the union of γ trees and even more general graphs have colouring number $\leqq 2\gamma$.

In Theorem 10 we state a corollary of our previous results, that a graph not containing a quadrilateral has an edge-decomposition into ω trees.

To state our theorems we need some concepts defined in [1].

DEFINITION 6. 1. *Let $\mathcal{G}$ be a graph, and let $\prec$ be an ordering of g. For any arbitrary $g'\subseteqq g$, $V(x, g')=\{y\colon \{x, y\}\in G$ and $y\in g'\}$ $\tau(x, g')=|V(x, g')|$. For $x\in g$ $g|\prec x$ is the set $\{y\in g\colon y\prec x\}$.*

The colouring number of $\mathcal{G}$ is the least cardinal γ for which g has a well-ordering $\prec$ such that $\tau(x, g|\prec x)<\gamma$ for every $x\in g$. The colouring number of $\mathcal{G}$ is denoted by Col $(\mathcal{G})$.

We prove

THEOREM 9. *Let $\gamma\geqq\omega$. The graph $\mathcal{G}$ has an edge-decomposition onto the union of γ trees if and only if* Col $(\mathcal{G})\leqq\gamma^+$.

As a corollary of this and a result of 1 we prove

THEOREM 10. *A graph $\mathcal{G}$ not containing quadrilaterals has an edge-decomposition of type ω where all the members are trees.*

* G. L. CHERLIN, R. L. GRAHAM, VAN LINT.

We need the following

LEMMA 8. *Let $\mathscr{G}$ be a graph and let $\mathscr{G}_\xi$, $\xi<\gamma$ be an edge-decomposition of type γ of $\mathscr{G}$ such that* $\operatorname{Col}(\mathscr{G}_\xi)\leqq\gamma^+$ *for every* $\xi<\gamma$. *Then* $\operatorname{Col}(\mathscr{G})\leqq\gamma^+$.

PROOF. By the assumption for every $\xi<\gamma$ there is a well-ordering $\prec_\xi$ of g such that $|V(x, g|\prec_\xi x)|\leqq\gamma$. Put

$$f(x)=\bigcup_{\xi<\gamma} V(x, g|\prec_\xi x).$$

Then $f(x)$ is a set mapping of order $\leqq\gamma^+$. It is obvious that $\{x, y\}\in G$, iff $y\in f(x)$ or $x\in f(y)$. Hence the statement follows from Theorems 6. 3 of [1].

PROOF OF THEOREM 9. Let $\mathscr{G}$ be a graph and assume that $\operatorname{Col}(\mathscr{G})\leqq\gamma^+$. Let $\prec$ be a well-ordering of g such that

$$\tau(x, g|\prec x)\leqq\gamma \quad \text{for every} \quad x\in g.$$

It is obvious that one can define the graphs $\mathscr{G}_\xi$, $\xi<\gamma$ on such a way that if $y\neq z$, and $y, z\in V(x, g|\prec x)$ then $\{y, x\}$, $\{z, x\}$ belong to different $\mathscr{G}_\xi$-s. Hence $\mathscr{G}$ is the union of γ trees.

On the other hand, assume now that $\mathscr{G}_\xi$, $\xi<\gamma$ is an edge-decomposition of type γ of $\mathscr{G}$ where all the $\mathscr{G}_\xi$-s are trees. Then $\operatorname{Col}(\mathscr{G}_\xi)=2$ for every $\xi<\gamma$. Hence the statement follows from Lemma 8.

PROOF OF THEOREM 10. Let $\mathscr{G}$ be a graph not containing quadrilaterals (or more generally $[i, \omega_1]$-complete even graphs) for some $i<\omega$. Then by Corollary 5. 6 of [1] $\operatorname{Col}(\mathscr{G})\leqq\omega$, hence the statement follows from Theorem 9.

Assume now $\gamma<\omega$. Using an argument similar to the one used in the proof of Lemma 8 and using Theorem 6.5 of [1] it would be easy to obtain that if $\mathscr{G}$ has an edge-decomposition to γ trees then it has colouring number $\leqq 2\gamma+1$. However this is not a best possible result. We will prove the following stronger

THEOREM 11. *Let $\gamma<\omega$. Assume that $\mathscr{G}$. has an edge-decomposition $\mathscr{G}_\xi$, $\xi<\gamma$ where the $\mathscr{G}_\xi$-s are trees. Then*

$$\operatorname{Col}(\mathscr{G})\leqq 2\gamma.$$

This is best possible since by a well-known result (see e.g. [9], p. 185) the complete 2γ-graph $[2\gamma]$ has an edge-decomposition to γ trees.

Theorem 11 can be proved with a similar argument to the one used for the proof of Theorem 9. 1 of [1] saying that for every $2\leqq\beta<\omega$ a graph all whose finite subgraphs have colouring number β has colouring number $\leqq 2\beta-2$. We mentioned in [1] that the same argument should be used to the proof of Theorem 6. 5 of [1].

For the convenience of the reader we outline here the proof of a more general theorem, which implies theorems 9.1 and 6.5 of [1] and Theorem 11 as well.

THEOREM 12. *Let $\mathscr{G}$ be a graph and $\varrho\in{}^g\omega$, $\varrho(x)>0$ for $x\in g$. For an arbitrary $A\in\mathscr{S}_\omega(g)$ put*

$$\nu(A, \mathscr{G})=2|G(A)| \quad \textit{and} \quad \varrho(A, \mathscr{G})=\sum_{x\in A}\varrho(x).$$

We briefly write $\nu(A, \mathscr{G})=\nu(A)$, $\varrho(A, \mathscr{G})=\varrho(A)$.

Assume that $v(A) < \varrho(A)$ *for every* $A \in \mathscr{S}_\omega(g)$, $A \neq 0$. *Then* g *has a well ordering* $\prec$ *such that*

$$\tau(x, g|\prec x) < \varrho(x) \quad \textit{for every} \quad x \in g.$$

COROLLARY 6. Let $\mathscr{G}$ be a graph, $1 \leqq \gamma < \omega$. Assume that every finite subgraph of $i > 0$ vertices has less than $i \cdot \gamma$ edges. Then $\mathscr{G}$ has colouring number $\leqq 2\gamma$.

Corollary 6 follows from the case $\varrho(x) = 2\gamma$ of Theorem 12 and obviously implies Theorem 11.

PROOF OF THEOREM 12 (in outline). If $\mathscr{G}$ is finite, the assumption implies that there is an $x \in g$ such that $\tau(x, g) < \varrho(x)$ since $v(g) = \sum_{x \in g} \tau(x, g) < \sum_{x \in g} \varrho(x) = \varrho(g)$. Hence the result follows by induction on $\alpha(\mathscr{G})$.

Assume $\alpha(\mathscr{G}) \geqq \omega$. Let $A \subseteqq g$. We shall briefly say that A is closed if for every $B \in \mathscr{S}_\omega(g \sim A)$

$$\sum_{x \in B} \tau(x, A \cup B) < \varrho(B). \tag{1}$$

The following assertions are evident:

(2) If A is not closed then there are $A' \in \mathscr{S}_\omega(A)$, $B \in \mathscr{S}_\omega(g \sim A)$ such that

$$\sum_{x \in B} \tau(x, A' \cup B) \geqq \varrho(B).$$

(3) If for every $A' \in \mathscr{S}_\omega(A)$ there is an $A'' \subseteqq A$ such that $A' \subseteqq A''$ and A'' is closed then A is closed.

(4) If $\{A_\xi\}_{\xi < \eta}$ is an increasing sequence of closed sets then $\bigcup_{\xi < \eta} A_\xi$ is closed.

We prove

(5) If $A \subseteqq g$ then there is an $A' \subseteqq g$, $A \subseteqq A'$ such that A' is closed

a) $|A'| < \omega$ if $|A| < \omega$

b) $|A'| = |A|$ if $|A| \geqq \omega$.

First we prove (5) a).* Assume that the assertion fails for some finite A. We define by induction on i an increasing sequence A_i of finite subsets of g as follows. Put $A_0 = A$. Assume $i > 0$ and that A_j is defined for every $j < i$ such that A_j is finite. Then $A \subseteqq \bigcup_{j<i} A_j$ is finite, hence it is not closed by the indirect assumption. Let then A_i be a finite subset of $g \sim \bigcup_{j<i} A_j$ such that

$$\sum_{x \in A_i} \tau\big(x, \bigcup_{j<i} A_j\big) \geqq \varrho(A_i). \tag{6}$$

Thus the sequence A_i is defined and (6) holds for every $i < \omega$. By the assumption $\tau(A_i) < \varrho(A_i)$, $\tau(x, \bigcup_{j<i} A_j) \neq 0$ holds for at least one $x \in A_i$. It follows by induction on i that

$$v\big(\bigcup_{j \leqq i} A_j\big) \geqq \sum_{j=1}^{i} \varrho(A_j) + i \quad \text{for} \quad i \geqq 1.$$

This is a contradiction since the right hand side is at least $\varrho\big(\bigcup_{j \leqq i} A_j\big)$ if $i \geqq \varrho(A_0)$.

* The proof is similar to the proof of Lemma 9. 4 of [1].

Thus (5) a) is proved. Let A' denote a set satisfying (5) a) for every $A \in \mathcal{S}_\omega(g)$. To prove (5) b) let $A \subseteq g$, $(A) \geqq \omega$. Put $A_0 = A$, $A_{i+1} = \bigcup_{B \in \mathcal{S}_\omega(A_i)} B'$, $A' = \bigcup_{i<\omega} A_i$. Then obviously $|A'| = |A|$ and by (3) A' is closed. This proves (5) b).

Put $\alpha(\mathcal{G}) = \alpha \geqq \omega$ and assume that the theorem is true for every graph $\mathcal{G}'$ with $\alpha(\mathcal{G}') < \alpha$.

Using (5) it is easy to define by transfinite induction an increasing sequence A_ξ, $\xi < \alpha$ of type α of subsets of g such that

(7) $g = \bigcup_{\xi<\alpha} A_\xi$, A_ξ is closed and $|A_\xi| < \alpha$ for every $\xi < \alpha$.

Put $B_\xi = \bigcup_{\eta<\xi} A_\eta$ for $\xi < \alpha$, $B_0 = 0$, $C_\xi = A_\xi - B_\xi$. By the assumption the empty set is closed, hence by (4) B_ξ is closed for every $\xi < \alpha$.

Put $\tau_1(x) = \tau(x, B_\xi)$ for every $x \in C_\xi$, $\xi < \alpha$.

Put $\mathcal{G}_\xi = \mathcal{G}(C_\xi)$ for $\xi < \alpha$. Then $\alpha(\mathcal{G}_\xi) < \alpha$ by (7) and $\tau_1(x) < \varrho(x)$ for every $x \in C_\xi$ since B_ξ is closed. Hence

$$\varrho(x, \xi) = \varrho(x) - \tau_1(x) > 0 \quad \text{for every} \quad C_\xi.$$

We will apply the induction hypothesis for the graphs $\mathcal{G}_\xi$ and the functions $\varrho(x, \xi)$. Let $B \in \mathcal{S}_\omega(C_\xi)$. Put briefly

$$v_\xi(B) = v(B, \mathcal{G}_\xi), \quad \varrho_\xi(B) = \varrho(B, \mathcal{G}_\xi) = \sum_{x \in B} \varrho(x, \xi).$$

Then $v_\xi(B) = \sum_{x \in B} \tau(x, B)$. Hence

$$v_\xi(B) + \sum_{x \in B} \tau_1(x) = \sum_{x \in B} \tau(x, B \cup B_\xi) < \varrho(B) = \sum_{x \in B} \varrho(x)$$

since B_ξ is closed and $B \subseteq C_\xi$. Hence

$$v_\xi(B) < \sum_{x \in B} \varrho(x) - \tau_1(x) = \varrho_\xi(B).$$

It follows that $\mathcal{G}_\xi$ satisfies the conditions of the theorem. It follows from the induction hypothesis that there is a well-ordering $\prec_\xi$ of C_ξ such that $\tau(x, C_\xi | \prec_\xi x) < \varrho(x) - \tau_1(x)$ for every $x \in C_\xi$. We define a well-ordering $\prec$ of g by the stipulation $x \prec y$ if $x \in C_\xi$, $y \in C_\eta$ and either $\xi < \eta$ or $\xi = \eta$ and $x \prec_\xi y$.

By (7) $\prec$ is obviously a well-ordering of g and

$$\tau(x, g | \prec x) = \tau_1(x) + \tau(x, C_\xi | \prec_\xi x)$$

for every $x \in C_\xi$, $\xi < \alpha$. Hence $\tau(x, g | \prec x) < \varrho(x)$ for every $x \in g$.

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