ON THE BOUNDEDNESS AND UNBOUNDEDNESS OF POLYNOMIALS

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Let

$$-1 \leqq x_1 < x_2 < \cdots < x_n \leqq 1$$

be n points in $(-1, +1)$. A well known theorem of Faber [5] states that there always is a polynomial of degree $n-1$ for which

$$\left|P_{n-1}(x_i)\right| \leqq 1, \quad 1 \leqq i \leqq n$$

but

$$\max_{-1 \leqq x \leqq 1} \left|P_{n-1}(x)\right| > c \log n.$$

Throughout this paper $P_n(x)$ will denote a polymonial of degree n; $c, c_1 c_2, \cdots$ will denote positive absolute constants not necessarily the same if they occur at different places. In other words: for no choice of the points $x_1, \cdots, x_n$ can we deduce from the boundedness of $\left|P_{n-1}(x_i)\right|$, $1 \leqq i \leqq n$ the boundedness of $\left|P_{n-1}(x)\right|$ in the whole interval $(-1, +1)$. Bernstein [1] asked himself the question whether one can deduce the boundedness of $\left|P_n(x)\right|$ in $(-1, +1)$ if we know that $\left|P_n(x)\right| < 1$ for $m > (1+c)n$ values of x. His answer was affirmative. In fact he showed that if $\left|P_n(x_i^{(m)})\right| \leqq 1$ for all roots of the m^{th} Tchebicheff polynomial $T_m(x)$ where $m > n(1+c)$, then

$$\max_{-1 \leqq x \leqq 1} \left|P_n(x)\right| < A(c) \tag{1}$$

where $A(c)$ depends only on c. Zygmund [7] proved that (1) holds if $T_m(x)$ is replaced by $P_m(x)$ the m^{th} Legendre polynomial.

We can now put the following question: Let

$$-1 \leqq x_1^{(m)} < \cdots < x_m^{(m)} \leqq 1, \qquad 1 \leqq m < \infty \tag{2}$$

be a triangular matrix. What is the necessary and sufficient condition on the matrix that if for $m > n(1+c)$.

$$\left| P_n(x_i^{(m)}) \right| < 1, \qquad 1 \leqq i \leqq m$$

then (1) holds. A priori it is not obvious that a reasonable necessary and sufficient condition can be formulated, but we will find such a condition which is not too complicated.

Put

$$\cos\, \theta_i^{(m)} = x_i^{(m)}, \qquad 0 \leqq \theta_i^{(m)} \leqq \pi.$$

Let $0 \leqq \alpha < \beta \leqq \pi$ and denote by $N_m(\alpha, \beta)$ the number of $\theta_i^{(m)}$ satisfying $\alpha < \theta_i^{(m)} < \beta$. Let

$$\alpha < \theta_i^{(m)} < \cdots < \theta_j^{(m)} < \beta$$

be the θ's in (α, β), for each η we define a subsequence $\theta_{i_1}^{(m)}, \cdots \theta_{i_k}^{(m)}$ of these θ's where $i_1 = i$ and if $i_1, \cdots i_{r-1}$ have already been defined then $\theta_{i_r}^{(m)}$ is the smallest $\theta_l^{(m)}$, $i_{r-1} < l \leqq j$ with $\theta_{i_r}^{(m)} - \theta_{i_{r-1}} > \dfrac{\eta}{m}$, thus the distance between any two $\theta_{i_r}^{(m)}$ is $> \eta/m$ and any other $\theta_l^{(m)}, \cdots, \alpha < \theta_l^{(m)} < \beta$ is at distance $\leqq \eta/m$ from at least one of the θ_{i_r}, $1 \leqq r \leqq k$. Put

$$N_m^{(\eta)}(\alpha, \beta) = k = k(\eta).$$

Now we formulate

Theorem 1. *Let $x_i^{(m)}$ satisfy* (2), *and assume that*

$$\left| P_n(x_i^m) \right| \leqq 1, \qquad 1 \leqq i \leqq m, \qquad m > n(1+c) \tag{3}$$

holds. Then the necessary and sufficient condition that (3) *should imply* (1) *is that there should be an $\eta > 0$ independent of m so that for every $\alpha_m < \beta_m$ satisfying $m(\beta_m - \alpha_m) \to \infty$*

$$N_m^{(\eta)}(\alpha_m, \beta_m) \geqq (1 + o(1)) \frac{m}{\pi} (\beta_m - \alpha_m). \tag{4}$$

Condition (4) means that every interval large compared to $\frac{1}{m}$ contains asymptotically at least as many points $\theta_i^{(m)}$, no two of which are "too close", as $T_m(x)$.

Before we give the fairly difficult proof I would like to call attention to a Theorem I proved 20 years ago [3].

Theorem 2. *Let $x_i^{(m)}$ satisfy (2). The necessary and sufficient condition that to every continuous function $f(x)$, $-1 \leqq x \leqq 1$ and to every $c > 0$ there should exists a sequence of polynomials $P_n(x)$, $n < m(1+c)$, such that*

$$P_n(x_i^{(m)}) = f(x_i^{(m)}), \quad 1 \leqq i \leqq m$$

and $P_n(x) \to f(x)$ uniformly in $(-1, +1)$ as $n \to \infty$, is that

$$\liminf_{n=\infty} m. \min_i (\theta_{i+1}^{(m)} - \theta_i^{(m)}) > 0 \tag{5}$$

and that if $m(\beta_n - \alpha_n) \to \infty$ then

$$N_m(\alpha_m, \beta_m) \leqq (1 + o(1)) \frac{m}{\pi} (\beta_m - \alpha_m). \tag{6}$$

Condition (6) means that every interval (in θ) which is large compared to $\frac{1}{m}$ contains asymptotically at most as many x_i's as $T_m(x)$. The classical orthogonal polynomials as is well known satisfy both (5) and (6), and also (4), thus our Theorem 1 contains the results of Bernstein and Zygmund as special cases.

In [3] the proof of Theorem 2 was only outlined. The proof of Theorem 2 is in fact similar to the proof of Theorem 1. It can be shown that Theorem 2 is substantially equivalent to the following

Theorem 3. *Let $x_i^{(m)}$ satisfy (2). The necessary and sufficient condition that there should exist to every $c>0$ and $A(c)$ so that to every $y_i^{(m)}, \left|y_i^{(m)}\right| \leqq 1,\ 1 \leqq i \leqq m,\ 1 \leqq m < \infty$ there should exist a polynomial $P_n(x)$, $n<(1+c)m$ satisfying*

$$P_n(x_i^{(m)}) = y_i^{(m)},\ \left|P_n(x)\right| < A(c), \qquad -1 \leqq x \leqq 1$$

is that (5) and (6) should be satisfied.

Theorem 3 (and therefore Theorem 2 too) is clearly related to Theorem 1. In this paper we do not further discuss Theorem 2 and 3.

Now we prove Theorem 1. First we show that (4) is sufficient, in other words if (4) holds then for every $c>0$ (3) implies (1). To show this it will clearly suffice to prove the following.

Theorem 4. *Let $\eta>0$, $c>0$ be arbitrary given numbers, $\varepsilon=\varepsilon(\eta,c)$ is sufficiently small and $B=B(\varepsilon)$ is given. Then there is an $A=A(\eta,c,\varepsilon,B)$ so that if*

$$-1 \leqq x_1 < \cdots < x_m \leqq 1, \quad \cos\theta_i = x_i, \quad i=1,\cdots,m$$

is a sequence for which for every $0 \leqq \alpha < \beta \leqq \pi$ satisfying

$$\beta-\alpha > \frac{B}{m} \tag{7}$$

we have

$$N_m^{(\eta)}(\alpha,\beta) > (1-\varepsilon)\,\frac{m}{\pi}\,(\beta-\alpha). \tag{8}$$

Then if $P_n(x)$, $n < \dfrac{m}{1+c}$ is any polynomial satisfying

$$\left|P_n(x_i)\right| < 1, \qquad 1 \leqq i \leqq m. \tag{9}$$

We have

$$\left|P_n(x)\right| < A(\eta,c,\varepsilon,B) \qquad \text{for } -1 \leqq x \leqq 1.$$

To prove Theorem 4 we will need two Lemmas:

Lemma 1. *Let* $z_1 <, \cdots < z_n$ *be the roots of the* n^{th} *Tchebicheff polynomial* $T_n(x)$. *Let* t *be a fixed integer and* $-1 < y_1 < \cdots < y_n < 1$ *where* $y_i = z_i$ *for* $1 \leqq i \leqq u$ *and* $u + t < i \leqq n$. *Assume further that for a fixed* $n > 0$.

$$\text{arc}\cos y_{u+s+1} - \text{arc}\cos y_{u+s} > \eta / n, \quad 1 \leqq s < t. \tag{10}$$

Then for an absolute constant $C = C(\eta, t)$ *we have*

$$\left| l_k(x) \right| \leqq C(\eta, t), \qquad -1 \leqq x \leqq 1, \qquad 1 \leqq k \leqq n$$

where

$$l_k(x) = \frac{\omega(x)}{\omega'(y_k)\,(x - y_k)}, \; \omega(x) = \prod_{k=1}^{n} \; (x - y_k)$$

are the fundamental polynomials of the Lagrange interpolation formula belonging to the y_k.

Denote by $L_k(x)$ the fundamental polynomials of the Lagrange interpolation belonging to the z_k. It is well known that [4]

$$\left| L_k(x) \right| < \frac{4}{\pi}, \qquad -1 \leqq x \leqq 1, \qquad 1 \leqq k \leqq n. \tag{11}$$

Lemma 1 follows from (11) by a simple computation by comparing the factors of $l_k(x)$ and $L_k(x)$ term by term and by using (10). We leave the simple details to the reader.

Before we state Lemma 2 (which will be the most difficult part of the paper) we introduce the following notations: Let $P_n(x) = \prod_{i=1}^{n} (x - x_i)$, $\cos \theta_0 = x_0$ is an arbitrary point in $(-1, +1)$. $I(\alpha)$ denotes the interval $\{\cos \theta_0, \cos(\theta_0 + \alpha)\}$ and $I(-\alpha, \beta)$ the interval $\{\cos(\theta_0 - \alpha), \cos(\theta_0 + \beta)\}$. $N_n(\alpha)$ respectively $N_n(-\alpha, \beta)$ denotes the number of the x_i in $I(\alpha)$ respectively in $I(-\alpha, \beta)$.

Lemma 2. *To every* t_1 *and* c_1 *there is a* $t_2 = t_2(t_1, c_1)$ *so that if* $n > n_0(t_1, t_2, c_1)$ *and for every* $t_1 < t < t_2$

$$N_n\left(\frac{t}{n}\right) > (1+c_1)\,t/\pi \text{ and } N_n\left(-\frac{t}{n}\right) > (1+c_1)\,t/\pi \tag{12}$$

then

$$\left|P_n(x_0)\right| < \frac{1}{2} \max_{-1 \leqq x \leqq 1} \left|P_n(x)\right|. \tag{13}$$

In other words qualitatively speaking if $P_n(x)$ has much more roots in every large neighborhood of x_0 than the n^{th} Tchebicheff polynomial then $\left|P_n(x_0)\right|$ is much smaller than the absolute maximum of $\left|P_n(x)\right|$ in $(-1, +1)$.

(13) would hold with an arbitrary c_2 instead of $\frac{1}{2}$, but then $t_2(t_1, c_1)$ has to be replaced by $t_2(t_1, c_1, c_2)$.

One further remark: In (12) we only consider those intervals for which $0 \leqq \theta_0 + \dfrac{t}{n} \leqq \pi$.

To prove Lemma 2 we replace our $P_n(x)$ by a new polynomial $Q_n(x)$. Outside of $I\left(-\dfrac{t_2}{n}, \dfrac{t_2}{n}\right)$ all the roots of $P_n(x)$ are also roots of $Q_n(x)$. $Q_n(x)$ has the further roots

$$\cos\left(\theta_0 + \frac{2i-1}{n}\pi\right),\ 1 \leqq i \leqq j_1 = N_n\left(\frac{t_2}{n}\right) \text{ and } \cos\left(\theta_0 - \frac{2i-1}{n}\pi\right), \quad 1 \leqq i \leqq j_2 = N_n\left(-\frac{t_2}{n}\right). \tag{14}$$

Our $Q_n(x)$ has now n roots. By (14), in the interval

$$I\left(-\frac{2j_2-1}{n}\pi, \quad \frac{2j_1-1}{n}\pi\right) \tag{15}$$

the roots of $Q_n(x)$ are congruent to those of $T_n(x)$ and by the well known theorem of M. Riesz [6] $Q_n(x)$ must assume its absolute maximum in $(-1, +1)$ outside the interval (15). By (12) $I\left(-\dfrac{t_2}{n}, \dfrac{t_2}{n}\right)$ is inside the interval (15).

Assume now that

$$|Q_n(z_0)| = \max_{-1 \leqq x \leqq 1} |Q_n(x)|. \tag{16}$$

By what has been just said we can assume that z_0 is outside the interval (15). Now we prove

$$|Q_n(x_0)/P_n(x_0)| > 2|Q_n(z_0)/P_n(z_0)|. \tag{17}$$

Assume that (17) has already been proved. By (16) we have $|Q_n(z_0)| \geqq |Q_n(x_0)|$, thus from (17)

$$|P_n(x_0)| < \frac{1}{2}| \; P_n(z_0)| \leqq \max_{-1 \leqq x \leqq 1} |P_n(x)|$$

which proves (13) and thus Lemma 2 is proved.

Thus to complete our proof we only have to prove (17). The proof of (17) is quite simple in principle and to avoid simple and routine computation we will not give all the details. Without loss of generality we can assume that z_0 lies to the right of the interval (15). Denote by $x_1 \leqq \cdots \leqq x_{j_1}$ the roots of $P_n(x)$ in $I\left(\frac{t_2}{n}\right)$ and by $y_1 < \cdots < y_{j_1}$ the roots of $Q_n(x)$ in $I\left(\frac{2j_1 - 1}{n}\right)$. $x'_1 \geqq \cdots \geqq x'_{j_2}$ are the roots of $P_n(x)$ in $I\left(-\frac{t_2}{n}\right)$ and $y'_1 < \cdots < y'_{j_2}$ those of $Q_n(x)$ in $I\left(-\frac{2j_2 - 1}{n}\pi\right)$.

Put

$$\left|\frac{Q_n(x_0)}{P_n(x_0)} \; \frac{P_n(z_0)}{Q_n(z_0)}\right| = \Pi_1 \; \Pi_2 \tag{18}$$

where

$$\Pi_1 = \prod_{i=1}^{j_1} \frac{(x_0 - y_i)(z_0 - x_i)}{(x_0 - x_i)(z_0 - y_i)}, \quad \Pi_2 = \prod_{i=1}^{j_2} \frac{(x_0 - y'_i)(z_0 - x'_i)}{(x_0 - x'_i)(z_0 - y'_i)}. \tag{19}$$

Now it immediately follows from (12) and the definition of the x's and y's that for every x_i and x'_i not in $I\left(-\frac{t_1}{n}, \frac{t_1}{n}\right)$,

(20) $$\frac{x_0 - y_i}{x_0 - x_i} > 1 + \delta \text{ and } \frac{x_0 - y_i'}{x_0 - x_i'} > 1 + \delta, \qquad \delta = \delta(c_1).$$

Also since z_0 is to the right of (15) we have for every x_i not in $I\left(\frac{t_1}{n}\right)$

(21) $$z_0 - x_i > z_0 - y_i.$$

From (12), (19), (20) and (21) we obtain by a simple computation that for $t_2 > t_2(t_1, c_1)$ and $n > n_0(t_2, t_1, c_1)$

(22) $$\Pi_1 > 2$$

since for sufficiently large $t_2 = t_2(t_1, c_1)$ and $n > n_0(t_2, t_1, c_1)$ the contribution to Π_1 of the x_i and y_i corresponding to the x_i in $I\left(\frac{t_1}{n}\right)$ *which do not satisfy* (20) can be ignored.

Similarly we see that for $t_2 = t_1(t_1, c_1)$ and $n > n_0(t_2, t_1, c_1)$

(23) $$\Pi_2 > 1$$

since for the x_i' and y_i' not in $I\left(-\frac{t_1}{n}\right)$ we have

(24) $$\frac{(x_0 - y_i')(z_0 - x_i')}{(x_0 - x_i')(z_0 - y_i')} > 1.$$

(24) can be deduced by a simple geometric (or analytic) argument from (20) and $z_0 > x_0$. The x_i in $I\left(\frac{t_1}{n}\right)$ can again be ignored for sufficiently large t_2.

(18), (22) and (23) prove (17) and hence the proof of Lemma 2 is complete.

It is an open question if (13) remains true if instead of (12) we assume only that for every $t_1 < t < t_2$ $N_n\left(-\frac{t}{n}, \frac{t}{n}\right) > (1 + c_1)2t/\pi$.

Now we are ready to prove Theorem 4. If Theorem 4 would be false then there would be a fixed c, ε, η and B so that for every D there would be arbitrarily large values of m for which there is a sequence $-1 \leqq x_1 < \cdots < x_m \leqq 1$ satisfying

$$\text{(25)} \qquad \arccos x_{i+1} - \arccos x_i > \eta/m$$

and for every α and β satisfying (7), (8) is satisfied. Finally this sequence would be such that there would exist a $P_n(x)$, $n < \frac{m}{1+c}$ satisfying (9) and

$$\text{(26)} \qquad \max_{-1 \leqq x \leqq 1} |P_n(x)| = D.$$

From these assumptions we have to derive a contradiction for sufficiently large D. Assume that $|P_n(x_0)| = D$, $-1 \leqq x_0 \leqq 1$ (i.e. $|P_n(x)|$ assumes its absolute maximum in $(-1, +1)$ at x_0). Put $\cos \theta_0 = x_0$, and let $B < t < T$ where T is sufficiently large and will be determined later (T is independent of m). By (25) and (8) (since (25) holds the η in (8) can be left out)

$$\text{(27)} \qquad N_m\left(\frac{t}{m}\right) > (1-\varepsilon)\frac{t}{\pi}\,.$$

$N_m\left(\frac{t}{m}\right)$ denotes the number of the x_i in $\left\{\cos \theta_0, \cos\left(\theta + \frac{t}{m}\right)\right\}$. On the other hand $T_n(x)$ has at most

$$\text{(28)} \qquad \frac{t}{\pi}\,\frac{n}{m} + 2 < \left(1 - \frac{c}{2}\right) N_m\left(\frac{t}{m}\right)$$

roots in $I\left(\frac{t}{m}\right)$ if $\varepsilon = \varepsilon(c)$ is sufficiently small.

Denote now by $-1 \leqq y_1 < \cdots < y_N \leqq 1$ the roots of $T_n(x)$ outside $I\left(-\frac{T}{m}, \frac{T}{m}\right)$ and our x_i in $I\left(-\frac{T}{m}, \frac{T}{m}\right)$. By (25) and (27), $N = n + O(1)$ where the error term $O(1)$ depends only on T. Denote by $S_{N-1}(x)$ the polynomial of degree at most $N-1$ which coincides with $P_n(x)$ on the x_i in $I\left(-\frac{t}{m}, \frac{t}{m}\right)$ and is 0 on the other y's. By the Lagrange interpolation formula

$$(29) \quad S_{N-1}(x) = \Sigma' P_n(x_i)\, l_k(x), l_k(x) = \frac{\omega(x)}{\omega'(y_k)(x-y_k)}, \omega(x) = \prod_{k=1}^{N} (x - y_k),$$

where in $\Sigma' k$ runs over the y_k (i.e. the x_k) in $I\left(-\frac{t}{m}, \frac{t}{m}\right)$. By (25) Lemma 1 can be applied for the $l_k(x)$ of (29) and we obtain for every $-1 \leqq x \leqq 1$

$$\left| S_{N-1}(x) \right| < C(\eta, T)\, \Sigma' \left| P_{n-1}(x_i) \right|.$$

By (25) the number of summands in Σ' is less than $\frac{2T}{\eta}$ hence by (9)

$$(30) \qquad \left| S_{N-1}(x) \right| < \frac{2T}{\eta}\, C(\eta, T), \qquad -1 \leqq x \leqq 1.$$

Choose now $D = \frac{6T}{\eta}\, C(\eta, T)$ and put

$$(31) \qquad R_{n_1}(x) = P_n(x) - S_{N-1}(x), \quad n_1 = \max(n, N-1) = n + O(1).$$

$R_{n_1}(x)$ vanishes at the $N_m\left(-\frac{T}{m}, \frac{T}{m}\right)$ x_i's in $I\left(-\frac{T}{m}, \frac{T}{m}\right)$. Thus by (31) (27) and (28), (12) (of Lemma 2) is satisfied by $R_{n_1}(x)$ with $B = t_1$, $T = t_2$ and $\frac{1}{1-c/2} = 1 + c_1$. At x_0 we have by (26), (29), (31) and the choice of D

$$(32) \qquad \left| R_{n_1}(x_0) \right| > \frac{1}{2} \max_{-1 \leqq x \leqq 1} \left| R_n(x) \right|$$

but this contradicts Lemma 2 for sufficiently large T. Hence the proof of Theorem 4 is complete and we showed that (4) is a sufficient condition that (3) should imply (1).

To complete the proof of Theorem 1 we have to prove the necessity of (4). In other words we shall show that if (4) is not satisfied then (3) does not imply (1). To show this it will suffice to show that the conditions of Theorem 4 are best possible. In other words we shall prove

Theorem 5. *Let A be an arbitrary positive number, $\eta = \eta(A)$ is suf-*

ficiently small, $\varepsilon > 0$ is fixed and $\delta < \varepsilon/2$ is arbitrary. Then there is a $B = B(A, \eta, \varepsilon, \delta)$ so that if

$$-1 \leqq x_1 < \cdots < x_m \leqq 1, \qquad m > m_0(A, \eta, \varepsilon, \delta, B)$$

is any sequence satisfying for some $0 \leqq \alpha < \alpha + \dfrac{B}{m} < \pi$

$$N_m^{(\eta)}\left(\alpha, \alpha + \frac{B}{m}\right) < B(1-\varepsilon). \tag{33}$$

Then there is a polynomial $P_n(x)$, $n < m(1-\delta)$ satisfying

$$|P_n(x_i)| \leqq 1, \quad \max_{-1 \leqq x \leqq 1} |P_n(x)| > A. \tag{34}$$

To make the idea of the proof more intelligible we first assume instead of (33) the stronger condition

$$N_m\left(\alpha, \alpha + \frac{B}{m}\right) < B(1-\varepsilon) \tag{35}$$

and deduce the existence of a polynomial satisfying (34) from (35). It will then be easy to modify our argument to show that (34) follows from (33) too.

First we define an auxiliary polynomial $Q_n(x)$. All the x_i in $\left(\alpha, \alpha + \dfrac{B}{m}\right)$ are roots of $Q_n(x)$ (the interval $\cos\beta < x < \cos\gamma$ will be denoted (β, γ)). In $\left(\alpha + \dfrac{B}{m}\left(1 - \dfrac{\varepsilon - \delta}{10}\right), 0\right)$ and in $\left(\pi, \alpha + \dfrac{B(\varepsilon-\delta)}{10m}\right)$ all the roots of $T_{[m(1-\delta)]}(x)$ are roots of $Q_n(x)$. By (33) and $\delta < \varepsilon$ we obtain that the degree of $Q_n(x)$ is less than $m(1-\delta)$. Thus by the theorem of M. Riesz [6] $Q_n(x)$ assumes its absolute maximum in $(-1, +1)$ in $\left(\alpha + \dfrac{B(\varepsilon-\delta)}{10m}, \alpha + \dfrac{B}{m}\left(1 - \dfrac{\varepsilon-\delta}{10}\right)\right)$, say at $x_0 = \cos\theta_0$.

Our polynomial $P_n(x)$ is obtained from $Q_n(x)$ as follows: All the common roots of $Q_n(x)$ and $T_{[m(1-\delta)]}(x)$ in $\alpha, \alpha + \dfrac{B(\varepsilon-\delta)}{10m}$ are moved to $\cos\alpha$ and all the common roots of $Q_m(x)$ and $T_{[m(1-\delta)]}(x)$ in $\left(\alpha+\frac{B}{m}\left(1-\frac{\varepsilon-\delta}{10}\right), \alpha+\frac{B}{m}\right)$ are moved to $\cos\left(\alpha+\frac{B}{m}\right)$.

Thus $\frac{B(\varepsilon-\delta)}{10}+O(1)$ roots are moved away from x_0 in both directions. We now show that $P_n(x)$ satisfies (34). First of all $P_n(x_i)=0$ for all the x_i in $(\alpha, \alpha+B/m)$, thus to complete our proof it will suffice to show that

$$|P_n(x_0)|/|P_n(z_0)| > A, \ |P_n(z_0)| = \max |P_n(x)| \tag{36}$$

where in (36) the maximum is taken over the x in $(-1,+1)$ which are not in $\left(\alpha, \alpha+\frac{B}{m}\right)$ (since (36) clearly implies $|P_n(x_0)| > A \max\limits_{1 \leqq i \leqq n} P_n(x_i)$ which is (34)). By assumption we have $|Q_n(x_0)| \geqq |Q_n(z_0)|$, hence (36) will follow if we can prove

$$\left|\frac{P_n(x_0)}{Q_n(x_0)}\right| > A \left|\frac{P_n(z_0)}{Q_n(z_0)}\right|. \tag{37}$$

The proof of (37) is almost identical with the proof of (17) and can be left to the reader. This completes the proof of Theorem 5 if we assume (35).

To complete our proof we now have to show that a polynomial $P_n(x)$, $n < m(1-\delta)$ exists satisfying (34) if we only assume (33) (instead of (35)). Choose $\eta = 2/\pi A$. By (33) and the definition of $N_m^{(\eta)}$ (see the introduction) there is a subsequence $x_{i_1}, \cdots, x_{i_r}$, $r < B(1-\varepsilon)$ of the x_i's in $\left(\alpha, \alpha+\frac{B}{m}\right)$ satisfying

$$\arccos x_{i_{j+1}} - \arccos x_{i_j} > \frac{2}{\pi A m} \left(\eta = \frac{2}{\pi A}\right)$$

so that for every x_u in $\left(\alpha, \alpha+\frac{B}{m}\right)$ there is an x_{i_j} satisfying

$$\left|\arccos x_u - \arccos x_{i_j}\right| \leqq \frac{2}{\pi A m}. \tag{38}$$

In view of our previous construction there is a polynomial $P_n(x)$, $n < m(1-\delta)$ satisfying

$$\max_{-1 \leqq x \leqq 1} \left| P_n(x) \right| = A, \quad P_n(x_{i_j}) = 0, \quad j = 1, \cdots, r \tag{39}$$

and

$$\max \left| P_n(x) \right| < 1 \tag{40}$$

where in (40) the maximum is taken over the $-1 \leqq x \leqq 1$ not in $\left(\alpha, \alpha + \frac{B}{m}\right)$. A well known theorem of Bernstein [2] states that if $f_n(\theta)$ is a trigonometric polynomial of degree n satisfying $\max\limits_{0 \leqq \theta \leqq 2\pi} \left| f_n(\theta) \right| = 1$ then $\max\limits_{0 \leqq \theta \leqq 2\pi} \left| f_n'(\theta) \right| \leqq n$, (thus from this theorem of Bernstein we easily obtain from (38) and (39) that for every x_u in $\left(\alpha, \alpha + \frac{B}{m}\right)$

$$\left| P_n(x_u) \right| < 1 . \tag{41}$$

(39), (40) and (41) prove (34) and hence the proof of Theorem 5 is complete, but this also finishes the proof of Theorem 1.

Finally we state without proof

Theorem 6. *To every A however large there is an $\varepsilon > 0$ so that if $n > n_0(A, \varepsilon)$, $m = [(1 + \varepsilon)n]$, then for every $-1 \leqq x_1 < \cdots < x_m \leqq 1$ there is a $P_n(x)$ satisfying*

$$\left| P_n(x_i) \right| \leqq 1 \quad i = 1, \cdots, m \quad \text{and} \quad \max_{-1 \leqq x \leqq 1} \left| P_n(x) \right| > A .$$

We do not give the proof of Theorem 6.

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