ON THE PARTIAL SUMS OF POWER SERIES*

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Abstract

A function $f(z)$ is said to belong to the class $\mathscr{F}$ if it is regular in $|z| < 1$ but not in any larger disc. If $f \in \mathscr{F}$ and $f(z) = \Sigma a_n z^n$ ($|z| < 1$) we investigate an aspect of the behaviour of the zeros of the partial sums

$$S_n(z) = \sum_{\nu=0}^{n} a_\nu z^\nu.$$

Let $\rho_n(f)$ be the largest number r such that $S_n(z)$ has a zero on $|z| = r$; and let

$$\rho(f) = \liminf_{n \to \infty} \rho_n(f), \quad P = \sup_{f \in \mathscr{F}} \rho(f).$$

It is well known, and not difficult to prove, that $1 \leqslant P \leqslant 2$. In this paper it will be proved that $\sqrt{2} < P < 2$. The proofs of $\sqrt{2} < P$ and $P < 2$ are by contradiction. By making the arguments more precise numerical constants $C_1 > 0$ and $C_2 > 0$ could be obtained such that $\sqrt{2} + C_1 < P$ and $P < 2 - C_2$. However, it was not thought worthwhile to obtain such constants C_1 and C_2 since they would almost certainly be far from sharp.

1. The function $f(z)$ is said to belong to the class $\mathscr{F}$ if $f(z)$ is regular in $|z| < 1$, but is not regular in $|z| < r$ for any $r > 1$. If $f \in \mathscr{F}$ and

$$f(z) = \sum_{0}^{\infty} a_n z^n \; (|z| < 1)$$

let $\rho_n(f)$ be the largest number r such that

$$S_n(z) = \sum_{0}^{n} a_\nu z^\nu$$

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has at least one zero on $|z| = r$. We assume that n is large enough for $\rho_n(f)$ to be well defined. Let

$$\rho(f) = \liminf_{n \to \infty} \rho_n(f)$$

and

$$P = \sup_{f \in \mathscr{F}} \rho(f).$$

We believe that it has been shown by Kakeya that $1 \leqslant P \leqslant 2$, but unfortunately we are unable to give any reference for this result. In this paper we shall show that $\sqrt{2} < P < 2$. It would be interesting to know what the correct value of P is, but the determination of P would appear to be rather difficult.

If for $f \in \mathscr{F}$ one considers the zeros of its partial sums $S_n(z)$ $(n \geqslant 0)$, then our result can be regarded as saying something about the possible extremal behaviour of these zeros. As regards the average behaviour of the zeros a number of results are known. The first of these seems to be due to Jentzsch [3, p. 238] who showed that each point of $|z| = 1$ is an accumulation point of the set of all zeros of $S_n(z)$ for $n \geqslant 0$. Later Szego [2] showed that if $\epsilon > 0$ is given then there is a sequence $\{n_k\}$, depending on the particular $f(z)$, such that all but $0(n_k)$ of the zeros of $S_{n_k}(z)$ lie in $1 - \epsilon \leqslant |z| \leqslant 1 + \epsilon$ and are uniformly distributed. The latter means that if $N(n_k; \alpha, \beta)$ is the number of zeros of $S_{n_k}(z)$ in $\alpha \leqslant \arg z \leqslant \beta$ $(0 < \beta - \alpha \leqslant 2\pi)$, then for fixed α, β we have

$$\frac{N(n_k;\, \alpha, \beta)}{n_k} \to \frac{\beta - \alpha}{2\pi}\, (k \to \infty).$$

More precise results of this kind were obtained by Erdös and Turán [1].

2. *Theorem* 1

$$P > \sqrt{2}.$$

We shall construct a function $g \in \mathscr{F}$ and prove that this function satisfies $\rho(g) > \sqrt{2}$. Let $a_0 = 1$ and suppose that for $n \geqslant 0$ we have defined $a_0, a_1, \ldots, a_n$ $(|a_\nu| = 1, 0 \leqslant \nu \leqslant n)$. If

$$S_n(z) = \sum_0^n a_\nu z^\nu$$

and

$$M(r, S_n) = \max_{|z| = r} |S_n(z)|,$$

then $M(1, S_n) > 1$ $(n \geqslant 1)$ and

$$\frac{M(r, S_n)}{r^{n+1}} \to 0 \ (r \to \infty).$$

Take $R = R_{n+1}$ as the largest value of r such that

$$\frac{M(r, s_n)}{r^{n+1}} = 1.$$

Then $R > 1$. There is in fact only one value of r such that

$$\frac{M(r, s_n)}{r^{n+1}} = 1,$$

but this fact plays no part in our arguments. Choose ζ ($|\zeta| = R$) so that $|S_n(\zeta)| = M(R, S_n)$ and put

$$a_{n+1} = -\frac{S_n(\zeta)}{\zeta^{n+1}}$$

Then $|a_{n+1}| = 1$ and $S_{n+1}(z)$ has a zero at $z = \zeta$. Let

$$g(z) = \sum_0^\infty a_n z^n \; (|z| < 1)$$

be the function obtained from this construction. Clearly $g \in \mathscr{F}$ and

$$\rho_n(g) \geqslant R_n \; (n \geqslant 1). \tag{1}$$

Lemma 1.

$$\liminf_{n\to\infty} R_n \geqslant \sqrt{2}.$$

Let ϵ satisfy $0 < \epsilon < 1$ and put $\lambda = \sqrt{2} - \epsilon$. By Parseval's Theorem [3, p. 84],

$$M^2(\lambda, S_n) \geqslant \frac{1}{2\pi}\int_0^{2\pi} |S_n(\lambda e^{i\theta})|^2 \, d\theta$$

$$= \sum_0^n |a_\nu|^2 \lambda^{2\nu}$$

$$= \frac{\lambda^{2n+2} - 1}{\lambda^2 - 1}.$$

Hence there is an $n_0 = n_0(\epsilon)$ such that

$$\frac{M(\lambda, S_n)}{\lambda^{n+1}} \geqslant \frac{(1 - \lambda^{-2n-2})^{\frac{1}{2}}}{1 - \epsilon} > 1 \; (n > n_0).$$

Consequently for $n > n_0$, $R_{n+1} > \lambda = \sqrt{2} - \epsilon$. As ϵ is arbitrary apart from satisfying $0 < \epsilon < 1$ the lemma follows.

From (1) and Lemma 1 it follows that $\rho(g) \geqslant \sqrt{2}$. We shall prove that $\rho(g) > \sqrt{2}$ by showing that the assumption $\rho(g) = \sqrt{2}$ leads to a contradiction. Suppose then that $\rho(g) = \sqrt{2}$. From (1) and Lemma 1 it follows that there is an increasing infinite sequence $\{n_k\}$ such that $R_{n_k+1} \to \sqrt{2}$ $(k \to \infty)$. We now consider the behaviour of the general nth partial sum $S_n(z)$ of $g(z)$ and introduce a number of auxiliary polynomials. By examining these as $n \to \infty$ through the sequence $\{n_k\}$ we show that the desired contradiction is obtained.

Henceforth assume that $n \geqslant 1$. We have

$$\frac{\mathrm{M}(R, S_n)}{R^{n+1}} = 1 \; (R = R_{n+1})$$

and so

$$|a_0 + a_1 R z + \dots\dots\dots + a_n R^n z^n| \leqslant R^{n+1} \; (|z| \leqslant 1).$$

From this it follows easily that

$$| a_n + \frac{a_{n-1}}{R} z + \dots\dots\dots + \frac{a_0}{R^n} z^n | \leqslant R \; (| z | \leqslant 1). \tag{2}$$

Let

$$\frac{a_{n-j}}{a_n} = b_j \; (0 \leqslant j \leqslant n).$$

Then $| b_j | = 1 \; (0 \leqslant j \leqslant n)$ and (2) can be written

$$| 1 + \frac{b_1}{R} z + \dots\dots\dots + \frac{b_n}{R^n} z^n | \leqslant R \; (| z | \leqslant 1). \tag{3}$$

Let $p(z)$ be the polynomial within the modulus bars on the left of (3) and define, with $R = R_{n+1}$,

$$h(z) = \frac{R\left(\frac{1}{R} - z\right).}{1 - \frac{z}{R}}$$

As $R > 1 \; (n \geqslant 1)$ it follows from (3) that the image domain of $| z | < 1$ by $p(z)$ is contained in that by $h(z)$. We also have that $p(0) = g(0) = 1$. Hence in $| z | < 1$, $p(z)$ is subordinate to $h(z)$ and so there is a function

$$w(z) = \sum_1^\infty w_n z^n \; (| z | < 1)$$

satisfying $| w(z) | < 1 \; (| z | < 1)$ such that

$$p(z) = \frac{R\left(\frac{1}{R} - w(z)\right)}{1 - \frac{w(z)}{R}}$$

$$= 1 + \sum_1^\infty c_n w^n(z) \quad (| z | < 1), \tag{4}$$

where

$$c_n = - \frac{(R^2 - 1)}{R^n} \; (n \geqslant 1). \tag{5}$$

Consequently

$$\frac{b_1}{R} = - \frac{(R^2 - 1) \; w_1}{R},$$

so that $w_1 \neq 0$ and

$$b_1 = - (R^2 - 1) \; w_1. \tag{6}$$

Let

$$q(z) = z^{n-1} \frac{S_{n-1}\left(\frac{1}{z}\right)}{a_n} = b_1 + b_2 z + \dots + b_n \; z^{n-1},$$

and if $w_1 = t \; e^{i\phi}$ put (7)

$$Q(z) = - e^{-i\phi} \; q \; (ze^{-i\phi}) = d_1 + d_2 z + - + d_n z^{n-1},$$

where $|d_j| = 1$ $(1 \leqslant j \leqslant n)$. Note that apart from the a_r each coefficient in the above depends not only on the displayed suffix but also on n, the order of the partial sum considered.

We now take the sequence $\{n_k\}$ such that $R_{n_k+1} \to \sqrt{2}$ $(k \to \infty)$ and examine the situation for $n = n_k$ in the above as $k \to \infty$. In what follows '$k \to \infty$' will always refer to the k of n_k. From (6) it follows that $|w_1| \to 1$ $(k \to \infty)$ since $R = R_{n_k+1} \to \sqrt{2}$ $(k \to \infty)$. Hence for each fixed $j > 1$, $w_j \to 0$ $(k \to \infty)$ since

$$w(z) = \sum_1^\infty w_n z^n$$

satisfies $|w(z)| < 1$ $(|z| < 1)$. Therefore given any fixed positive integer N we obtain, from (4) and (5),

$$b_j \sim -w_1{}^j \ (k \to \infty;\ 1 \leqslant j \leqslant N).$$

Hence it follows from (7) that

$$d_j \to 1 \ (k \to \infty;\ 1 \leqslant j \leqslant N) \tag{8}$$

To complete the proof we require the following result.

Lemma 2. *Let* γ $(0 < \gamma < 1)$ *be given. Then there is an* $\eta = \eta(\gamma)$ $(0 < \eta < 1)$ *such that any polynomial*

$$A(z) = A_0 + A_1 z + \ldots + A_n z^n \quad \left(n \geqslant \frac{1}{\eta}\right)$$

satisfying $|A_j| = 1$ $(0 \leqslant j \leqslant n)$ *and*

$$|A_j - 1| \leqslant \eta \left(0 \leqslant j \leqslant \frac{1}{\eta}\right)$$

has all its zeros in $|z| > \gamma$.

Choose $\eta = \eta(\gamma)$ so that $0 < \eta < 1$ and

$$\frac{1 - \gamma^{\frac{1}{\eta}}}{1 + \gamma} > \eta \, \frac{1 - \gamma^{\frac{1}{\eta}+1}}{1 - \gamma} + \frac{\gamma^{\frac{1}{\eta}}}{1 - \gamma}. \tag{9}$$

As the right hand side of (9) tends to 0 and the left hand side tends to

$$\frac{1}{1 + \gamma}$$

as $\eta \to 0+$ such a choice is possible. Now for $|z| < 1$, with

$$\left[\frac{1}{\eta}\right] = N - 1,$$

$$A(z) = \frac{1 - z^N}{1 - z} + B_1(z) + B_2(z),$$

where

$$|B_1(z)| \leqslant \eta \, \frac{1 - |z|^N}{1 - |z|}, \quad |B_2(z)| \leqslant \frac{|z|^N}{1 - |z|}.$$

Hence if $z(| z | < 1)$ is a zero of $A(z)$ we find that

$$\frac{1 - | z |^N}{1 + | z |} \leqslant \eta \frac{1 - | z |^N}{1 - | z |} + \frac{| z |^N}{1 - | z |.}$$

As

$$\left[\frac{1}{\eta}\right] = N - 1$$

this implies that

$$\frac{1 - | z |^{\frac{1}{\eta}}}{1 + | z |} \leqslant \eta \frac{1 - | z |^{\frac{1}{\eta}+1}}{1 + | z |} + \frac{| z |^{\frac{1}{\eta}}}{1 - | z |} \quad . \tag{10}$$

The left hand side of (10) decreases and the right hand side increases as $| z | < 1$ increases and so, if $| z | \leqslant \gamma$ then

$$\frac{1 - \gamma^{\frac{1}{\eta}}}{1 + \gamma} \leqslant \eta \frac{1 - \gamma^{\frac{1}{\eta}+1}}{1 - \gamma} + \frac{\gamma^{\frac{1}{\eta}}}{1 - \gamma} \quad .$$

But this inequality contradicts (9) which defines η in terms of γ. Consequently any zero of $A(z)$ must satisfy $| z | > \gamma$. This proves the lemma.

Take γ of Lemma 2 to be

$$\frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

and η_0 to be a corresponding value of η. Let

$$\left[\frac{1}{\eta_0}\right] = N_0 - 2.$$

From (8) there is an integer k_0 such that

$$| d_j - 1 | \leqslant \eta_0 \; (k > k_0;\, 1 \leqslant j \leqslant N_0).$$

By Lemma 2 all the zeros of $Q(z)$ lie in

$$| z | > \frac{1}{2}\left(1 + \frac{1}{\sqrt{2}}\right)$$

for $k > k_0$. Hence, from the definition of $Q(z)$ in (7), all the zeros of $S_{n_k - 1}(z)$ lie in

$$| z | < 2\left(1 + \frac{1}{\sqrt{2}}\right)^{-1} = \frac{2}{1 + \sqrt{2}} \sqrt{2}.$$

However, this contradicts the result that $\rho(g) \geqslant \sqrt{2}$ which has already been established. Consequently $\rho(g) > \sqrt{2}$ and Theorem 1 is proved.

3. *Theorem* 2.

$$P < 2$$

In order to prove Theorem 2 one has to show that there is an absolute constant $c > 0$ such that for any $f \epsilon \mathscr{F}$ we have $\rho(f) < 2 - c$. We shall in fact only prove the weaker result that for any $f \epsilon \mathscr{F}$ we have $\rho(f) < 2$. We have adopted this approach since it will be clear how our proof of the weaker result can be modified to give the stronger result. At the same time we shall avoid a number of rather complicated, but nevertheless straightforward, technical difficulties that such a modification would entail.

We assume that there is an

$$f(z) = \sum_0^\infty a_n z^n \ (\mid z \mid < 1)$$

belonging to $\mathscr{F}$ for which $\rho(f) \geqslant 2$ and show that this leads to a contradiction.

Lemma 3. *If*

$$f(z) = \sum_0^\infty a_n z^n \ (\mid z \mid < 1)$$

belongs to $\mathscr{F}$ then there is an increasing infinite sequence of integers depending on f, which we denote by σ, such that

$$\limsup_{\substack{n \to \infty \\ n \epsilon \sigma}} \left\{ \max_{0 \leqslant \nu \leqslant n} \left| \frac{a_\nu}{a_n} \right|^{\frac{1}{n-\nu}} \right\} = 1.$$

Since $f \epsilon \mathscr{F}$ it follows that for any $l > 1$ the sequence $\{ \mid a_n \mid l^n \}$ is unbounded. Hence there is an increasing infinite sequence of integers n such that

$$\mid a_\nu \mid l^\nu \leqslant \mid a_n \mid l^n \quad (0 \leqslant \nu \leqslant n)$$

and so

$$\left| \frac{a_\nu}{a_n} \right|^{\frac{1}{n-\nu}} \leqslant l \ (0 \leqslant \nu \leqslant n).$$

From this result it is clear that a sequence σ for which Lemma 3 is true can be constructed.

Lemma 4. *Suppose that*

$$f(z) = \sum_0^\infty a_n z^n \ (\mid z \mid < 1)$$

belongs to $\mathscr{F}$ and satisfies $\rho(f) \geqslant 2$. If σ is any sequence for which Lemma 3 is true then, for any fixed positive integer k we have

$$\lim_{\substack{n \to \infty \\ n \epsilon \sigma}} \left| \frac{a_{n-k}}{a_n} \right| = 1$$

Corollary. Under the conditions of Lemma 4, if σ^* is the sequence of integers n such that $n + 1 \epsilon \sigma$, then

$$\limsup_{\substack{n \to \infty \\ n \epsilon \sigma^*}} \left\{ \max_{0 \leqslant \nu \leqslant n} \left| \frac{a_\nu}{a_n} \right|^{\frac{1}{n-\nu}} \right\} = 1$$

Let ζ_n be a zero of largest modulus of

$$\sum_0^n a_\nu z^\nu,$$

assuming that n is large enough for ζ_n to be well defined. Then

$$a_n \zeta_n{}^n = -\sum_0^{n-1} a_\nu \zeta_n{}^\nu$$

and so, with $\lambda_n = | \zeta_n |$,

$$| a_n | \lambda_n{}^n \leqslant \sum_0^{n-1} | a_\nu | \lambda_n{}^\nu \cdot \tag{11}$$

Suppose now that $n \epsilon \sigma$ and that

$$\max_{0 \leqslant \nu \leqslant n} \left| \frac{a_\nu}{a_n} \right|^{\frac{1}{n-\nu}} = l_n. \tag{12}$$

Then

$$1 \leqslant \sum_{\nu=0}^{n-1} \left(\frac{l_n}{\lambda_n}\right)^{n-v} = \frac{l_n}{\lambda_n} \frac{1 - (l_n/\lambda_n)^n}{1 - l_n/\lambda_n} \quad .$$

As $n \to \infty$ through σ we have lim sup $l_n = 1$ and lim inf $\lambda_n \geqslant 2$ so that

$$1 \leqslant \liminf \frac{1}{\lambda_n - 1}$$

Hence it follows that lim sup $\lambda_n \leqslant 2$ as $n \to \infty$ through σ. Consequently

$$\lim_{\substack{n\to\infty \\ n\epsilon\sigma}} \lambda_n = 2. \tag{13}$$

For any fixed positive integer k and $n > k$ $(n \epsilon \sigma)$ it follows from (11) and (12) that

$$1 \leqslant \sum_{0}^{n-1} \left(\frac{l_n}{\lambda_n}\right)^{n-\nu} + \left\{ \left|\frac{a_{n-k}}{a_n}\right| \lambda_n^{-k} - \left(\frac{l_n}{\lambda_n}\right)^k \right\}$$

$$= \frac{l_n}{\gamma_n} \frac{1-(l_n/\lambda_n)^n}{1-l_n/\lambda_n} + \left\{ \left|\frac{a_{n-k}}{a_n}\right| \lambda_n^{-k} - \left(\frac{l_n}{\lambda_n}\right)^k \right\} .$$

Letting $n \to \infty$ through σ we obtain, using (13),

$$1 \leqslant 1 + \frac{1}{2^k} \liminf \left(\left|\frac{a_{n-k}}{a_n}\right| - 1 \right),$$

so that

$$\liminf_{\substack{n\to\infty \\ n\epsilon\sigma}} \left|\frac{a_{n-k}}{a_n}\right| = 1.$$

From this and the definition of the sequence σ the result of Lemma 4 follows.

The corollary is an immediate consequence of Lemmas 3 and 4.

Lemma 5. *Under the conditions of Lemma* 4 *we have*

$$\frac{a_{n-k}}{a_n} \sim - e^{ik\psi_n} \text{ (as } n \to \infty \text{ through } \sigma)$$

where, in the previous notation, $\psi_n = \arg \zeta_n$.

We assume that $k > 1$ and that we have proved that as $n \to \infty$ through σ then

$$\frac{a_{n-\nu}}{a_n} \sim - e^{i\nu\psi_n} \ (1 \leqslant \nu \leqslant k-1).$$

From

$$a_n \zeta_n^n + \ldots + a_{n-k} \zeta_n^{n-k} = - \sum_{0}^{n-k-1} a_\nu \zeta_n^\nu$$

it follows that, using (12) and $\lambda_n = |\zeta_n|$,

$$\left| 1 + \frac{a_{n-1}}{a_n} \zeta_n^{-1} + \ldots + \frac{a_{n-k}}{a_n} \zeta_n^{-k} \right| \leqslant \sum_{0}^{n-k-1} \left(\frac{l_n}{\lambda_n}\right)^{n-\nu}$$

$$= \left(\frac{l_n}{\lambda_n}\right)^{k+1} \frac{1-(l_n/\lambda_n)^{n-k}}{1-l_n/\lambda_n} .$$

Hence

$$\limsup_{\substack{n\to\infty \\ n\epsilon\sigma}} \left| 1 + \frac{a_{n-1}}{a_n} \zeta_n^{-1} + \ldots + \frac{a_{n-k}}{a_n} \zeta_n^{-k} \right| \leqslant \frac{1}{2^k}$$

By the induction hypothesis and (13) we obtain

$$\limsup_{\substack{n\to\infty \\ n\epsilon\sigma}} \left| 1 - \frac{1}{2} - \frac{1}{2^2} \cdots\cdots \frac{1}{2^{k-1}} + \frac{1}{2^k} \frac{a_{n-k}}{a_n} e^{-ik\psi n} \right| \leqslant \frac{1}{2^k}$$

or

$$\limsup_{\substack{n\to\infty \\ n\epsilon\sigma}} \left| \frac{1}{2^{k-1}} + \frac{1}{2^k} \frac{a_{n-k}}{a_n} e^{-ik\psi n} \right| \leqslant \frac{1}{2^k}.$$

Consequently

$$\frac{1}{2^{k-1}} + \frac{1}{2^k} \limsup_{\substack{n\to\infty \\ n\epsilon\sigma}} Re \left(\frac{a_{n-k}}{a_n} e^{-ik\psi n} \right) \leqslant \frac{1}{2^k}$$

or

$$\limsup_{\substack{n\to\infty \\ n\epsilon\sigma}} \mathrm{Re} \left(\frac{a_{n-k}}{a_n} e^{-ik\psi n} \right) \leqslant -1.$$

From the above and Lemma 4 it follows that

$$\frac{a_{n-k}}{a_n} \sim -e^{ik\psi n} \ (n \to \infty;\ n \epsilon \sigma).$$

This completes the proof of the lemma apart from showing the result to be true for $k = 1$ so that the induction argument is valid. It is clear that this can be done by taking $k = 1$ in the above discussion.

The result of Lemma 5 is valid when $\rho(f) \geqslant 2$ and $n \to \infty$ through any sequence σ for which Lemma 3 is satisfied. By the corollary to Lemma 4 we see that Lemma 5 is also true when the sequence σ is replaced by σ^*. Hence if $\rho(f) \geqslant 2$, then

$$\frac{a_{n-1-k}}{a_{n-1}} \sim -e^{ik\psi n-1} \ (n \to \infty;\ n \epsilon \sigma). \tag{14}$$

From (14) and Lemma 5 it follows that as $n \to \infty$ through σ, then

$$\begin{cases} \dfrac{a_{n-2}}{a_{n-1}} & e^{i\psi n} \\ \dfrac{a_{n-3}}{a_{n-1}} & \sim e^{i2\psi n}, \end{cases} \qquad \begin{cases} \dfrac{a_{n-2}}{a_{n-1}} & \sim -e^{i\psi n-1} \\ \dfrac{a_{n-3}}{a_{n-1}} & \sim -e^{i2\psi n-1}. \end{cases}$$

Hence as $n \to \infty$ through σ, then

$$e^{i\psi n} \sim -e^{i\psi n-1},\ e^{2i\psi n} \sim -e^{i2\psi n-1}.$$

But clearly these two asymptotic results are mutually incompatible and so we have arrived at a contradiction. Therefore the assumption which led to this contradiction, viz. $\rho(f) \geqslant 2$, is false so that $\rho(f) < 2$.

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