PARTITION RELATIONS AND TRANSITIVITY DOMAINS OF BINARY RELATIONS

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1. *Introduction*

Let α, β and γ be order types and r a positive integer. The *partition* relation [1]

$$\alpha \to (\beta, \gamma)^r \tag{1}$$

expresses, by definition, the following condition. If S is an ordered set, of order type tp $S = \alpha$, and if the set $[S]^r$ of all subsets of S of exactly r elements is arbitrarily expressed as the union of two sets K_0, K_1, then there always exists a set $X \subset S$ such that either $\operatorname{tp} X = \beta$ and $[X]^r \subset K_0$, or $\operatorname{tp} X = \gamma$ and $[X]^r \subset K_1$. The following result is known [1; Theorem 25] involving the least infinite ordinal $\omega_0 = \omega$ and the negation $\alpha \not\to (\beta, \gamma)^r$ of (1).

THEOREM 1. *Given positive integers m and n, there is a positive integer $l_0(m, n)$ such that*

$$\omega_0 l_0(m, n) \to (m, \omega_0 n)^2,$$

$$\gamma \not\to (m, \omega_0 n)^2$$

for every ordinal $\gamma < \omega_0 l_0(m, n)$. The number $l_0(m, n)$ is the least positive integer l *such that, whenever $\rho(\lambda, \mu) \in \{0, 1\}$ for $0 \leqslant \lambda, \mu < l$,* then *there* always *exists either* (i) a *system $\lambda_0, \ldots, \lambda_{m-1}$ of m distinct numbers out of 0, 1, . . , $l-1$. such that $\rho(\lambda_i, \lambda_j) = 0$ for $0 \leqslant i < j < m$, or* (ii) a system $\lambda_0, \ldots, \lambda_{n-1}$ *of n distinct numbers out of 0, ... $l-1$ such that $\rho(\lambda_i, \lambda_j) = \rho(\lambda_j, \lambda_i) = 1$ for $0 \leqslant i < j < n$.*

It will be seen that $l_0(m, n)$ is characterized by a finite combinatorial property and can therefore be determined for every given pair m, n. We have $l_0(1, n) = l_0(m, 1) = 1$ for all m and n, and $l_0(m, 2) = 2^{m-1}$ for $m \leqslant 4$. In Theorem 2 of this note we show that, more generally, there is a positive integer $l(m, n)$ such that, for every ordinal α and positive integers m, n,

$$\omega_\alpha l(m, n) \to (m, \omega_\alpha n)^2.$$

We recall that ω_α is the least ordinal whose cardinal is $\aleph_\alpha$. We give an explicit upper estimate for $l(m, 12)$. We conjecture that $l_0(m, n)$ can be taken as l (m, n) but have only been able to prove this when $m \leqslant 4$ and n $\leqslant 2$.

Theorem 3 is a " stepping-up " result of the general form : if $\alpha_\nu \to (\beta)_k^1$ for all $\nu < n$ and certain k, and if every α_ν is a power of ω, then $\Sigma\, \alpha_\nu \to (3, \beta)^2$. The symbols involved here will be defined in §6.

Theorem 4 was suggested by the following corollary of Theorem 2. Let the binary relation $x \prec y$ be defined on a set S and have the property

Received 1 January, 1966.

[JOURNAL LONDON MATH. SOC., 42 (1967), 624–633]

that for $x, y \in S$ exactly one of the relations $x = y$; $x \prec y$; $y \prec x$ holds. Then, given a positive integer a, there always exists a subset X of S, of cardinal a, such that the given relation is transitive on X, provided that S has at least 2^{a-1} elements. This result was first obtained by R. Stearns [7]. His proof is reproduced in [8; p. 126] and is very simple indeed. In the present note we establish a similar result for infinite cardinals a.

We should like to thank the referee for his helpful suggestions and, above all, for having pointed out two errors in the first version of this paper.

2. Small letters denote ordinals unless another convention is introduced.

THEOREM 2. *Given positive integers m and n, there is a positive integer $l(m, n)$ such that, for every α,*

$$\omega_\alpha l(m, n) \to (m, \omega_\alpha n)^2. \tag{2}$$

If $l_\alpha(m, n)$ denotes the least number $l(m, n)$ such that (2) holds for a given α, then†

$$l_\alpha(m, n) \leqslant (2n-3)^{-1}[2^{m-1}(n-1)^m + n - 2], \tag{3}$$

$$\gamma \not\to (m, \omega_\alpha n)^2 \tag{4}$$

for every $\gamma < \omega_\alpha l_\alpha(m, n)$. Also, (4) *holds for every* α and every $\gamma < \omega_\alpha l_0(m, n)$.

Remarks. (i) We conjecture that $l_\alpha(m, n) = l_0(m, n)$. This has so far only been proved when $m \leqslant 4$ and $n \leqslant 2$.

(ii) If in the first relation of Theorem 1 we make $n \to \omega$ we obtain, formally,

$$\omega^2 \to (m, \omega^2)^2 \quad (m < \omega),$$

a relation which was, in fact, proved by Specker [3]. It is not known whether the same process, when applied to the relation (2) of Theorem 2, leads to a correct relation. This has not even been decided for $\alpha = 1$ and $m = 3$ when we are led to the relation

$$\omega_1 \omega \to (3, \omega_1 \omega)^2.$$

3. Before proving Theorem 2 we introduce some notation and conventions. For $\alpha \leqslant \beta$ we put $[\alpha, \beta) = \{\xi : \alpha \leqslant \xi < \beta\}$. Capital letters denote sets, and $|A|$ denotes the cardinal of A. By $|\alpha|$ we denote the cardinal of a set ordered according to the type α. By $A \subset B$, $A+B$, $A-B$ and AB we denote inclusion in the wide sense, union, difference and intersection respectively. Also, $\Sigma(\nu \in N) A$, and $\Pi(\nu \in N) A$, are alternative ways of denoting unions and intersections. If S is ordered and $[S]^r = \Sigma(\lambda \in L) K_\lambda$,

† The right hand side of (3) is a positive integer, equal to $1 + (n-1)\,\Sigma(\mu < m-1)\,(n-1)^\mu$.

then we put

$$[K_\lambda] = \{\mathrm{tp}\, X : X \subset S;\ [X]^r \subset K_\lambda\} \quad (\lambda \in L).$$

If, in addition, $r = 2$ then we put

$$U_\lambda(x) = \{y : \{x, y\} \in K_\lambda\} \quad (\lambda \in L;\ x \in S).$$

We use the obliteration operator $\hat{}$ whose effect on a well-ordered sequence consists in removing the member above which it is placed. The symbol $\{x_0, \ldots, \hat{x}_n\}_{\neq}$ denotes the set $\{x_0, \ldots, \hat{x}_n\}$ and also expresses the condition that $x_\mu \neq x_\nu$ for $\mu < \nu < n$. Following Tarski, we denote by $\mathrm{cf}(\alpha)$ the least β such that $\aleph_\alpha$ is the sum of $\aleph_\beta$ cardinals less than $\aleph_\alpha$. We put $a' = \aleph_\beta$ if $a = \aleph_\alpha$ and $\mathrm{cf}(\alpha) = \beta$. The symbol $A +' B$ denotes the set $A + B$ and, at the same time, expresses the condition that $AB = \emptyset$. More generally, $\Sigma'(\nu < n)\, A_\nu$ denotes the set $\Sigma(\nu < n)\, A_\nu$ and also expresses the condition that $A_\mu A_\nu = \emptyset$ for $\mu < \nu < n$. If, for $\nu < n$, S_ν is an ordered set then $\Sigma(\nu < n)\, S_\nu(\mathrm{tp})$ denotes the set $S = \Sigma'(\nu < n)\, S_\nu$ and expresses the fact that S is ordered in such a way that the order in each S_ν is preserved and every member of S_μ precedes every member of S_ν for $\mu < \nu < n$.

4. We need the following result in the theory of graphs due to de Bruijn and Erdős [2]. Let $c < \omega$, and let Γ be a finite directed graph such that from every node of Γ there start fewer than c edges. Then the chromatic number of Γ is less than $2c$. For convenience we state this result without using the language of graphs and give the very simple proof.

Lemma. *Let* $p < \omega;\ Q \subset \{(\beta, \gamma) : \beta, \gamma < p;\ \beta \neq \gamma\}$,

$$v(\beta) = |\{\gamma : (\beta, \gamma) \in Q\}| < c < \omega \quad (\beta < p).$$

Then there are numbers $k_0, \ldots, \hat{k}_p < 2c - 1$ *such that* $k_\beta \neq k_\gamma$ *whenever* $(\beta, \gamma) \in Q$.

Proof. The case $p = 0$ is trivial. Let $p \geq 1$ and use induction over p. Put $w(\beta) = |\{\gamma : (\beta, \gamma) \in Q$ or $(\gamma, \beta) \in Q\}|\ (\beta < p)$. We may assume that $w(0) \geq \ldots \geq w(p-1)$. By induction hypothesis there are numbers $k_0, \ldots, \hat{k}_{p-1} < 2c - 1$ such that $k_\beta \neq k_\gamma$ whenever $\beta, \gamma < p - 1$ and $(\beta, \gamma) \in Q$. Since

$$w(p-1) \leq p^{-1} \Sigma(\beta < p)\, w(\beta) = 2p^{-1} \Sigma(\beta < p)\, v(\beta) \leq 2c - 2,$$

there is $k_{p-1} < 2c - 1$ such that $k_{p-1} \neq k_\beta$ whenever $(\beta, p-1) \in Q$ or $(p-1, \beta) \in Q$. This proves the lemma.

5. *Proof of Theorem 2.* Let $\alpha \geq 0$. Then, clearly, for $1 \leq n < \omega$, we have $l_\alpha(1, n) = 1$ and $l_\alpha(2, n) = n$. It is known ([5], also [1; Theorem 44]) that

$$\omega_\alpha \to (\omega_0, \omega_\alpha)^2. \tag{5}$$

Hence $l_\alpha(m, 1) = 1$ for $1 \leq m < \omega$. It is now easily verified that (3) holds for $m < 3$ and arbitrary n, and also for $n = 1$ and arbitrary m. Therefore it suffices to consider the *case* when $m \geqslant 3$ and $n \geqslant 2$, and we may assume that $l_\alpha(\mu, n)$ exists for $1 \leqslant \mu < m$.

Let $1 \leqslant p < \omega$ and $\omega_\alpha p \not\to (m, \omega_\alpha n)^2$. This is, for instance, true if $p = 1$. Let $S = \Sigma(\pi < p)\, S_\pi(\mathrm{tp})$, where $\mathrm{tp}\, S_\pi = \omega_\alpha$ for $\pi < p$. Then $\mathrm{tp}\, S = \omega_\alpha p$, and there is a partition $[S]^2 = K_0 + K_1$ such that

$$m \notin [K_0];\ \omega_\alpha n \notin [K_1]. \tag{6}$$

By (5) and the relations $\omega \to m \notin [K_0]$ there is a set $S_\pi' \subset S_\pi$ such that $|S_\pi'| = \aleph_\alpha$ and $[S_\pi']^2 \subset K_1$ for $\pi < p$. Put $N = \{(\beta, \gamma) : \{\beta, \gamma\}_{\neq} \subset [0, p)\}$. We define operators O_0, O_1 which operate on systems $(A_0, \ldots, A_{p-1})$ of p sets $A_\pi \subset S_\pi'$ and are defined as follows. We have, for $\lambda < 2$, $O_\lambda(A_0, \ldots, A_{p-1}) = (A_0^\lambda, \ldots, A_{p-1}^\lambda)$. *We* now define A_π^λ.

(i) If there is a pair $(\beta, \gamma) \in N$ such that, for suitable sets $A_\beta' \subset A_\beta$ and $A_\gamma' \subset A_\gamma$, we have $|A_\beta'| = |A_\gamma'| = \aleph_\alpha$ and $|U_0(x) A_\gamma'| < \aleph_\alpha$ for $x \in A_\beta'$, then we choose such β, γ and put $A_\beta^0 = A_\beta'$; $A_\gamma^0 = A_\gamma'$; $A_\pi^0 = A_\pi$ for $\pi \in [0, p) - \{\beta, \gamma\}$. Then $|A_\beta^0| = |A_\gamma^0| = \aleph_\alpha$ and $|U_0(x) A_\gamma^0| < \aleph_\alpha$ for $x \in A_\beta^0$. If there is no such pair (β, γ) then we put $A_\pi^0 = A_\pi$ for $\pi < p$.

(ii) If there is a pair $(\beta, \gamma) \in N$ such that, for a suitable element $\bar{x} \in A_\beta$, we have $|U_0(\bar{x}) A_\gamma| = \aleph_\alpha$, then we choose such β and γ and put $A_\beta^1 = A_\beta - \{x : |U_0(x) A_\gamma| < \aleph_\alpha\}$; $A_\pi^1 = A_\pi$ for $\pi \in [0, p) - \{\beta\}$. Then $|U_0(x) A_\gamma^1| = \aleph_\alpha$ for $x \in A_\beta^1$. If there is no such pair (β, γ) then we put $A_\pi^1 = A_\pi$ for $\pi < p$.

We now iterate these operators O_λ and put in particular

$$O_1^{p(p-1)} O_0^{p(p-1)} (S_0', \ldots, S_{p-1}') = (S_0'', \ldots, S_{p-1}'').$$

Then $S_\pi'' \subset S_\pi' \subset S_\pi$ and $|S_\pi''| = \aleph_\alpha$ for $\pi < p$.

Denote by P the set of all pairs (β, γ) such that $\beta, \gamma < p$ and $|U_0(x) S_\gamma''| < \aleph_\alpha$ for $x \in S_\beta''$. Then, by definition of S_π', $(\pi, \pi) \in P$ for $\pi < p$. We have, for $\beta < p$ and all $x \in S_\beta''$,

$$\begin{aligned} |U_0(x) S_\gamma''| &< \aleph_\alpha \text{ if } (\beta, \gamma) \in P, \\ &= \aleph_\alpha \text{ if } (\beta, \gamma) \in N - P. \end{aligned}$$

In order to see this we need only observe that the relations between cardinals of sets of the form $U_0(x)\, A$ which have been established by an application of O_0 and O_1 are not destroyed by any further applications of the operators O_λ.

Let $\beta, \gamma < p$ and $x \in S_\beta''$. Then $\mathrm{tp}\, U_0(x) S_\gamma'' = \omega_\alpha$ if $(\beta, \gamma) \notin P$. Moreover, by (6) and the definition of $l_\alpha(m-1, n) = c$, *say,* we have $\mathrm{tp}\, U_0(x) < \omega_\alpha c$. Hence $v(\beta) < c$, where

$$v(\beta) = |\{\gamma : (\beta, \gamma) \in N - P\}| \quad (\beta < p).$$

By applyinng the lemma to the set $Q = N - P$ we find numbers

$k_0, \ldots, k_{p-1} < 2c-1$ such that $k_\beta \neq k_\gamma$ whenever $(\beta, \gamma) \in N - P$. Put $M_\rho = \{\beta : k_\beta = \rho\}$ *for* $\rho < 2c-1$. We may assume that $|M_0| \geqslant p(2c-1)^{-1}$. We have $(\beta, \gamma) \in P$ whenever $\beta, \gamma \in M_0$.

Case 1. $|M_0| \geqslant n$. Then $\Sigma(\pi \in M_0) S_\pi' \supset T_0 + \ldots + T_{n-1}$ (tp), where tp $T_\nu = \omega_\alpha$ *for* $\nu < n$ and

$$|U_0(x) T_\nu| < \aleph_\alpha \quad (\mu, \nu < n;\ x \in T_\mu).$$

Case 1a. $\mathrm{cf}(\alpha) = \alpha$. Then we write

$$\{(\nu, \sigma) : \nu < n;\ \sigma < \omega_\alpha\} = \{(\nu_\rho, \sigma_\rho) : \rho < \omega_\alpha\}.$$

We can choose $x_0, \ldots, \hat{x}_{\omega_\alpha}$ such that

$$x_\rho \in T_{\nu_\rho} - \Big(\{x_0, \ldots, \hat{x}_\rho\} + \Sigma(\tau < \rho)\ U_0(x_\tau)\Big) \quad \textit{for} \quad \rho < \omega_\alpha.$$

Put $X = \{x_\rho : \rho < \omega_\alpha\}$. Then $[X]^2 \subset K_1$ and tp $X \geqslant \omega_\alpha n$ which contradicts *(6)*.

Case **1b.** $\mathrm{cf}(\alpha) < \alpha$. Let $r = \omega_{\mathrm{cf}(\alpha)}$. We can write

$$\{(\nu, \tau) : \nu < n;\ \tau < r\} = \{(\nu_\rho, \tau_\rho) : \rho < r\}. \tag{7}$$

Also, $\aleph_\alpha = \Sigma\ (\rho < r)\ a(\rho)$, where

$$\left.\begin{aligned} |r| < a(0) < \ldots < \hat{a}(r) < \aleph_\alpha, \\ \big(a(\rho)\big)' = a(\rho) \quad (\rho < r). \end{aligned}\right\} \tag{8}$$

The $a(\rho)$ can be found by the following standard procedure. There are cardinals $b(\rho) < \aleph_\alpha$ such that $\aleph_\alpha = \Sigma\ (\rho < r)\ b(\rho)$. Then, by definition of r, we have

$$\sup(\rho < \bar{\rho})\, b(\rho) < \aleph_\alpha = \sup(\rho < r)\, b(\rho) \text{ for } \bar{\rho} < r.$$

Hence there is $\rho_0 < r$ such that $b(\rho_0) > |r|$, and we can find inductively ordinals $\rho_1, \ldots, \hat{\rho}_r$ such that

$$b(\rho_{\lambda_0}) > \Sigma(\lambda < \lambda_0)\, b(\lambda) + \sup(\lambda < \lambda_0)\, b(\rho_\lambda) \text{ for } \lambda_0 < r.$$

Now we may put $a(\lambda) = \big(b(\rho_\lambda)\big)^+$ for $\lambda < r$, where b^+ denotes the least cardinal greater than b.

We can write $T_\nu = \Sigma(\rho < r)\, T_\nu(\rho)$ (tp), where $|T_\nu(\rho)| = a(\rho)$ $(\nu < n;\ \rho < r)$. Let $\nu, \nu' < n;\ \rho < r;\ x \in T_\nu(\rho)$. Then $|U_0(x) T_{\nu'}| < \aleph_\alpha$. There is $\pi(x, \nu') < r$ such that $|U_0(x) T_{\nu'}| < a\big(\pi(x, \nu')\big)$. Put $\pi(x) = \max(\nu' < n)\, \pi(x, \nu')$. Then, by (8), there are a set $T_\nu'(\rho) \subset T_\nu(\rho)$ and a number $\pi_\nu(\rho) < r$ such that

$$|T_\nu'(\rho)| = a(\rho), \tag{9}$$

$$\pi(x) = \pi_\nu(\rho) \quad \big(\nu < n;\ \rho < r;\ x \in T_\nu'(\rho)\big).$$

Then

$$|U_0(x) T_{\nu'}| < a\big(\pi_\nu(\rho)\big) \quad \big(\nu, \nu' < n;\ \rho < r;\ x \in T_\nu'(\rho)\big).$$

There is $\tau(\rho)$ such that $\rho \leqslant \tau(\rho) < r$ and

$$\left|\Sigma\left(x \in T_\nu'(\rho)\right) U_0(x) T_{\nu'}\right| < a\left(\tau(\rho)\right) \quad (\nu, \nu' < n;\ \rho < r). \tag{10}$$

We now define ordinals s(p) for $\rho < r$. Let $\rho_0 < r$, and let the ordinals s(p) be defined for $\rho < \rho_0$ and satisfy s(p) $< r$ for $\rho < \rho_0$. Then we can choose $s(\rho_0)$ such that $\rho_0 \leqslant s(\rho_0) < r$ and

$$\Sigma(\rho < \rho_0)\, a\left(\tau\left(\tau\left(s(\rho)\right)\right)\right) < a\left(\tau\left(s(\rho_0)\right)\right). \tag{11}$$

This defines s(p) for $\rho < r$. Now consider the sets A,, ... $\hat{A}_r$ defined inductively by

$$A_{\rho_0} = T'_{\nu\rho_0}\left(\tau\left(s(\rho_0)\right)\right) - \Sigma(\rho < \rho_0;\ x \in A_\rho)\ U_0(x)\ (\rho_0 < r).$$

Let $\rho_0 < r$. Then

$$\left|\Sigma(\rho < \rho_0;\ x \in A_\rho)\, U_0(x)\, T'_{\nu\rho_0}\left(\tau\left(s(\rho_0)\right)\right)\right|$$

$$\leqslant \Sigma(\rho < \rho_0)\left|\Sigma(x \in A_\rho)\, U_0(x)\, T'_{\nu\rho_0}\left(\tau\left(s(\rho_0)\right)\right)\right|$$

$$\leqslant \Sigma(\rho < \rho_0)\left|\Sigma\left(x \in T'_{\nu\rho}\left(\tau\left(s(\rho)\right)\right)\right) U_0(x)\, T_{\nu\rho_0}\right|$$

$$\leqslant \Sigma(\rho < \rho_0)\, a\left(\tau\left(\tau\left(s(\rho)\right)\right)\right) \quad \text{(by (10))}$$

$$< a\left(\tau\left(s(\rho_0)\right)\right) \quad \text{(by (11))}$$

$$= \left|T'_{\nu\rho_0}\left(\tau\left(s(\rho_0)\right)\right)\right| \quad \text{(by (9))}.$$

Hence

$$|A_{\rho_0}| = a\left(\tau\left(s(\rho_0)\right)\right) \geqslant a(\rho_0)\ (\rho_0 < r).$$

It now follows from (7) that tp $\Sigma(\rho < r)$ A, $\geqslant \omega_\alpha$ n. Since, in addition, $[\Sigma\ (\rho < r)\ A_\rho]^2 \subset K_1$ we have a contradiction against (6).

Case 2. $|M_0| < n$. Then $p(2c-1)^{-1} \leqslant |M_0| \leqslant n-1$; $p \leqslant (n-1)(2c-1)$. Hence $l_\alpha(m, n)$ exists, and we may put in all foregoing relations $p = l_\alpha(m,$?a)-1. w e note that $p \geqslant 0$. We have thus shown that

$$l_\alpha(m, n) - 1 \leqslant (n-1)\left(2l_\alpha(m-1, n) - 1\right).$$

Put n-l=q and $l_\alpha(\mu, n) - 1 = d_\mu$ $(2 \leqslant \mu \leqslant m)$. Then $d_m \leqslant q(2d_{m-1}+1)$, *i.e.*, $d_m + e \leqslant 2q(d_{m-1}+e)$, where $e = q(2q-1)^{-1}$. Hence

$$d_m + e \leqslant (2q)^{m-2}(d_2+e) = (2q)^{m-2}(q+e)$$

which is the same as (3).

We now prove (4). Let tp $S = \gamma < \omega_\alpha l_\alpha(m, n)$. Then $\gamma = \omega_\alpha l' + s'$, where $l' < l_\alpha(m, n)$ and $s' < \omega_\alpha$. We have $S = S_0 + S_1(\text{tp})$; tp $S_0 = \omega_\alpha l'$; tp $S_1 = s'$. By definition of $l_\alpha(m, n)$ there is a partition $[S_0]^2 = K_0 + K_1$

such that (6) holds. Then we can write $[S]^2 = K_0 + 'L_1$ and we have, obviously, $\omega_\alpha n \notin [L_1]$. Hence (4) follows.

To complete the proof of Theorem 2, let us suppose that $\gamma \lhd \omega_\alpha l_0(m, n)$. Then $\gamma = \omega_\alpha l'' +$ s", where $l'' < l_0(m,$ n) and s" $< \omega_\alpha$. By the property of $l_0(m,$ n) stated in Theorem 1, there is a function $\rho(\lambda, \mu) \in \{0, 1\}$, defined for $\lambda, \mu < l''$, such that

(i) there is no set $\{\lambda_0, \ldots, \lambda_{m-1}\}_{\neq} \subset$ [0, Z") such that

$$\rho(\lambda_i, \lambda_j) = 0 \text{ for } i < j < m;$$

(ii) there is no set $\{\lambda_0, \ldots, \lambda_{n-1}\}_{\neq} \subset$ [0, Z") such that

$$\rho(\lambda_i, \lambda_j) = \rho(\lambda_j, \lambda_i) = 1 \text{ for } i < j < n.$$

Let S= [0, γ) and order S by magnitude, so that tp S = γ. Then $[S]^2 = K_0 + 'K_1$, where

$$K_0 = \{\{\omega_\alpha \lambda + \tau, \omega_\alpha \lambda' + \tau'\} : \text{A}, \lambda' < l''; \lambda \neq \lambda'; \rho(\lambda, \lambda') = 0; \tau < \tau' < \omega_\alpha\}.$$

Then it follows from the property (i) of $\rho(\lambda, \mu)$ that $m \notin [K_0]$ and from the property (ii) that ω_α n $\notin [K_1]$. Hence (6) holds, and Theorem *2* is established.

6. Before stating our next theorem we introduce another kind of partition relation. Let α and β be order types and let k be an ordinal. Then the relation†

$$\alpha \to (\beta)_k^1$$

expresses the following condition. Let S be an ordered set and tp S= α. Let S = $\sum (\kappa \lhd k) K_\kappa$. Then there always exists a number $\kappa < k$ such that $\operatorname{tp} K_\kappa \geqslant \beta$. We recall that ***initial ordinals*** are ordinals δ such that $\epsilon \lhd \delta$ implies $|\epsilon| < |\delta|$.

THEOREM ***3. Let n be an ordinal.*** Let ***a= a,+. . . +*** $\hat{\alpha}_n$ and $\beta = \beta_0 + \ldots + \hat{\beta}_n$, ***where, for*** $\nu \lhd$ ***n, a, is such that‡***

$$\alpha_\nu \to (\alpha_\nu, \alpha_\nu)^1, \tag{12}$$

and β_ν ***is an initial ordinal. Suppose that***

$$\alpha_\nu \to (\beta)_k^1 \quad (\nu < n; \ |k| < |\beta|). \tag{13}$$

Then

$$\alpha \to (3, \beta)^2. \tag{14}$$

COROLLARY. *If* $\operatorname{cf}(\alpha) = \alpha$, *then*

$$\omega_\alpha^{2p+1} \to (3, \omega_\alpha^{p+1})^2 \quad (p < \omega). \tag{15}$$

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† This relation is a special case of the relation $\alpha \to (\beta)_k^r$ which is defined in the obvious way.

‡ As is well known, (12) holds if and only if α_ν is either zero or a power of ω.

Remarks. (i) For $\alpha = 0$ the relation (15), in fact, the stronger result

$$\omega^{1+ph} \rightarrow (2^h, \omega^{1+p})^2 \quad (p < \omega_1 ; h < \omega)$$

has already been obtained by E. C. Milner [4].

(ii) It is not known whether (15) remains valid when $\mathrm{cf}(\alpha) < \alpha$.

7. Let us begin by deducing the corollary from the theorem. We apply Theorem 3 with $n = \omega_\alpha{}^p$; $\alpha_\nu = \omega_\alpha{}^{p+1}$ and $\beta_\nu = \omega_\alpha$ for $\nu < n$. Then (14) becomes (15), and we need only verify (12) and (13) which amounts to showing that $\omega_\alpha{}^{p+1} \rightarrow (\omega_\alpha{}^{p+1})_k{}^1$ ($k < \omega_\alpha$). Let k be fixed, $k < \omega_\alpha$. Then, clearly, $\omega_\alpha{}^0 \rightarrow (\omega_\alpha{}^0)_k{}^1$. Let $q < \omega$, and suppose that

$$\omega_\alpha{}^q \rightarrow (\omega_\alpha{}^q)_k{}^1. \tag{16}$$

It suffices to deduce that

$$\omega_\alpha{}^{q+1} \rightarrow (\omega_\alpha{}^{q+1})_k{}^1. \tag{17}$$

Let $\mathrm{tp}\, S = \omega_\alpha{}^{q+1}$ and $S = \Sigma(\kappa < k)\, K_\kappa$. Then $S = \Sigma(\nu < \omega_\alpha)\, S_\nu(\mathrm{tp})$, $\mathrm{tp}\, S_\nu = \omega_\alpha{}^q$ $(\nu \lhd \omega_\alpha)$. Then, for $\nu < \omega_\alpha$, we have $S_\nu = \Sigma\, (\kappa \lhd k)\, S_\nu K_\kappa$ and therefore, by (16), there is $\kappa_\nu \lhd k$ such that $\mathrm{tp}\, S_\nu K_{\kappa_\nu} = \omega_\alpha{}^q$. Since $\mathrm{cf}\,(\alpha) = \alpha$, there are a number $\kappa < k$ and a set $M \subset [0, \omega_\alpha)$ such that $|M| = \aleph_\alpha$ and $\kappa_\nu = \kappa$ for $\nu \in M$. Then

$$\mathrm{tp}\, K_\kappa \geqslant \Sigma(\nu \in M)\, \mathrm{tp}\, S_\nu K_\kappa = \Sigma(\nu \in M)\, \omega_\alpha{}^q = \omega_\alpha{}^{q+1},$$

and (17) follows.

8. *Proof of Theroem 3.* Let $\mathrm{tp}\, A_\nu = \alpha_\nu$ ($\nu < n$) and order the set $P = \{(\nu, x) : \nu < n ; x \in A_\nu\}$ lexicographically. Then $\mathrm{tp}\, P = \alpha$. Let $\alpha \not\rightarrow (3, \beta)^2$. Then there is a partition $[P]^2 = K_0 + K_1$ such that

$$3 \notin [K_0], \tag{18}$$

$$\beta \notin [K_1]. \tag{19}$$

We have to deduce a contradiction. We can write

$$\{(\nu, t) : \nu < n ; t < \beta_\nu\} = \{(\nu(\lambda), t(\lambda)) : \lambda < l\}, \tag{20}$$

where l is the initial ordinal satisfying $|l| = |\beta|$. We now define elements $p_0, \ldots, \hat{p}_l$ of P. Let $\lambda_0 < l$, and suppose that $p_0, \ldots, \hat{p}_{\lambda_0} \in P$. We shall define p_{λ_0}. Put $Q_\lambda = U_0(p_\lambda)$ for $\lambda < \lambda_0$. Then, by (18), $[Q_\lambda]^2 \subset K_1$ and hence, by (19), $\mathrm{tp}\, Q_\lambda \not\geqslant \beta$ for $\lambda < \lambda_0$. Since $|\lambda_0| < |\beta|$ we deduce from (13) that

$$\mathrm{tp}\, \Sigma(\lambda < \lambda_0)\, Q_\lambda \not\geqslant \alpha_{\nu(\lambda_0)}. \tag{21}$$

We have $|\lambda_0| < |l| = |\beta|$ and so $|\beta| \geqslant 1$ and $n \geqslant 1$. If $\beta = 1$ then $\alpha \not\rightarrow (3, 1)^2$, so that $a = 0$ and therefore, by (13), $0 = \alpha_0 \rightarrow (1)_0{}^1$ which is false. Hence $|\beta| \geqslant 2$ and we may put $k = l$ in (13) and obtain $\alpha_\nu \geqslant \beta$ ($\nu < n$). Therefore $|\{p_0, \ldots, \hat{p}_\lambda\}| \leqslant |\lambda_0| < |\beta| \leqslant |\alpha_{\nu(\lambda_0)}|$ and hence

$$\mathrm{tp}\{p_0, \ldots, \hat{p}_{\lambda_0}\} \not\geqslant \alpha_{\nu(\lambda_0)}. \tag{22}$$

It follows from (21), (22) and (12) that

$$\mathrm{tp}\,\Sigma(\lambda<\lambda_0)(Q_\lambda+\{p_\lambda\})\not\geqslant\alpha_{\nu(\lambda_0)}=\mathrm{tp}\,P\big(\nu(\lambda_0)\big),$$

where $$P(\mu) = \{(\mu, x) : x\in A_\mu\} \quad (\mu \lhd n).$$

Hence we can choose

$$p_{\lambda_0}\in P\big(\nu(\lambda_0)\big)-\Sigma(\lambda<\lambda_0)(Q_\lambda+\{p_\lambda\}).$$

This defines a set $X=\{p_0, \ldots, p_l\}_{\neq}$ which satisfies $[X]^2\subset K_1$. We have, by (20), $|XP(\nu)| \geqslant |\beta_\nu|$ and hence, since β_ν is an initial ordinal, $\mathrm{tp}\,XP(\nu)\geqslant\beta_\nu$ $(\nu<n)$. But then $\mathrm{tp}\,X=\Sigma(\nu<n)\,\mathrm{tp}\,XP(\nu)\geqslant\Sigma(\nu<n)\beta_\nu=\beta$ which contradicts (19). This proves Theorem 3.

9. In this final section we consider a binary relation $x\prec y$ defined on a set S, such that for all $x, y\in S$ exactly one of the three relations $x = y$; $x\prec y$; $y\prec x$ holds. The relation is said to be *transitive* on a subset X of S if, for $x, y, z\in X$, whenever $x\prec y$ and $y\prec z$, then $x\prec z$.

THEOREM 4. *Let a be a cardinal. Then the relation $x\prec y$ is transitive on some subset X of S such that $|X| = a$, provided that*

(i) $|S|\geqslant 2^{a-1}$ if $a<\aleph_0$,

(ii) $|S|\geqslant\aleph_0$ if $a=\aleph_0$,

(iii) $|S|>\Sigma(b<a)\,2^b$ if $a>\aleph_0$,

where the summation extends over all cardinals b less than a.

Remarks 1. If $a>\aleph_0$, and if the following weak version of the generalised continuum hypothesis is assumed: $2^b\leqslant a$ for $b<a$, then (iii) is the same as $|S|>a$.

2. The condition under (i) is best possible for $1\leqslant a\leqslant 3$.

Proof of Theorem 4. Case 1. $a<\aleph_0$. Although, as was mentioned in the introduction, there is a very simple proof for this case in [8], it is of interest to show that the conclusion can be deduced from Theorem 2. We may assume that $a\geqslant 2$ and $S = [0, 2^{a-1})$. Let $R = [0, \omega 2^{a-1})$. Then $[R]^2=K_0+'K_1$, where

$$K_0=\Big\{\{\omega\lambda+\tau, \omega\lambda'+\tau'\} : \lambda, \lambda' < 2^{a-1};\ \lambda\prec\lambda';\ \tau<\tau'<\omega\Big\}.$$

We order R be magnitude. Then $\omega 2\notin[K_1]$. By (3) we have $l_0(a, 2)\leqslant 2^{a-1}$ and therefore, by (2), $\mathrm{tp}\ R\to(a, \omega 2)^2$. Hence $a\in[K_0]$, and there is a set

$$T=\{\omega\lambda_0+\tau_0, \ldots, \omega\lambda_{a-1}+\tau_{a-1}\}_{\neq}\subset R$$

such that $[T]^2\subset K_0$. We can choose the notation in such a way that $\tau_0\leqslant\ldots\leqslant\tau_{a-1}<\omega$. Then, by definition of K_0, $\tau_p<\tau_q$ and $\lambda_p\prec\lambda_q$ for $p<q<a$, so that we may put $X=\{\lambda_0, \ldots, \lambda_{a-1}\}$.

Case 2. $a \geq \aleph_0$. We first show that

$$|S| \to (4, a)^3. \tag{23}$$

If $a = \aleph_0$ then $|S| \geq \aleph_0$, and (23) follows from [6]. Now let $a = \aleph_n > \aleph_0$. By (5), $\omega_n \to (3, \omega_n)^2$. Hence, by [1], Theorem 39 (i), we have, for every ordinal β such that $|\beta| > \Sigma\,(\lambda < \omega_n)\, 2^{|\lambda|^2}$, the relation $\beta \to (4, \omega_n + 1)^3$ and therefore

$$|\beta| \to (4, a)^3. \tag{24}$$

Now

$$\Sigma(\lambda < \omega_n)\, 2^{|\lambda|^2} \leq \Sigma(\lambda < \omega)\, \aleph_0 + \Sigma(\nu < n)\, \Sigma(\omega_\nu \leq \lambda < \omega_{\nu+1})\, 2^{\aleph_\nu}$$
$$= \aleph_0 + \Sigma(\nu < n)\, 2^{\aleph_\nu} \aleph_{\nu+1} = \Sigma(b < a)\, 2^b < |S|.$$

Hence in (24) we may replace $|\beta|$ by $|S|$, and (23) follows.

We apply (23) to the partition $[S]^3 = K_0 + {}' K_1$, where

$$K_0 = \{\{x, y, z\} : x \prec y \prec z \prec x\}.$$

Then we have the following cases.

Case 2a. There is a set $A = \{x_0, x_1, x_2, x_3\}_{\neq} \subset S$ such that $[A]^3 \subset K_0$. Then we can choose the notation in such a way that $x_0 \prec x_1 \prec x_2 \prec x_0$. Then, by definition of K_0, $x_0 \prec x_1 \prec x_3 \prec x_0$.

Case 2a1. $x_2 \prec x_3$. Then $x_2 \prec x_3 \prec x_1 \prec x_2$ which contradicts $x_1 \prec x_3$.

Case 2a2. $x_3 \prec x_2$. Then $x_3 \prec x_2 \prec x_0 \prec x_3$ which contradicts $x_3 \prec x_0$.

Case 2b. There is a set $B \subset S$ such that $|B| = a$ and $[B]^3 \subset K_1$. Let $x, y, z \in B$ and $x \prec y \prec z$. Then $x \neq z$. If $z \prec x$, then $\{x, y, z\} \in K_0$ which is false. Hence $x \prec z$, and we may put $X = B$. This proves Theorem 4.

References

1. P. Erdős and R. Rado, "A partition calculus in set theory", *Bull. American Math. Soc.*, 62 (1956), 427-439.
2. N. G. de Bruijn and P. Erdős, "A colour problem for infinite graphs and a problem in the theory of relations", *Nederl. Akad. Wetensch. Proc.* (A), 54, 371-373.
3. E. Specker, *Comment. Math.*, 31 (1957), 302-314.
4. E. C. Milner, *Some combinatorial problems in set theory*, Thesis (Reading University, 1961).
5. B. Dushnik and E. W. Miller, "Partially ordered sets", *American J. of Math.*, 63 (1941), 605.
6. F. P. Ramsey, "On a problem of formal logic", *Proc. London Math. Soc.* (2), 30 (1930), 264–286.
7. R. Stearns, "The voting problem", *American Math. Monthly*, 66 (1959). 761-763.
8. P. Erdős and L. Moser, "On the representation of directed graphs as union of orderings", *Publ. Math. Inst. Hung. Acad. Sc.*, 9 (1964), 125–132.

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Printed by C. F. Hodgson & Son, Ltd., 23 Pakenham Street, London, W.C.1.