ON SOME NEW INEQUALITIES CONCERNING EXTREMAL PROPERTIES OF GRAPHS

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Denote by $G(n;\, l)$ a graph of n vertices and l edges. $\varkappa(G)$ will denote the chromatic number of G. $K_r(p_1, \ldots, p_r)$ denotes the complete r-chromatic graph with p_i vertices of the i-th colour where any two vertices of different colour are joined. $K_1(p)$ is a graph consisting of p isolated vertices. $(G : K_r(p_1, \ldots, p_r))$ is obtained from G by adjoining a $K_r(p_1, \ldots, p_r)$, and by joining every new vertex to all the vertices of G. Clearly $\varkappa\big((G : K_r(p_1, \ldots, p_r)\big) = \varkappa(G) + r . f(n;\, G)$ is the smallest integer so that every $G_1\big(n;\, f(n;\, G)\big)$ contains G as a subgraph. The graphs $G'(n) = G'(n;\, f(n;\, G) - 1)$ which do not contain G as a subgraph are called the extremal graphs belonging to G.

The vertices of G will be denoted by $x, x_1, \ldots, y, \ldots,$ the edges will be denoted by $(x,\, y)$. The valence of a vertex x of G is the number of edges incident to x. $\pi(G)$ denotes the number of vertices, $\nu(G)$ the number of edges of G. If G' is a graph and $x_1, \ldots, x_k$ are some of the vertices of G' then $G'(x_1, \ldots, x_k)$ is the subgraph of G' spanned by $x_1, \ldots, x_k$. $c, c_1, \ldots$ denote absolute constants not necessarily the same if they occur in different formulas.

In a previous paper [1] I stated without proof that

$$f(n\,;\, K_r(t, \ldots, t)) < \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + cn^{2-1/t}. \tag{1}$$

In the present paper I will prove that (1) is a special case of a more general theorem. A recent result of SIMONOVITS and myself states [2] ($\varkappa(G) = r$)

$$f(n\,;\, G) = \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + o(n^2)\,. \tag{2}$$

In this paper I will prove

THEOREM 1. *Let* $\varkappa(G) = 2$. *Then for* $n > n_0(t)$

$$f(n\,;\, (G : K_{r-2}(t, \ldots, t))) < \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + \big(1 + o(1)\big)\,(r-1)\, f\left(\left[\frac{n}{r-1}\right];\; G\right) + cn\,.$$

(c independent of t!).

First we deduce (1) from Theorem 1. A well known result of KŐVÁRI and the TURÁNS [5] states that

$$f(n; K_2(t,t)) < cn^{2-1/t}. \tag{3}$$

Clearly $K_r(t, \ldots, t) = \big(K_2(t, t) : K_{r-2}(t, \ldots, t)\big)$. Thus from Theorem 1 $\big(G = K_2(t, t)\big)$ we immediately obtain (1). (1) is probably best possible for every r and t but I can prove this only for $t \leq 3$.

Theorem 1 immediately implies that for $n > n_0(l)$

$$f\big(n; K_r(t, t, l, \ldots, l)\big) - \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) < c_1(r-1)\, n^{2-1/t} + c_2\, n. \tag{4}$$

where both c_1 and c_2 are independent of l. In fact perhaps for $n > n_0(l_1, l_2)$

$$\Big| f(n; K_r(t, t, l_1, \ldots, l_1)) - f(n; K_r(t, t, l_2, \ldots, l_2))\Big| < cn, \tag{5}$$

but I am very far from being able to prove (5).

It seems likely that in contrast to (4) and (5)

$$c_l' n^{2-1/t} < \left| f(n; K_r(t, l, \ldots, l)) - \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)\right| < c_l'' n^{2-1/t}$$

where $c_l' \to \infty$ and $c_l'' \to \infty$ as $l \to \infty$. The upper bound follows easily from Theorem 1 and the known result

$$K_2(t, l) < c_l''\, n^{2-1/t}. \tag{6}$$

$\big($(6) follows e.g. by the method of [5]$\big)$, but I can not prove the lower bound.

By more complicated methods I can prove the following strengthening of Theorem 1.

THEOREM 2. *Let $\varkappa(G) = r$ and put*

$$f(n; G) = \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + h(n; G)^1$$

Let $\delta = \delta(G)$ be sufficiently small. Then for $n > n_0(G, \delta)$

$$f(n; (G : K_1([\delta n])) < \frac{n^2}{2}\left(1 - \frac{1}{r}\right) + c_1\, h(n; G) + c_2\, n.$$

Theorem 2 in particular implies ($\varkappa(G) = 2$)

$$f\big(n; (G : K_{r-2}(t, \ldots, t, [\delta n]))\big) < \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + \\ + \big(1 + o(1)\big)(r-1)\, f\left(\left[\frac{n}{r-1}\right]; G\right) + cn.$$

We do not prove Theorem 2 in this paper.

[1] By [2] $h(n; G) = o(n^2)$.

In a recent paper [3] I proved the following sharpening of (2):

THEOREM A. *Let* $l = (1 + o(1))\frac{n^2}{2}\left(1 - \frac{1}{r-1}\right)$ *and assume that* $G(n;\ l)$ *does not contain a* $K_r(t, \ldots, t)$ *as a subgraph. Then there is a*

$$(7) \quad K_{r-1}(p_1, \ldots, p_{r-1}), \sum_{i=1}^{r-1} p_i = n, p_i = (1 + o(1))\frac{n}{r-1}, \ i = 1, \ldots, r-1$$

which differs from our $G(n;\ l)$ *by* $o(n^2)$ *edges.*

The principal tool in the proof of Theorem 1 will be

THEOREM 3. *Let* $G'(n)$ *be any extremal graph belonging to* $G\left(\varkappa(G) = r\right)$. *Then the vertices* $x_1, \ldots, x_n$ *of our* $G'(n)$ *can be partitioned into* $r - 1$ *classes each containing* $\left(1 + o(1)\right)\frac{n}{r-1}$ *of the* x_i *so that for every* $\varepsilon > 0$ *all but* c_ε *of the* x_i *are joined to all but* εn *of the* x*'s which do not belong to the same class as* x_i.

Observe that Theorem 3 does not contain Theorem A, though the conclusion of Theorem 3 is stronger its assumption is also more stringent.

To prove Theorem 3 we need a lemma which is of independent interest.

LEMMA. *Let* $G'(n)$ *be one of the extremal graphs belonging to* G. *Then every vertex of* $G'(n)$ *has valence greater than* $(1 + o(1))\, n\left(1 - \frac{1}{r-1}\right)$.

Assume that the lemma is not true and let y be a vertex of $G'(n)$ whose valence is less than $(1 - \varepsilon)n\left(1 - \frac{1}{r-1}\right)$. It easily follows from Theorem A that for every k, if $n > n_0(k)$, $G'(n)$ has k vertices $x_1, \ldots, x_k$ each of which is joined to $y_1, \ldots, y_s$, $s = \left(1 + o(1)\right)n\left(1 - \frac{1}{r-1}\right)$. The existence of these vertices is clear since by Theorem A all but $o(n)$ vertices of the first colour in $K(p_1, \ldots, p_{r-1})$ are joined in our $G'(n)$, to all but $o(n)$ other vertices of different colours. Delete now all the edges incident to y and replace them by the edges (y, y_i), $i = 1, \ldots, s$. The new graph has more than $G_1\left(n;\ f(n;\ G)\right)$ edges and clearly can not contain G as a subgraph since if it would contain G and if k would be greater than $\pi(G)$ then the subgraph $G'(x_1, \ldots, x_k, y_1, \ldots, y_s)$ of $G'(n)$ would also contain G as a subgraph, which contradicts our assumption. This contradiction proves our lemma.

Not to complete the proof of Theorem 3 assume for the sake of simplicity that $r = 3$ the case $r > 3$ can be settled similarly. Let

$$K_2(p_1, p_2), p_1 + p_2 = n,\ p_i = (1 + o(1))\,\frac{n}{2}, \qquad i = 1, 2$$

be the graph (7) and let $x_1, \ldots, x_{p_1}, y_1, \ldots, y_{p_2}$ be the vertices of colour one and two, respectively. By Theorem A all but $o(n^2)$ of the edges (x_i, y_j) occur in our $G'(n)$. By our Lemma we can further assume that the valence (in

$G'(n))$ of all the x_i and y_j is $\geq (1+o(1))\dfrac{n}{2}$ and that each x is joined with at least as many y's than x's and each y is joined with at least as many x's than y's (for if say x_1 is joined to more x's than y's we put it amongst the y's). Thus, each vertex is joined with at least $(1+o(1))\dfrac{n}{4}$ vertices of the opposite colour.

Assume now that Theorem 3 is not true. Then we can assume that for a fixed $\varepsilon > 0$ and for every k if $n > n_0(k)$ there are vertices $x_1, \ldots, x_k$, $k > k_0(\varepsilon)$ each of which are joined to fewer than $(1-\varepsilon)\dfrac{n}{2}$ y's. But then by our lemma each x_i, $i = 1, \ldots, k$ is joined to at least $\dfrac{\varepsilon}{2}n$ x's. I now show that this leads to a contradiction, since then our $G'(n)$ will contain G as a subgraph, in fact for large enough $k > k_0(\varepsilon, t)$ it contains a $K_3(t, t, t)$ which of course contains our G if $t \geq \pi(G)$.

Applying twice the lemma on p. 185 of [4] it easily follows that if $k > k_0(\varepsilon, t)$ there are t x's say $x_1, \ldots, x_t$ and more than $\eta\, n$, $\eta = \eta(\varepsilon, k, t)$ other x's and $> \eta\, n$ y's say $x_{u_1}, \ldots, x_{u_s}$; $y_1, \ldots, y_s$, $s > \eta\, n$ so that every x_i, $i = 1, \ldots, t$ is joined to every x_{u_i}, $i = 1, \ldots, s$ and to every y_j, $j = 1, \ldots s$. By Theorem A all but $o(s^2)$ of the edges (x_{u_i}, y_j) occur in $G'(n)$, hence by the theorem of KŐVÁRI and the TURÁNS [5] there are vertices say $x_{u_1}, \ldots, x_{u_t}$; $y_1, \ldots, y_t$ so that all the edges (x_{u_i}, y_j), $1 \leq i, j \leq t$ occur in $G'(n)$ but then clearly $G'(x_1, \ldots, x_t, x_{u_1}, \ldots, x_{u_t}, y_1, \ldots, y_t)$ contains a $K_3(t, t, t)$. This contradiction completes the proof of Theorem 3.

Theorem 1 follows easily from Theorem 3. Let G'_{r-2} be an extremal graph of n vertices with respect to $(G : K_{r-2}(t, \ldots, t))$. To prove Theorem 1 we only have to show

$$(8) \qquad \nu(G'_{r-2}) < \frac{n^2}{2}\left(1 - \frac{1}{r-1}\right) + (1+o(1))\,(r-1)\,f\left(\left[\frac{n}{r-1}\right]; G\right) + cn\,.$$

We now use Theorem 3. Let $x_1, \ldots, x_l$, $l < c_\varepsilon$ be the exceptional vertices of G'_{r-2} whose existence is permitted by Theorem 3. The other $n - l$ vertices of G'_{r-2} can by Theorem 3 be partitioned into $r-1$ classes each of which has $p_i = (1+o(1))\dfrac{n}{r-1}$ vertices and each of these vertices is joined to all but εn vertices which belong to different classes. The graphs spanned by the p_i vertices of the i-th class can not contain G as a subgraph, for if this statement would be false let $y_1, \ldots, y_m$ $m = \pi(G)$ be the vertices of the i-th class which span a graph containing G as a subgraph. By what has been just said the y_i, $i = 1, \ldots, m$ are joined to all but εn vertices of the other classes, and since each of these vertices are again joined to all but εn vertices of the other classes we obtain by a simple but not quite short argument that for $n > n_0\,(r, t, l)$ our G'_{r-2} contains a $(G : K_{r-2}\,(t, \ldots, t))$ which contradicts our assumption.

Thus, the number of edges which join two vertices belonging to the same class is less than

$$\sum_{i=1}^{r-1} f(p_i\,; G) < (1+o(1))\,(r-1)\,f\left(\left[\frac{n}{r-1}\right]; G\right). \tag{9}$$

In (9) we used that if $u_1 = (1+o(1))u_2$ then

$$f(u_1\,; G) = (1+o(1))\,f(u_2\,; G)\,, \tag{10}$$

the proof of (10) is easy and can be left to the reader.

The number of edges which join vertices belonging to different classes is clearly not greater than

$$\sum_{1\le i<j\le n} p_i\,p_j \le \binom{r-1}{2}\frac{n^2}{(r-1)^2} = \frac{n^2}{2}\left(1-\frac{1}{r-1}\right). \tag{11}$$

The number of edges incident to the $l < c_\varepsilon$ exceptional vertices is clearly less than $c_\varepsilon n$, hence (9) and (11) imply (8), which proves Theorem 1.

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