ON A COMBINATORIAL PROBLEM III

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A family of sets $\{A_\alpha\}$ is said by Miller [3] to have property B if there exists a set S which meets all the sets A_α and contains none of them. Property B has been extensively studied in several recent papers (see the references in [2] and the last chapter of P. Erdős and A. Hajnal, On chromatic number of graphs and set systems, Acta. Math. Acad. Sci. Hung. 17 (1966) 61-99). Hajnal and I define m(n) as the smallest integer for which there is a family of m(n) sets A_k, $|A_k| = n$, $1 \le k \le m(n)$, which do not have property B [1]. Trivially $m(n) \le \binom{2n-1}{n}$ (take all subsets taken n at a time of a set of 2n-1 elements), m(2) = 3, m(3) = 7, m(4) is not known. It is known [2], [4] that for $n > n_0(\epsilon)$

$$(1) \qquad 2^n\left(1+\frac{4}{n}\right)^{-1} \le m(n) < (1+\epsilon)\, e \log 2\, n^2 2^{n-2}$$

$m_N(n)$ is the smallest integer for which there are $m_N(n)$ sets A_k, $|A_k| = n$, $1 \le k \le m_N(n)$ which are all subsets of a set S, $|S| = N$ and which do not have property B. I conjectured in [2] that for $N < c_1 n$, $m_N(n) > (2 + c_2)^n$. In this note we prove this conjecture and get fairly good upper and lower bounds for $m_N(n)$. In fact we prove that if $N = (c + o(1))n$

$$(2) \qquad \begin{cases} \lim\limits_{n=\infty} m_N(n)^{1/n} = 2(c-2)^{\frac{1}{2}(c-2)}(c-1)^{1-c}c^{\frac{1}{2}c} \text{ for } c > 2 \text{ and} \\ \lim\limits_{n=\infty} m_N(n)^{1/n} = 4 \text{ if } N = (2 + o(1))n. \end{cases}$$

THEOREM 1.

$$(3)\qquad m_{2N-1}(n) \geq m_{2N}(n) \geq 2^{n-1} \prod_{i=0}^{n-1} \left(1 + \frac{i}{2N-2i}\right).$$

Let $|S| = 2N$ and $|A_k| = n, 1 \leq k \leq m_{2N}(n)$ where $\{A_k\}$ is a family of subsets of S which does not have property B. Clearly S can be split in $\frac{1}{2}\binom{2N}{N}$ ways as the union of two disjoint sets $S_1^{(t)}$ and $S_2^{(t)}$, $1 \leq t \leq \frac{1}{2}\binom{2N}{N}$ for every t, $|S_1^{(t)}| = |S_2^{(t)}| = N$. By assumption the family $\{A_k\}$, $1 \leq k \leq m_{2N}(n)$ does not have property B. Thus for every t, $1 \leq t \leq \frac{1}{2}\binom{2N}{N}$, at least one of the sets $S_i^{(t)}$, $i = 1$ or 2, contains one of our A_k's. A fixed A_k can clearly be contained for only $\binom{2N-n}{N-n}$ values of t in one of the sets $S_i^{(t)}$, $i = 1$ or 2 (i.e. there are $\binom{2N-n}{N-n}$ subsets of S having N elements which contains a given A_k). Thus clearly

$$m_{2N}(n) \geq \frac{1}{2}\binom{2N}{N}\Big/\binom{2N-n}{N-n} = \frac{1}{2} \prod_{i=0}^{n-1} \frac{2N-i}{N-i} = 2^{n-1} \prod_{i=0}^{n-1} \left(1 + \frac{i}{2N-2i}\right).$$

Thus since $m_{k+1}(n) \leq m_k(n)$ is obvious, Theorem 1 is proved.

THEOREM 2.

$$(4)\qquad m_{2N+1}(n) \leq m_{2N}(n) \leq \left[N2^n \prod_{i=0}^{n-1} \left(1 - \frac{i}{2N-i}\right)^{-1}\right]$$

$$= N2^n \prod_{i=0}^{n-1} \left(1 + \frac{i}{2N-2i}\right) = f(N, n)$$

The proof of Theorem 2 follows very closely the proof in [2]. Let $|S| = 2N$. We shall construct our $f(N,n)$ sets A_k, $1 \leq k \leq f(N,n)$, $A_k \subset S$, $|A_k| = n$, not having property B by induction. Suppose I have already chosen ℓ of the sets A_j, $1 \leq j \leq \ell < f(N,n)$ and suppose that there are u_ℓ pairs of subsets of S $\{K_i, \overline{K}_i\}$, $1 \leq i \leq u_\ell$ so that no set A_j, $1 \leq j \leq \ell$ is contained either in K_i or in $\overline{K}_i$.

If $u_\ell = 0$ Theorem 2 is proved. Assume henceforth $u_\ell > 0$. We shall prove that we can find a set $A_{\ell+1}$ so that

$$(5) \qquad u_{\ell+1} \leq u_\ell \left(1 - \prod_{i=0}^{n-1} \left(1 - \frac{i}{2N-i}\right)/2^{n-1}\right).$$

For each i, $1 \leq i \leq u_\ell$, consider all subsets of n elements of K_i and $\overline{K}_i$. For fixed i the number of these subsets is clearly

$$\binom{|K_i|}{n} + \binom{|\overline{K}_i|}{n} \geq 2\binom{N}{n} \qquad (|K_i| + |\overline{K}_i| = |S| = 2N).$$

Thus the total number of subsets under consideration $(1 \leq i \leq u_\ell)$ is at least $2u_\ell\binom{N}{n}$. The total number of subsets of S taken n at a time is $\binom{2N}{n}$. Hence at least one of those sets say $A_{\ell+1}$ occurs either in K_i or in $\overline{K}_i$ for at least

$$\frac{2u_\ell\binom{N}{n}}{\binom{2N}{n}} = 2u_\ell \prod_{i=0}^{n-1} (N-i)\left(\prod_{i=0}^{n-1} (2N-i)\right)^{-1} = \frac{u_\ell}{2^{n-1}} \prod_{i=0}^{n-1} \left(1 - \frac{i}{2N-i}\right)$$

values of i, which proves (5).

Clearly $u_0 = 2^{2N-1}$ (since S has 2^{2N} subsets). Hence from (5)

$$(6) \qquad u_r \leq 2^{2N-1} \Big/ \left(1 - \frac{\prod_{i=0}^{n-1} \left(1 - \frac{i}{2N-i}\right)}{2^{n-1}}\right)^r$$

Thus by (6) if $r = f(N,n)$, $u_r < 1$ and our sets A_j, $1 \leq j \leq f(N,n)$, do not have property B, which completes the proof of Theorem 2.

(2) follows easily from Theorems 1 and 2 by Stirling's formula.

For large values of N instead of $m_N(n)$ it seems more appropriate to consider $m'_N(n)$ where $m'_N(n)$ is the smallest integer for which there is a family $\{A_k\}$ $1 \leq k \leq m'_N(n)$ not having property B and satisfying $A_k \subset S$, $|S| = N$ and the further property that the set of A'_ks contained in any proper subset of S has property B. For $n = 2$, $m'_{2N+1}(n) = 2N+1$, and, for even N, $m'_N(n)$ is not defined; this is just a restatement of the fact that the only critical three chromatic graphs are the odd circuits.

It is easy to see that $m_{2n-1}(n) = m_{2n}(n) = \binom{2n-1}{n}$. I can not compute $m_{2n+1}(n)$ and in fact do not know the value of $m_9(4)$.

It would be interesting to find an asymptotic formula for $m_N(n)$ and $m'_N(n)$, but I have not been able to do so. The upper and lower bounds for $m_N(n)$ given by Theorems 1 and 2 differ by 2N; I could not even decrease this to $o(N)$.

I wish to thank the referee for some very useful remarks.

REFERENCES

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