On the sum $\sum d_4(n)$

By P. ERDŐS and I. KÁTAI in Budapest

1. Introduction

Let $d(n)$ denote the number of divisors of n, and $d_k(n)$ be the k-fold iterate of $d(n)$, i.e. $d_1(n)=d(n)$ and $d_k(n)=d(d_{k-1}(n))$ for $k\geqq 2$. Let

$$D_k(x)=\sum_{n\leqq x} d_k(n). \tag{1.1}$$

BELLMAN and SHAPIRO [1] conjectured that $D_k(x)=(1+o(1))\,c_k\,x\log_k x$ for all $k\geqq 1$, where $\log_k$ denotes the k-fold iterated logarithm.

This conjecture was proved for $k=2$ and 3 by KÁTAI [2], [3]. The aim of this paper is to prove it for $k=4$. The cases $k>4$ seem to be essentially more difficult.

Theorem 1. *We have*

$$D_4(x)=(1+o(1))c\,x\log_4 x$$

as $x\to\infty$, *where* c *is a positive constant.*

2. Notations and decomposition of the sum $D_4(x)$

The letters $p, p_1, \ldots, q, q_1, \ldots$ stand for prime numbers. Let $\omega(n)$ denote the number of the different, and $\Omega(n)$ the number of all prime factors of n, i.e. for $n=p_1^{\alpha_1}\ldots p_r^{\alpha_r}$ let $\omega(n)=r$ and $\Omega(n)=\alpha_1+\ldots+\alpha_r$. Let $\lambda(n)=(-1)^{\Omega(n)}$ and let $\mu(n)$ denote the Moebius function. ($|\mu(n)|=1$ or 0 according as n is square-free or not.) Let $\sigma_a(n)=\sum_{d|n} d^a$.

The letters $c, c_1\ldots$ denote suitable positive constants, and $\varepsilon, \varepsilon_1\ldots$ are arbitrary small positive constants not necessarily the same in every place.

We use the symbol $\ll$ in VINOGRADOV's sense.

For the sake of brevity denote $x_1=\log x$, $x_{i+1}=\log x_i$, $y_1=\log y$, $y_{i+1}=\log y_i$ $(i\geqq 1)$ and set

$$a_j(x)=\frac{(\log\log x)^{j-1}}{(j-1)!}\qquad (j=1,2,\cdots). \tag{2.1}$$

Denote by $\mathscr{K}$ the set of integers all whose prime factors occur with an exponent greater than 1. Clearly every integer can be uniquely written in the form

$$n = Km \quad \text{with} \quad (K, m) = 1, \quad K \in \mathscr{K}, \quad m \quad \text{square-free.} \tag{2. 2}$$

K will denote the quadratic part, m the square-free part of n. $\mathscr{A}_K$ is the set of integers whose quadratic part is K.

For $K \in \mathscr{K}$ let the numbers k, k_1, k_2, α be defined as follows:

$$k = d(K), \quad k = 2^\alpha k_1, \quad k_1 \quad \text{odd}, \quad k_2 = d(k_1). \tag{2. 3}$$

Then for an n in (2. 2) we have

$$d_2(n) = (\alpha + 1 + \omega(m))k_2. \tag{2. 4}$$

Set

$$\Sigma_K = \sum_{\substack{n \leq x \\ n \in \mathscr{A}_K}} d_4(n). \tag{2. 5}$$

Then

$$D_4(x) = \sum_{K \in \mathscr{K}} \Sigma_K. \tag{2. 6}$$

Furthermore

$$\Sigma_K = \sum_{r=1}^{\infty} \Sigma_K^r, \tag{2. 7}$$

where in $\sum_K^r$ we sum over those n for which $\omega(m) = r$ (see (2. 2)). Let further

$$Z(y, K, r) = \sum_{\substack{(n, K) = 1 \\ \omega(n) = r \\ n \leq y}} |\mu(n)|; \quad Z(y, K) = \sum_{\substack{n \leq y \\ (n, K) = 1}} |\mu(n)|. \tag{2. 8)-(2. 9}$$

So by (2. 4) we have

$$\Sigma_K^r = d_2(k_2(\alpha + 1 + r))Z\left(\frac{x}{K}, K, r\right). \tag{2. 10}$$

For a general natural number n let $\mathscr{B}_n$ denote the set of those positive integers all prime factors of which occur in n. Let further

$$\tau(n) = \prod_{p|n} \left(1 + \frac{1}{p}\right)^{-1} = \sum_{v \in \mathscr{B}_n} \frac{\lambda(v)}{v}. \tag{2. 11}$$

Let $\pi_r(x)$ be the number of those integers not exceeding x which contain exactly r prime factors.

3. Lemmas

Lemma 1. *For all $r \geqq 1$ we have*

$$\pi_r(x) < c_1 \frac{x(x_2+c_2)^{r-1}}{x_1(r-1)!}. \tag{3.1}$$

This is a known theorem of HARDY and RAMANUJAN [4].
Hence we easily deduce the well-known

Lemma 2. *For all constant $\delta > 0$ the inequalities*

$$\sum_{\substack{n \leqq Y \\ \Omega(n) < (1-\delta)\log_2 Y}} 1 \ll Y(\log Y)^{-\gamma_\delta}, \qquad \sum_{\substack{n \leqq Y \\ \Omega(n) > (1+\delta)\log_2 Y}} 1 \ll Y(\log Y)^{-\gamma_\delta} \tag{3.2)-(3.3}$$

hold with a suitable positive constant γ_δ. Further we have $\gamma_9 = 2$ in (3. 3).

Lemma 3. *Let $h(x)$ denote an increasing function of x, tending to infinity with x. Then*

$$\frac{x}{x_1} \sum_{|j-x_2| \leqq h(x)\sqrt{x_2}} a_j(x) = \big(1+o(1)\big)x, \tag{3.4}$$

and consequently

$$\sum_{j \leqq x_2 - h(x)\sqrt{x_2}} a_j(x) = o(x_1), \qquad \sum_{j \geqq x_2 + h(x)\sqrt{x_2}} a_j(x) = o(x_1). \tag{3.5}$$

Lemma 3 is well known and can be proved by a simple computation.

Lemma 4. *Let $\beta < 1$ be an arbitrary positive constant, $Y_1 \geqq Y_2 \geqq Y_1^\beta$. Then*

$$\sum_{Y_1 \leqq n \leqq Y_1 + Y_2} \{\omega(n) - \log_2 Y_1\}^2 \ll Y_2 \log_2 Y_1. \tag{3.6}$$

This lemma can be proved by the method of TURÁN (see [5]).
Let

$$D_h(x, t) = \sum_{\substack{x < mn^2 \leqq x+h \\ n > t}} \sum 1. \tag{3.7}$$

Lemma 5. *For $0 < t \leqq x^{1/3}$ and $0 < h \leqq x^{2/3}$ we have*

$$D_h(x, t) \ll x^{\vartheta_1} + ht^{-1} \quad \textit{with} \quad \vartheta_1 = 0{,}23. \tag{3.8}$$

Lemma 6. *We have*

$$Z(x, 1) = \frac{6}{\pi^2} x + O(x^{1/2}). \tag{3.9}$$

Furthermore, for $0 \leqq h \leqq x^{2/3}$,

$$Z(x+h, 1) - Z(x, 1) = \frac{6}{\pi^2} x + O(h^{1/2}) + O(x^{\vartheta_1}) \tag{3.10}$$

holds.

For the proof of (3. 8) and (3. 10) see RICHERT [6]. (3. 9) is well known.

Let $I(y, c)$ denote the interval $[y_2 - c\sqrt{y_2},\, y_2 + c\sqrt{y_2}]$. Let further A be an arbitrary but fixed constant and

$$yy_1^{-A} \leqq y^* \leqq y. \tag{3.11}$$

Lemma 7. *For a suitable increasing function $g(y)$ with $\lim_{y\to\infty} g(y) = \infty$ we have*

$$Z(y^*, 1, r) = \frac{6}{\pi^2}\big(1+o(1)\big)\frac{y}{y_1} a_r(y) \quad (y \to \infty) \tag{3.12}$$

uniformly in $I(y, 4g(y))$.

This is a slightly modified form of a result of P. ERDŐS [7].

4. Further lemmas

Lemma 8. *Let $b_K \ll K^\varepsilon$. Then we have*

$$\sum_{\substack{K>u \\ K\in\mathscr{K}}} \frac{b_K}{K} \ll u^{-1/3} \quad (u \to \infty). \tag{4.1}$$

Proof. This is an immediate consequence of the simple and known fact that

$$\sum_{\substack{K\leqq x \\ K\in\mathscr{K}}} 1 \ll x^{1/2+\varepsilon}.$$

Lemma 9. *For fixed $\beta > 0$ we have*

$$\sum_{v\in\mathscr{B}_n} v^{-\beta} \ll d(n), \tag{4.2}$$

furthermore

$$\sum_{\substack{v\in\mathscr{B}_n \\ v>u}} v^{-\beta} \ll d(n) u^{-\gamma} \tag{4.3}$$

when $\gamma < \beta$ and γ is constant.

Proof. Since

$$\sum_{v\in\mathscr{B}_n} v^{-\beta} = \prod_{p|n}\left(1-\frac{1}{p^\beta}\right)^{-1} = \prod_{p^\beta<2} \cdot \prod_{p^\beta\geqq 2} \leqq C(\beta)\, d(n)$$

which proves (4. 2). Now (4. 2) implies (4. 3) since $v^{-\beta} \leqq u^{-\gamma} v^{-\beta+\gamma}$ for $v \geqq u$.

Lemma 10. *We have*

$$Z(y, K, r) = \frac{6}{\pi^2}\big(1+o(1)\big)\tau(K)\frac{y}{y_1} a_r(y) \quad (y\to\infty) \tag{4.4}$$

uniformly for $K \leqq y_2^4$, $r \in I(y, 2g(y))$. [$g(y)$ as in Lemma 7.]

Proof. The identity

$$Z(y,K,r)=\sum_{v\in\mathcal{B}_K}\lambda(v)Z\left(\frac{y}{v},1,r-\Omega(v)\right) \tag{4. 5}$$

can be proved elementarily or by using the uniqueness of Dirichlet series expansions.

Suppose that $K\leqq y_2^4$, $r\in I(y, 2g(y))$. Let $\Delta=y_2^6$. For $v<\Delta$ we have $\Omega(v)<$ $>c\log 2v<cy_3\leqq g(y)y_2^{1/2}$. Hence $r-\Omega(v)\in I(y, 2g(y))$, if y is sufficiently large. From (4. 5) we obtain

$$Z(y,K,r)=\sum_{\substack{v\in\mathcal{B}_K\\ v\leqq\Delta}}\lambda(v)Z\left(\frac{y}{v},1,r-\Omega(v)\right)+ \tag{4. 6}$$

$$+O\left(\sum_{\substack{v>\Delta\\ v\in\mathcal{B}_K}}Z\left(\frac{y}{v},1,r-\Omega(v)\right)\right)=\Sigma_1+O(\Sigma_2).$$

Using Lemma 7 we deduce

$$\Sigma_1=\frac{6}{\pi^2}\big(1+o(1)\big)\frac{y}{y_1}\sum_{\substack{v\in\mathcal{B}_K\\ v\leqq\Delta}}\frac{\lambda(v)}{v}a_{r-\Omega(v)}(y). \tag{4. 7}$$

Since $a_{r-\Omega(v)}(y)=\big(1+o(1)\big)a_r(y)$ in $r\in I\big(y, 2g(y)\big)$ we have

$$\Sigma_1=\frac{6}{\pi^2}\big(1+o(1)\big)\tau(K)\frac{ya_r(y)}{y_1}+o(1)a_r(y)\frac{y}{y_1}\sum_{v\in\mathcal{B}_K}1_v+O\left(\frac{y}{y_1}a_r(y)\sum_{\substack{v\geqq\Delta\\ v\in\mathcal{B}_K}}\frac{1}{v}\right).$$

Hence by $\sum_{v\in\mathcal{B}_K}v^{-1}\ll\tau(K)$ and (4. 3) we obtain

$$\Sigma_1=\frac{6}{\pi^2}\big(1+o(1)\big)\tau(K)\frac{ya_r(y)}{y_1}.$$

Now we estimate the sum Σ_2. We have by (4. 2) that

$$\Sigma_2\leqq y\sum_{v>\Delta}v^{-1}\ll yd(K)\Delta^{-1/2}\ll yy_2^{-2}$$

and so $\Sigma_2=o(\Sigma_1)$. Hence (4. 4) follows.

Lemma 11. *We have*

$$Z(y,K)=\frac{6}{\pi^2}\tau(K)y+O\big(d(K)y^{1/2}\big). \tag{4. 8}$$

Proof. Summing in (4. 5) for $1\leqq r<\infty$ we deduce

$$Z(y,K)=\sum_{\substack{v\leqq y\\ v\in\mathcal{B}_K}}\lambda(v)Z\left(\frac{y}{v},1\right). \tag{4. 9}$$

Hence by (3. 9) we have

$$Z(y, K) = \frac{6}{\pi^2}\tau(K)y + O\left(y \sum_{\substack{v \geqq y \\ v \in \mathcal{B}_K}} \frac{1}{v}\right) + O\left(y^{1/2} \sum_{v \in \mathcal{B}_K} \frac{1}{\sqrt{v}}\right).$$

By Lemma 9 we deduce (4. 8).

Lemma 12. *Let* $z_1^{2/3} \geqq z_2 \geqq z_1^{1/4}$, $L = O(z_1^{1/4})$. *Then we have*

$$Z(z_1+z_2, L) - Z(z_1, L) = \frac{6}{\pi^2}\tau(L)z_2 + O\big(d(L)(z_1^{1/4} + z_2^{1/2})\big). \tag{4. 10}$$

Proof. Using the identity (4. 9) we have

$$\Delta \overset{\text{def}}{=} Z(z_1+z_2, L) - Z(z_1, L) = \sum_{\substack{v \in \mathcal{B}_L \\ v < z_1+z_2}} \lambda(v)\left\{Z\left(\frac{z_1+z_2}{v}, 1\right) - Z\left(\frac{z_1}{v}, 1\right)\right\}.$$

Hence by (3. 10) we obtain

$$\Delta = \frac{6}{\pi^2}\tau(L)z_2 + O\left(z_2 \sum_{\substack{v > z_2 \\ v \in \mathcal{B}_L}} \frac{1}{v}\right) + O\big(z_1^{1/4} \sum_{v \in \mathcal{B}_L} v^{-1}\big) +$$

$$+ O\big(\sqrt{z_2} \sum_{v \in \mathcal{B}_L} v^{-1/2}\big) + O\big(\sum_{\substack{z_2 < v < z_1+z_2 \\ v \in \mathcal{B}_L}} 1\big).$$

For the last sum we have

$$\sum_{\substack{z_2 < v < z_1+z_2 \\ v \in \mathcal{B}_L}} 1 < (2z_1)^{0,1} \sum_{v \in \mathcal{B}_L} v^{-0,1} \ll d(L) z_1^{0,1}.$$

Using Lemma 9 for the other remainder terms we have (4. 10).

5. For a general integer S let

$$T_S(Y_1, Y_1+Y_2) = \sum_{Y_1 < r \leqq Y_1+Y_2} d_2(Sr). \tag{5. 1}$$

Every integer r can be represented in the form

$$r = R_1 R_2 \varrho, \quad R_1 \in \mathcal{B}_S, (R_2\varrho, S) = 1, \quad R_2 \in \mathcal{K}, \quad |\mu(\varrho)| = 1 \tag{5. 2}$$

and this representation is unique.

Let $L = R_1 R_2$ and D_L be the set of those r in (5. 2) for which $L = R_1 R_2$. Let

$$d(SL) = l = 2^\beta l_1, \quad \text{with } l_1 \text{ odd and } d(l_1) = l_2, \tag{5. 3}$$

and let

$$\Lambda(S) = \sum_{R_1, R_2} \frac{l_2 \tau(SR_2)}{R_1 R_2}. \tag{5. 4}$$

Lemma 13. *Let* $Y_1^{1/2} \geqq Y_2 \geqq Y_1^{1/3}$, $S \leqq Y_1^{0,01}$. *Then*

$$(5.5) \qquad T_S(Y_1, Y_1+Y_2) = \frac{6}{\pi^2} A(S) Y_2 \log\log Y_1 + O\big(Y_2 (\log_2 Y_1)^{1/2} S^{\varepsilon}\big).$$

Proof. Using the notations in (5.3) we have for an r in (5.2)

$$(5.6) \qquad d_2(Sr) = \big(\omega(\varrho)+\beta+1\big) l_2 = \omega(r) l_2 + \big(\beta+1-\omega(L)\big) l_2.$$

Hence

$$(5.7) \quad T_S(Y_1, Y_1+Y_2) = \sum_{Y_1<r\leqq Y_1+Y_2} \omega(r) l_2 + \sum_{Y_1<r\leqq Y_1+Y_2} \big(\beta+1-\omega(L)\big) l_2 = \Sigma_1 + \Sigma_2.$$

Furthermore

$$(5.8) \qquad \Sigma_1 = \log_2 Y_1 \sum_{Y_1<r\leqq Y_1+Y_2} l_2 + O\Big(\sum_{Y_1<r\leqq Y_1+Y_2} |\omega(r) - \log_2 Y_1| l_2\Big) = \\ = \log_2 Y_1 \Sigma_3 + O(\Sigma_4).$$

By the Cauchy inequality we have

$$(5.9) \quad \Sigma_4 \leqq \Big\{\sum_{Y_1<r\leqq Y_1+Y_2} (\omega(r) - \log_2 Y_1)^2\Big\}^{1/2} \Big\{\sum_{Y_1<r\leqq Y_1+Y_2} l_2^2\Big\}^{1/2} = \Sigma_5^{1/2} \Sigma_6^{1/2}.$$

By Lemma 4

$$(5.10) \qquad \Sigma_5 \ll Y_2 \log_2 Y_1.$$

Using (5.3) we obtain $\big(d(m) < m^{\varepsilon}\big)$

$$l_2^2 = O\big((SL)^{\varepsilon}\big) \quad \big(\beta+1-\omega(L)\big) l_2 = O\big((SL)^{\varepsilon}\big).$$

Consequently,

$$(5.11) \qquad \Sigma_2 = O(S^{\varepsilon} \Sigma_7), \quad \Sigma_6 = O(S^{\varepsilon} \Sigma_7),$$

where

$$(5.12) \qquad \Sigma_7 = \sum_{Y_1<r\leqq Y_1+Y_2} L^{\varepsilon}.$$

We have

$$(5.13) \qquad \Sigma_7 \ll \sum_{\substack{Y_1<r=\varrho L\leqq Y_1+Y_2 \\ L\leqq Y_2}} L^{\varepsilon} + Y_1^{\varepsilon} \sum_{\substack{Y_1<r=\varrho L\leqq Y_1+Y_2 \\ L>Y_2}} 1 = \Sigma_8 + Y_1^{\varepsilon} \Sigma_9.$$

Furthermore

$$(5.14) \qquad \Sigma_8 \ll Y_2 \sum_{R_1 R_2 \leqq Y_2} (R_1 R_2)^{\varepsilon-1} \ll Y_2 \Big\{\sum_{R_1 \in \mathscr{B}_S} R_1^{\varepsilon-1}\Big\}\Big\{\sum_{R_2 \in \mathscr{K}} R_2^{\varepsilon-1}\Big\} \ll \\ \ll Y_2 \prod_{p|S} \left(1 - \frac{1}{p^{1-\varepsilon}}\right)^{-1} \ll d(S) Y_2 \ll Y_2 S^{\varepsilon}.$$

Now we estimate the sum Σ_9.

Let u^2 and v^2 denote the greatest square divisors of the numbers R_1 and R_2. Since $R_1 \in \mathscr{B}_S$, so $u^2 \geqq R_1/S \geqq R_1 Y_1^{-0,01} \geqq R_1 Y_2^{0,03}$ holds. Furthermore, since all prime factors of R_2 occur with an exponent greater than 1, we have $R_2 = v^2 l$

and l/v, i.e. $v \geqq R_2^{1/3}$. Hence the in $r\Sigma_9$ have the form $r=n^2m$, where $n \geqq R_1 R_2^{1/3} Y_2^{-0,03} \geqq$ $\geqq (R_1R_2)^{1/3} Y_2^{-0,03} \geqq Y_2^{0,3}$. Thus

$$\Sigma_9 \ll \sum_{\substack{Y_1 \leqq n^2 m \leqq Y_1+Y_2 \\ n \geqq Y_2^{0,3}}} 1.$$

Applying Lemma 5 we obtain

$$\Sigma_9 \ll Y_1^{1/4} + Y_2^{0,7} \ll Y_2^{3/4}. \tag{5. 15}$$

Combining this with (5. 14) we deduce

$$T_S(Y_1, Y_1+Y_2) = \log_2 Y_1 \cdot \Sigma_3 + O\big(Y_2(\log_2 Y_1)^{1/2} S^\varepsilon\big). \tag{5. 16}$$

Now we estimate the sum Σ_3. We have by (5. 15)

$$\Sigma_3 = \sum_{\substack{Y_1 \leqq r \leqq Y_1+Y_2 \\ L \leqq Y_2}} l_2 + O\big(Y_1^\varepsilon \sum_{\substack{Y_1 < r \leqq Y_1+Y_2 \\ L > Y_2}} 1\big) = \Sigma_{10} + O(Y_1^\varepsilon \Sigma_9) = \Sigma_{10} + O(Y_2^{0,8}). \tag{5. 17}$$

Furthermore by (5. 2)

$$\Sigma_{10} = \sum_{L \leqq Y_2} l_2 \left\{ Z\left(\frac{Y_1+Y_2}{L}, SL\right) - Z\left(\frac{Y_1}{L}, SL\right)\right\} = \tag{5. 18}$$

$$= \sum_{L \leqq Y_2^{0,01}} + \sum_{L > Y_2^{0,01}} = \Sigma_{11} + \Sigma_{12}.$$

For Σ_{12} we have

$$\Sigma_{12} \ll Y_2 \sum_{L \geqq Y_2^{0,01}} \frac{l_2}{L} \ll Y_2 \sum_{L \geqq Y_2^{0,01}} L^{-1+\varepsilon} \ll Y_2 \{\sum_{R_1} R_1^{-1+\varepsilon}\}\{\sum_{R_2 > Y_2^{0,005}} R_2^{-1+\varepsilon}\} + \tag{5. 19}$$

$$+ Y_2 \{\sum_{R_1 > Y_2^{0,005}} R_1^{-1+\varepsilon}\}\{\sum_{R_2} R_2^{-1+\varepsilon}\} \ll Y_2^{1-0,0001}.$$

For Σ_{11} we use Lemma 12 and deduce

$$\Sigma_{12} = \frac{6}{\pi^2} Y_2 \sum_{L \leqq Y_2^{0,01}} \frac{l_2 \tau(LS)}{L} + O\left(Y_1^{1/4} \sum_{L \leqq Y_2^{0,01}} \frac{d(L) l_2}{L^{1/4}}\right) + \tag{5. 20}$$

$$+ O\left(Y_2^{1/2} \sum_{L \leqq Y_2^{0,01}} \frac{d(L) l_2}{L^{1/2}}\right) = \frac{6}{\pi^2} Y_2 \Lambda(S) + O\left(\sum_{L \geqq Y_2^{0,01}} \frac{l_2 \tau(LS)}{L}\right) + O(Y_2^{3/4}).$$

Further, by elementary calculation,

$$\sum_{L \geqq Y_2^{0,01}} l_2 \frac{\tau(LS)}{L} \leqq \frac{\tau(S)}{(Y_2^{0.01})^{1/4}} Y_2^\varepsilon \sum_L \frac{\tau(L)}{L^{3/4}} \ll Y_2^{-0,001}. \tag{5. 21}$$

Combining our inequalities (5. 17)—(5. 21), (5. 5) follows and hence the lemma is proved.

Putting $S=1$ in Lemma 13 we obtain by a simple calculation

$$D_2(x) = \sum_{n \leqq x} d_2(n) = cx \log_2 x + O\big(x\sqrt{\log_2 x}\big).$$

6. The proof of the theorem

First we prove that

(6. 1) $$\Sigma_1 \overset{\text{def}}{=} \sum_{\substack{K > x_2{}^3 \\ K \in \mathcal{K}}} \Sigma_K \ll x.$$

Indeed by (2. 3), (2. 4) we have

$$d_4(n) \leqq d_2(n) = \big(\alpha + 1 + \omega(m)\big)k_2, \ \alpha \ll \log K.$$

Let $\Sigma_1 = \Sigma_2 + \Sigma_3$, with $\omega(m) \leqq 10x_2$ in Σ_2 and $\omega(m) > 10x_2$ in Σ_3. So by Lemma 8 we have

$$\Sigma_2 \ll x \sum_{\substack{K > x_2{}^3 \\ K \in \mathcal{K}}} \frac{(\log K + x_2)k_2}{K} \ll x.$$

Furthermore, using that $\omega(m) \ll x_1$, we have

$$\Sigma_3 \ll x_1 \sum_{\omega(n) \geqq 10x_2} k_2 \ll x_1 \Big\{\sum_{n \leqq x} k_2^2\Big\}^{1/2} \Big\{\sum_{\substack{\omega(n) \geqq 10x_2 \\ u \leqq x}} 1\Big\}^{1/2} \ll xx_1 \left\{\sum \frac{k_2^2}{K}\right\}^{1/2} x_1^{-1} \ll x,$$

by Lemma 8 and Lemma 2.

Suppose now that $K \leqq x_2^3$. Let

(6. 2)-(6. 3) $$\Sigma_K^{(-)} = \sum_{r \leqq \frac{1}{2} x_2} \Sigma_K^r, \quad \Sigma_K^{(+)} = \sum_{r \geqq 2x_2} \Sigma_K^r,$$

(6. 4) $$\Sigma_K^{(0)} = \sum_{\frac{1}{2} x_2 < r < 2x_2} \Sigma_K^r.$$

We prove that

(6. 5) $$\Sigma^{(-)} = \sum_{\substack{K \leqq x_2{}^3 \\ K \in \mathcal{K}}} \Sigma_K^{(-)} = O(x),$$

and that

(6. 6) $$\Sigma^{(+)} = \sum_{\substack{K \leqq x_2{}^3 \\ K \in \mathcal{K}}} \Sigma_K^{(+)} = O(x).$$

Since $K \leqq x_2^3$ we have $\omega(K) \ll x_3$ and so in the sums $\Sigma_K^{(-)}$ $\omega(n) \ll \frac{3}{4} x_2$. Furthermore we have $d_4(n) \leqq G(\varepsilon) d^\varepsilon(n)$. So by the Hölder inequality

$$\Sigma^{(-)} \ll \sum_{\substack{n \leqq x \\ \omega(n) \leqq \frac{3}{4} x_2}} d^\varepsilon(n) \ll \Big\{\sum_{\omega(n) \leqq \frac{3}{4} x_2} 1\Big\}^{1-\varepsilon} \{\Sigma\, d(n)\}^\varepsilon \ll x \cdot x_1^{\varepsilon - \gamma_{1/4}(1-\varepsilon)} \ll x,$$

if ε is small enough (see (3. 2)). The proof of (6. 6) is very similar.

Finally we prove that

$$\Sigma^{(0)} \stackrel{\text{def}}{=} \sum_{K<x_2{}^3} \Sigma_K^{(0)} = c\bigl(1+o(1)\bigr)\cdot xx_4. \tag{6. 7}$$

Since $D_4(x) = \Sigma^{(0)}+\Sigma^{(+)}+\Sigma^{(-)}+\Sigma_1$, the theorem will immediately follow.
By (2. 10) we have

$$\Sigma_K^{(0)} = \sum_{\frac{x_2}{2}<r<2x_2} d_2\bigl(k_2(\alpha+1+r)\bigr)Z\left(\frac{x}{K}, K, r\right) = \Sigma_K^{(1)}+\Sigma_K^{(2)}, \tag{6. 8}$$

where in $\Sigma_K^{(1)}$ $|r-x_2|\leqq g(x)\sqrt{x_2}$ and in $\Sigma_K^{(2)}$ $|r-x_2|\geqq g(x)\sqrt{x_2}$ holds. Here $g(x)$ is a sequence which tends to infinity with α monotonically and for which the Lemma 10 holds.

Let $A=[x_2^{1/3}]$, $A_x = x_2-g(x)\sqrt{x_2}$, $B_x = x_2+g(x)\sqrt{x_2}$ and split the interval into consecutive subintervals with lengths A. Let

$$G_j=[A_x+(j-1)A,\ A_x+jA],\quad j=1,2,\cdots,T;\quad T=\left[\frac{2g(x)\sqrt{x_2}}{A}\right]+1.$$

Thus we have by (4. 4) that

$$\Sigma_K^{(1)} = \sum_{j=1}^{T}\Sigma_K^{(1,j)}+O(\Sigma_K^{(1,T)}), \tag{6. 9}$$

where

$$\Sigma_K^{(1,j)} = \sum_{r\in G_j} d_2\bigl(k_2(\alpha+1+r)\bigr)Z\left(\frac{x}{K}, K, r\right).$$

By Lemma 10 we have

$$\Sigma_K^{(1,j)} = \frac{6}{\pi^2}\bigl(1+o(1)\bigr)\frac{\tau(K)}{K}\frac{x}{\log x/K}\sum_{r\in G_j} d_2\bigl(k_2(\alpha+1+r)\bigr)a_r\left(\frac{x}{K}\right).$$

Taking into account that $a_r(x/K) = \bigl(1+o(1)\bigr)a_r(x)$ for $K\leqq x_2^4$, $r\in I\bigl(x, 2g(x)\bigr)$ and that $a_{r_1}(x)/a_{r_2}(x) = 1+o(1)$ for $|r_1-r_2|\leqq A$, $r_1, r_2\in I\bigl(x, 2g(x)\bigr)$ we have

$$\Sigma_K^{(1,j)} =$$

$$= \frac{6}{\pi^2}\bigl(1+o(1)\bigr)\frac{\tau(K)}{K}\frac{x}{x_1}T_{k_2}\bigl(A_x+(j-1)A+\alpha+1,\ A_x+jA+\alpha+1\bigr)\frac{1}{A}\sum_{r\in G_j}a_r(x).$$

Observing that the conditions of Lemma 13 are satisfied, we deduce

$$\Sigma_K^{(1,j)} = \left(\frac{6}{\pi^2}\right)^2\bigl(1+o(1)\bigr)\frac{\tau(K)}{K}\Lambda(k_2)\frac{x}{x_1}x_4\sum_{r\in G_j}a_r(x)+O\left(\frac{\tau(K)}{K}k_2^{\varepsilon}\frac{x}{x_1}x_4^{1/2}\sum_{r\in C_j}a_r(x)\right).$$

Hence by (6. 9) using (3. 4) we have

$$\Sigma_K^{(1)} = \bigl(1+o(1)\bigr)\left(\frac{6}{\pi^2}\right)^2\frac{\tau(K)}{K}\Lambda(k_2)xx_4+O\left(xx_4^{1/2}\frac{\tau(K)}{K}k_2^{\varepsilon}\right). \tag{6. 10}$$

Now we consider the sum $\Sigma_K^{(2)}$. From (3. 1) we easily deduce

$$Z\left(\frac{x}{K}, K, r\right) < c_1 \frac{x}{Kx_1} \frac{(x_2+c_2)^{r-1}}{(r-1)!} < c \frac{x}{Kx_1} a_r(x)$$

for $r < 2x_2$ and $K < x_2^3$. Hence we have

$$\Sigma_K^{(2)} \ll \frac{x}{Kx_1}\Big\{ \sum_{\frac{x_2}{2} \leqq r \leqq A_x} d_2\big(k_2(\alpha+1+r)\big)a_r(x) + \\ + \sum_{B_x \leqq r \leqq 2x_2} d_2\big(k_2(\alpha+1+r)\big)a_r(x)\Big\}. \tag{6. 11}$$

Let Σ_4 and Σ_5 denote the first and the second sum in the right hand side of (6. 11). Taking into account that $a_r(x)$ is monotonically increasing in Σ_4 and decreasing in Σ_5 in r, we have

$$\Sigma_4 \leqq \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x) \sum_{A_x - jA \leqq r \leqq A_x + (j+1)A} d_2\big(k_2(\alpha+1+r)\big)$$

and similarly

$$\Sigma_5 \leqq \sum_{j=1}^{\left[\frac{x_2}{2A}\right]} a_{B_x + jA}(x) \sum_{B_x + (j-1)A \leqq r \leqq B_x + jA} d_2\big(k_2(\alpha+1+r)\big).$$

Hence by Lemma 13 we have

$$\Sigma_4 \ll \{x_4 \Lambda(k_2) + O(x_4^{1/2} k_2^{\varepsilon})\} A \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x).$$

Since

$$A \sum_{j=0}^{\left[\frac{x_2}{2A}\right]} a_{A_x - jA}(x) \leqq \sum_{r \leqq A_x + A} a_r(x) = o(x_1),$$

we have

$$\Sigma_4 = o(x_1)\{x_4 \Lambda(k_2) + x_4^{1/2} k_2^{1/2}\}. \tag{6. 12}$$

Using similar arguments we can deduce for Σ_5 the same inequality.

Hence by (6. 11) and (6. 10)

$$\Sigma_K^{(2)} = o(1)\Sigma_K^{(1)} \quad \text{i.e.} \quad \Sigma_K^{(0)} = \big(1+o(1)\big)\Sigma_K^{(1)}.$$

Summing over K we have

$$\Sigma^{(0)} = \big(1+o(1)\big)\left(\frac{6}{\pi^2}\right)^2 xx_4 \sum_{K \leqq x_2^3} \frac{\tau(K)}{K} \Lambda(k_2) + O\left(xx_4^{1/2} \sum_{K \leqq x_2^3} \frac{\tau(K)k_2^{\varepsilon}}{K}\right).$$

Observing that the sums are convergent we deduce (6. 7).

This completes the proof of our theorem.

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