AN EXTREMAL PROBLEM ON THE SET OF NONCOPRIME DIVISORS OF A NUMBER

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ABSTRACT

A combinatorial theorem is established, stating that if a family $A_1, A_2, .., A_s$ of subsets of a set M contains every subset of each member, then the complements in M of the A's have a permutation $C_1, C_2, .., C_s$ such that $C_i \supset A_i$. This is used to determine the minimal size of a maximal set of divisors of a number N no two of them being coprime.

1. Introduction and results

Many theorems on intersections of sets have been generalized for entities more general than sets. A first such result is that of De Brujn, Van Tengbergen and Kruijswijk [1]. They established a theorem on *maximal sets of divisors of a number N, no member of which divides another member.* If N is square free, this is equivalent to Sperner's theorem on *the maximal set of subsets of a given set, no subset containing another one.* Other results in the same direction have been obtained in [2, 3, 4]. Two of us [6] generalized in the same sense the following result of [5]:

THEOREM 1. *If $\mathscr{A} = \{A_1, A_2, \cdots, A_m\}$ is a family of (different) subsets of a given set M, $|M| = n$, such that*

$$A_i \cap A_j \neq \phi, \text{ for every } i,j \tag{1}$$

then

a) $m \leqq 2^{n-1}$

and for every n there are $m = 2^{n-1}$ such subsets.

b) *if $m < 2^{n-1}$ then additional members may be included in $\mathscr{A}$, the enlarged family still satisfying* (1).

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REMARK 1. If $m = 2^{n-1}$, then the set $\mathcal{M}$ of all subsets of M is partitioned into $\mathcal{M} = \mathcal{A} \cup \mathcal{F}$, where $\mathcal{F}$ consists of the complements with respect to M of the members of $\mathcal{A}$.

The result in [6] mentioned above is the following:

THEOREM 2. *If* $\mathcal{D} = \{D_1, D_2, \cdots, D_m\}$ *is a set of divisors of an integer* N *whose decomposition into primes is* $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ *and*

$$(D_i, D_j) > 1, \text{ for every } i, j \tag{2}$$

then, denoting $\alpha_1 \alpha_2 \cdots \alpha_n = \alpha$

a) $$m \leq f(N) = \tfrac{1}{2} \sum_I \max \left\{ \prod_{\nu=1}^{\mu} \alpha_{i\nu};\ \alpha / \prod_{\nu=1}^{\mu} \alpha_{i\nu} \right\},$$

where the summation is over all subsets $I = \{i_1, i_2, \cdots, i_\mu\}$ *of* $\{1, 2, \cdots, n\}$, *the product corresponding to the empty set being comsidered as I; and for every N there are* $f(N)$ *such divisors.*

b) *If*

$$m < g(N) = \alpha - 1 + \tfrac{1}{2} \sum_I \min \left(\prod_{\nu=1}^{\mu} \alpha_{i\nu};\ \alpha / \prod_{\nu=1}^{\mu} \alpha_{i\nu} \right) \tag{3}$$

then additional members may be included in $\mathcal{D}$, *the enlarged set still satisfying* (2).

REMARK 2. If N is square free this result is equivalent to Theorem 1. Then $\alpha_1 = \alpha_2 = \cdots = \alpha_n = \alpha = 1$ and $f(N) = g(N) = 2^{n-1}$

REMARK 3. The example of the divisors of 180 which are multiples of 5 shows that for certain N's $g(N)$ is best possible. But $\mathcal{D} = \{2^2 \cdot 3 \cdot 5 \cdot 7; 2 \cdot 3 \cdot 5 \cdot 7; 2^2 \cdot 3 \cdot 5; 2 \cdot 3 \cdot 5; 2^2 \cdot 3 \cdot 7; 2 \cdot 3 \cdot 7;\ 3 \cdot 5 \cdot 7;\ 2^2 \cdot 5 \cdot 7;\ 2 \cdot 5 \cdot 7;\ 3 \cdot 5;\ 3 \cdot 7;\ 5 \cdot 7\}$ contains 12 members while $g(420) = 9$. In both examples the number of members in $\mathcal{D}$ is $\alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1)$ i.e. equals the number of divisors of N which are multiples of p_n—and in the second example not every member is divisible by $p_n = 7$. In both examples the α_i's are supposed to be ordered as in Lemma 1.

Remark 3 makes part 6 of Theorem 2 appear not too illuminating. This is remedied in the present paper by establishing the minimal size of a set $\mathcal{D}$ which satisfies the assumptions of Theorem 2 and cannot be enlarged. This is formulated in the following theorem:

THEOREM 4. *If* $\mathcal{D}, |\mathcal{D}| = m$, *is a set of divisors of* $N = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$,

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n, \tag{4}$$

no two members of the set being coprime and if no additional member may be included in $\mathscr{D}$ without contradicting this requirement then

$$m \geqq \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1). \tag{5}$$

REMARK 4. (5) is best possible, the right side representing the number of divisors of N being multiples of p_n. Two such divisors are clearly not coprime. The final observation in Remark 3 shows that there are other sets of divisors satisfying (5) with equality.

The proof of Theorem 4 depends on the following combinatorial theorem and on Lemma 1.

THEOREM 3. *Let A and M be sets, $A \subset M$. Denote $\bar{A} = M - A$. If $\mathscr{F} = \{A_1, A_2, \cdots, A_s\}$ is a family of sets satisfying*

(i) $A_i \subset M$, $i = 1, 2, \cdots, s$

(ii) $X \subset A_i \Rightarrow X \in \mathscr{F}$

then there exists a permutatuion $C_1, C_2, \cdots, C_s$ of $\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_s$ such that

$$C_i \supset A_i.$$

DEFINITION. A family of sets $\mathscr{F} = \{A_1, A_2, \cdots, A_s\}$ has the property $\mathscr{P}(M)$ if (i) and (ii) hold.

LEMMA 1. *Let M be the set $M = \{1, 2, \cdots, n\}$ and let $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_n$ be positive integers. Denote $\alpha = \alpha_1 \alpha_2 \cdots \alpha_n$, $\bar{A} = M - A$.*

If $\mathscr{F}$ is a family of sets having property $\mathscr{P}(M)$ and if

$$A \in \mathscr{F} \Rightarrow \bar{A} \notin \mathscr{F}, \tag{6}$$

then

$$\alpha_n \sum \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_t} \leqq \sum \alpha/\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_t} \tag{7}$$

where the summation is over $\{i_1, \cdots, i_t\} \in \mathscr{F}$.

2. Proofs

PROOF OF THEOREM 3. For $s = 1, 2$ the theorem is true. Let $s = s_0 > 2$ and suppose by induction that it is true for $s \leqq s_0 - 1$. Let a be a fixed element contained in at least one member of $\mathscr{F}$. Denote by $B'_1, B'_2, \cdots, B'_r$ the members of $\mathscr{F}$ containing the element a, then $B_i = B'_i - a$, $i = 1, 2, \cdots, r$ are also members of $\mathscr{F}$. Denote by $B_{r+1}, B_{r+2}, \cdots, B_{r+q}$ the other members of $\mathscr{F}$, if any. Since

$s_0 = 2r + q$ the families $B_1, B_2, \cdots, B_r$ and $B_1, B_2, \cdots, B_{r+q}$ have fewer members than s_0, and since both have the property $\mathscr{P}(M)$, by the induction hypothesis, there is a permutation of $\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_r$ say $C_1, C_2, \cdots, C_r$ and a permutation of $\bar{\bar{B}}_1, \bar{B}_2, \cdots, \bar{B}_{r+q}$ say $D_1, D_2, \cdots, D_{r+q}$ such that $C_i \supset B_i$ $(i = 1, 2, \cdots, r)$ and $D_i \supset B_i'$ $(i = 1, 2, \cdots, r+q)$. It follows that $D_i \supset B_i' (i = 1, 2, \cdots, r)$, $C_i - a \supset B_i (i = 1, \cdots, r)$ and since $C_i = \bar{B}_j$ implies $C_i - a = \bar{B}_j'$

$$D_1, D_2, \cdots D_r, C_1 - a, \cdots, C_r - a, D_{r+1}, \cdots, D_{r+q}$$

is the required permutation of the complements of the members of $\mathscr{F}$.

PROOF OF LEMMA 1. By Theorem 3 each term of the first sum in (7) divides a corresponding term of the second sum. Moreover, by (6) each such factor is proper and therefore by (4) each term may be multiplied by α_n.

PROOF OF THEOREM 4. Define $\mathscr{A} = \{(j_1, j_2, \cdots, j_k) \mid p_{j_1}^{\beta_1} \cdots p_{j_k}^{\beta_k} \in \mathscr{D}$ for some $\beta_i > 0$, $i = 1, \cdots, k\}$ and let $\mathscr{M}$ be the set of all subsets of $M = \{1, 2, \cdots, n\}$. Then by the maximum property of $\mathscr{D}$,

$$m = \sum_{\mathscr{A}} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_k},$$

where the summation is over $\{j_1, j_2, \cdots, j_k\} \in \mathscr{A}$, and

$$|\mathscr{A}| = 2^{n-1} \text{ by Theorem 1.}$$

Furthermore, since $\mathscr{A}$ cannot contain a set and its complement, the set $\mathscr{F}$ of all complements of members of $\mathscr{A}$ has no member in common with $\mathscr{A}$ and

$$\mathscr{M} = \mathscr{A} \cup \mathscr{F} \tag{8}$$

is a partition of $\mathscr{M}$. It follows also that

$$m = \sum_{\mathscr{A}} \alpha_{j_1} \alpha_{j_2} \cdots \alpha_{j_k} = \sum_{\mathscr{F}} \alpha / \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_t}$$

where the second summation is over $\{i_1, i_2, \cdots i_t\} \in \mathscr{F}$. We have to prove

$$\sum_{\mathscr{F}} \alpha / \alpha_{i_1} \cdots \alpha_{i_t} \geqq \alpha_n \prod_{n-1}^{i=1} (\alpha_i + 1). \tag{9}$$

If $p_n \in \mathscr{D}$, (9) holds obviously with equality, while $p_n \notin \mathscr{D}$ means $n \in \mathscr{F}$. Denote by $\mathscr{A}_n$ and by $\mathscr{F}_n$ the families of sets in $\mathscr{A}$ and $\mathscr{F}$ respectively containing n, and by $\mathscr{F}^*$ the family obtained by deleting n from each member of $\mathscr{F}_n$. Denote also by $\mathscr{A}'$ and $\mathscr{F}'$ the families of sets in $\mathscr{A}$ and $\mathscr{F}$ respectively not containing n.

$$m = \sum_{\mathscr{F}'} \alpha / \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_t} + \sum_{\mathscr{F}_n} \alpha / \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_t},$$

and since

$$\sum_{\mathscr{F}'} \alpha/\alpha_{i_1}\cdots\alpha_{i_t} + \sum_{\mathscr{F}_-} \alpha_{i_1}\cdots\alpha_{i_t} = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1),$$

in order to show (9) it is sufficient to prove

$$\sum_{\mathscr{F}} \alpha/\alpha_{i_1}\cdots\alpha_{i_t} \geqq \sum_{\mathscr{F}_n} \alpha_{i_1}\cdots\alpha_{i_t}$$

i.e.

$$\sum_{\mathscr{F}^*} \alpha/\alpha_{i_1}\cdots\alpha_{i_t}\alpha_n \geqq \alpha_n \sum_{\mathscr{F}^*} \alpha_{i_1}\cdots\alpha_{i_t}.$$

Observe that (10) $\mathscr{F} \in \mathscr{P}(M)$ and hence $\mathscr{F}^* \in \mathscr{P}(M-n)$. For (10), let $B \in \mathscr{F}$ then by (8) $\bar{B} \in \mathscr{A}$, so $\mathscr{D} \subset B$ implies $X \in \mathscr{F}$. The assumptions of Lemma 1 are satisfied by $\mathscr{F}^*$. It follows that

$$\sum_{\mathscr{F}^*} (\alpha/\alpha_n)/\alpha_{i_1}\cdots\alpha_{i_t} \geqq \alpha_{n-1} \sum_{\mathscr{F}^*} \alpha_{i_1}\cdots\alpha_{i_t} \geqq \alpha_n \sum_{\mathscr{F}^*} \alpha_{i_1}\cdots\alpha_{i_t}$$

and the proof is complete.

Final remark

It would be of intetest to determine all sets $\mathscr{D}$ satisfying the assumptions of Theorem 4 with $m = \alpha_n \prod_{i=1}^{n-1} (\alpha_i + 1)$.

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