ELEMENTE DER MATHEMATIK

Revue de mathématiques élémentaires – Rivista di matematica elementare

Zeitschrift zur Pflege der Mathematik

und zur Förderung des mathematisch-physikalischen Unterrichts

Publiziert mit Unterstützung des Schweizerischen Nationalfonds
zur Förderung der wissenschaftlichen Forschung

El. Math. Band 25 Heft 6 Seiten 121-144 10. November 1970

Distinct Distances Between Lattice Points

How many points (x_i, y_i), $1 \leqslant i \leqslant k$, with integer coordinates $0 < x_i, y_i \leqslant n$, may be chosen with all mutual distances distinct? By counting such distances, and pairs of differences of coordinates, we have

$$\binom{k}{2} \leqslant \binom{n+1}{2} - 1\,, \tag{1}$$

so that $k \leqslant n$, and for $2 \leqslant n \leqslant 7$ such a bound can be attained; e.g. for $2 \leqslant n \leqslant 5$, by the points (1,1), (1,2), (3,1), (4,4) and (5,3); for $n = 6$ by (1,1), (1,2), (2,4), (4,6), (6,3) and (6,6); and for $n = 7$ by (1,1), (1,3), (2,3), (3,7), (4,1), (6,6) and (7,7).

However, the fact that numbers may be expressed in more than one way as the sum of two squares indicates that this bound cannot be attained for $n > 15$. A result of LANDAU [4] states that the number of integers less than x expressible as the sum of two squares is asymptotically $c_1\, x\, (\log x)^{-1/2}$, so we can replace the right member of (1) by $c_2\, n^2\, (\log n)^{-1/2}$ and we have the upper bound

$$k < c_3\, n\, (\log n)^{-1/4}\,, \tag{2}$$

where c_i is in each case a positive constant.

A heuristic argument can be given to support the conjecture

$$(?) \qquad k < c_4\, n^{2/3}\, (\log n)^{1/6}\,, \tag{3}$$

but it lacks conviction since the corresponding argument in one dimension gives a false result.

On the other hand we can show

$$k > n^{2/3-\varepsilon} \tag{4}$$

for any $\varepsilon > 0$ and sufficiently large n, by means of the following construction. Choose points successively; when k points have been chosen, take another so that

(a) it does not lie on any circle having one of the k points as centre and one of the $\binom{k}{2}$ distinct distances determined by these points as radius.

(b) it does not form, with any of the first k points, a line with slope b/a, $(a, b) = 1$, $|a| < n^{1/3}$, $|b| < n^{1/3}$. Note that in particular no two points determine a distance less than $n^{1/3}$.

(c) it is not equidistant from any pair of the first k points.

We may choose such a point provided that all n^2 points are not excluded by these conditions.

Condition (a) excludes at most $k\binom{k}{2} n^{c_5/\log\log n}$ points, since there are $\binom{k}{2}$ circles round each of k points, and each circle contains at most $n^{c_5/\log\log n}$ lattice points[1]).

Condition (b) excludes at most

$$k \sum_{a=1}^{n^{1/3}} 4\,\varphi(a)\,\frac{n}{a} < c_6\, k\, n^{4/3}$$

points, since a line with slope b/a, $b < a$, $(a, b) = 1$, contains at most n/a lattice points.

Condition (c) excludes at most $\binom{k}{2} n^{2/3}$ points, since there are $\binom{k}{2}$ lines of equidistant points, each of which has slope b/a, $(a, b) = 1$, $|a| \geqslant n^{1/3}$ and such a line contains at most $n/|a| \leqslant n^{2/3}$ lattice points.

Hence, so long as

$$\frac{1}{2}\, k^3\, n^{c_5/\log\log n} + c_6\, k\, n^{4/3} + \frac{1}{2}\, k^2\, n^{2/3} < n^2\,,$$

there remain eligible points, and this is the case if $k \leqq n^{2/3-\varepsilon}$. The lower bound (4) is thus established.

For the corresponding problem in one dimension, the existence of perfect difference sets [6] shows that for n an even power of a prime,

$$k \geqslant n^{1/2} + 1\,,$$

so that generally

$$k > n^{1/2}\,(1-\varepsilon)\,. \tag{5}$$

On the other hand it is known [2, 5] that

$$k < n^{1/2} + n^{1/4} + 1\,. \tag{6}$$

In d dimensions, $d \geqslant 3$, we may replace Landau's theorem by the theorems on sums of three or four squares, giving an upper bound

$$k < c_7\, d^{1/2} n\,, \tag{7}$$

while the corresponding heuristic argument suggests the conjecture

$$(?) \qquad k < c_8\, d^{2/3} n^{2/3} (\log n)^{1/3}\,. \tag{8}$$

The construction, with (hyper)spheres and (hyper)planes, corresponding to that given above, yields the same lower bound (4) as before.

One can also ask for configurations containing a *minimum* number of points, determining distinct distances, so that *no* point may be added without duplicating

[1]) It is well known that the number of solutions of $n = x^2 + y^2$ is less than or equal to $d(n)$, the number of divisors of n [3] and $d(n) < n^{c/\log\log n}$ by a well known result of Wigert [3].

a distance. Can this be done with as few as $O(n^{1/2})$ points; or with $O(n^{1/3})$ points in one dimension?

Another open problem [1] is given any n points in the plane (not necessarily lattice points) [or in d dimensions], how many can one select so that the distances which are determined are all distinct?

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