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On a lemma of Hajnal–Folkman

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HAJNAL and FOLKMAN [3], [2], independently of each other, proved the following Lemma: Let $|\mathcal{S}| = 2n-1$, $A_i \subset \mathcal{S}$, $|A_i| = n$ be subsets of $\mathcal{S}$ so that to every element x of $\mathcal{S}$ there is an A_i not containing x. We define now a graph as follows: $x \in \mathcal{S}$, $y \in \mathcal{S}$ are joined if for some A_i they are both contained in A_i. The Lemma asserts that this graph contains a complete graph of $n+1$ vertices. We are going to generalise and extend this Lemma in various directions and establish some connections with RAMSEY's theorem. First we have to introduce some notations.

The basic elements of an r-graph are its vertices and the r-tuples formed from some of its vertices. $K_r(n)$ is the complete r-graph of n vertices and all its $\binom{n}{r}$ r-tuples.

For $r=2$ we obtain the ordinary graphs. Let $\mathcal{S}$ be a set. A family of subsets $A_i \subset \mathcal{S}$ defines an r-graph $\mathcal{G}^{(r)}(\mathcal{S}; A_1, \dots)$ as follows: The vertices are the elements of $\mathcal{S}$, an r-tuple belongs to our graph if and only if it is a subset of one of our A's. Such r-graphs were, as far as I know, first studied in [1] in a context that differs from this. We say that the family can be represented by i vertices if there are i elements $x_1, \dots, x_i$ of $\mathcal{S}$, so that all the A's contain one of the x_j's, $1 \le j \le i$. The symbol $(m,n,i,r) \longrightarrow u$ means that if $|\mathcal{S}| = m \ge n$ and $A_i \subset \mathcal{S}, |A_i| \ge n$ is any family of subsets which cannot be represented by i vertices, then $\mathcal{G}^{(r)}(\mathcal{S}; A_1, \dots)$ contains a $K^{(r)}(u)$. $(m,n,i,r) \not\rightarrow u$ means that $(m,n,i,r) \longrightarrow u$ does not hold (i.e. there are sets $A_j \subset \mathcal{S}$, $|\mathcal{S}| = m$, $|A_j| \ge n$ which cannot be represented by i vertices but $\mathcal{G}(\mathcal{S}; A_1, \dots)$ does not contain a $K^{(r)}(u)$). $f(k,\ell)$ are the so-called Ramsey numbers, $f(k,\ell)$ is the smallest integer, so that every graph of $f(k,\ell)$ vertices either contains a $K(\ell)$ or its complementary graph contains a $K(\ell)$ (in the complementary graph, two vertices are joined if and only if they are not joined in the graph).

Trivially $(m,n,i,r) \longrightarrow n$ always holds and the only interesting cases occur for $u > n$. Clearly the following monotonicity relations hold:

(1) $(m,n,i,r) \longrightarrow u$ implies $(m,n,i',r) \longrightarrow u$ if $i' > i$

(2) $(m,n,i,r) \longrightarrow u$ implies $(m',n,i,r) \longrightarrow u$ if $m > m' \ge u$

(3) $(m,n,i,r) \longrightarrow u$ implies (m,n,i,r') if $r' < r$

The Lemma of Hajnal and Folkman can be expressed in our notation as $(2n-1, n, 1, 2) \rightarrow n+1$. Clearly $(2n, n, 1, 2) \not\rightarrow n+1$ (it suffices to take two disjoint n-tuples in $\mathcal{S}$, $|\mathcal{S}| = 2n$), also for every $m \ge n$ $(m,n,1,2) \not\rightarrow n+2$ (take all n-element subsets of $\mathcal{S}$, $|\mathcal{S}| = n+1$). On the other hand we prove the following generalisation of the Lemma of HAJNAL and FOLKMAN:

THEOREM. Let $i \ge 1$. Then

(4) $$(2n+i-2, n, i, 2) \longrightarrow n+i .$$

For $i = 1$ this is the Lemma of Hajnal and Folkman. To prove (4) for $i > 1$, we use induction with respect to i.

Assume that (4) holds for $i-1$ and every n. Let $|\mathcal{S}| = 2n+i-2$ and let x_j be any element of $\mathcal{S}$. Consider the family of all the A's contained in $\mathcal{S}-x_j$. They cannot be represented by $i-1$ elements, hence by our induction hypothesis $(2n+i-3, n, i-1, 2) \rightarrow n+i-1$, thus for every x_j $\mathcal{G}^{(2)}(\mathcal{S}-x_j, A_1, \dots)$ contains a complete graph $K_2(n+i-1)$ which is contained in $\mathcal{G}^{(2)}(\mathcal{S}, A_1, \dots)$. Denote the set of vertices of this graph by F_j, $F_j \subset \mathcal{S}-x_j$. Clearly the family of sets F_j cannot be represented by one element, thus by the Lemma of HAJNAL and FOLKMAN (and (2)) we have $(2n+i-2, n+i-1, 1, 2) \longrightarrow n+i$, or $\mathcal{G}^{(2)}(\mathcal{S}; F_1, \dots)$ contains a $K_2(n+i)$, but since $\mathcal{G}^{(2)}(\mathcal{S}; F_1, \dots)$ is clearly a subgraph of $\mathcal{G}^{(2)}(\mathcal{S}; A_1, \dots)$, this completes the proof of our Theorem.

Our Theorem is the best possible. To show this observe that

(5) $$(2n+i-2, n, i, 2) \not\rightarrow n+i+1,$$

(6) $$(2n+i-1, n, i, 2) \not\rightarrow n+\left[\frac{i+1}{2}\right],$$

(7) $$(2n+i-2, n, i-1, 2) \not\rightarrow n+i.$$

(5) is obvious, it suffices to consider all n-element subsets of $\mathcal{S}$, $|\mathcal{S}| = n+i$

(7) immediately follows from (6). (6) is slightly less obvious. Assume first that $i = 2j+1$. Let the elements of $\mathcal{S}$ be x_t, y_t, $t = 1, \dots, n+1$. Let the A_j, $|A_j| = n$ be all subsets of $\mathcal{S}$ which contain at most one of the elements x_t, y_t, $t = 1, \dots, n+j$.

Clearly $\mathcal{G}^{(2)}(\mathcal{S}; A_1, \dots)$ does not contain a $K_2(n+j+1)$ and the A's can not be represented by $2j+1$ elements; this proves (6) for odd i. Assume next $i = 2j+2$. We then have to show $(2n+2j+1, n, 2j+2, 2) \not\rightarrow n+j+1$. Let the elements of $\mathcal{S}$ be the residues $\bmod (2n+2j+1)$. The sets A_t, $A_t \subset \mathcal{S}$, $|A_t| = n$ are those n-element subsets of $\mathcal{S}$ which do not contain two consecutive residues. Clearly $\mathcal{G}^{(2)}(\mathcal{S}; A_1, \dots)$ does not contain a $K_2(n+j+1)$ thus, to complete our proof, we only have to show that the A's are not represented by $2j+2$ residues.

Let $|\mathcal{U}| = 2j+2$ be a set of $2j+2$ residues; we show that $\mathcal{S} - \mathcal{U}$ must contain an A. Without loss of generality, we can assume that 1 is in $\mathcal{U}$. Let $\mathcal{S}_1$ be the set of odd residues excluding 1 and $\mathcal{S}_2$ is the set of even residues. $|\mathcal{U}| = 2j+2$ implies that either $|\mathcal{S}_1 \cap \mathcal{U}| \leq j$ or $|\mathcal{S}_2 \cap \mathcal{U}| \leq j$. Assume without loss of generality $|\mathcal{S}_1 \cap \mathcal{U}| \leq j$. But then $|\mathcal{S}_1 - \mathcal{S}_1 \cap \mathcal{U}| \geq n$ or $\mathcal{S}_1 - \mathcal{U}$ contains an A, as stated. This completes the proof of (6). It seems certain that (6) is not the best possible.

Several unsolved problems can be posed. Denote by $A(n,i)$ the smallest integer for which $(A(n,i), n, i, 2) \not\rightarrow n+1$. (6) and our Theorems show $A(n,1) = 2n$, $A(n,2) = 2n+1$. I conjectured $A(n,3) > 2n+2$, in other words I conjectured

(8) $$(2n+2, n, 3, 2) \rightarrow n+1 .$$

For $n=2$ (8) is Ramsey's theorem (a graph of 6 vertices either contains a triangle or a set of three independent vertices). HAJNAL and I proved (8) for $n=3$ and recently SZEMERÉDI has proved (8) for all n.

HAJNAL and I proved $A(n,3) \leq 3n$ i.e. we proved

(9) $$(3n+1, n, 3, 2) \not\rightarrow n+1 .$$

To prove (9), let $\mathcal{S}$ be the set of residues $\bmod (3n+1)$ and the A's are all the sets of n consecutive residues. Perhaps $(3n, n, 3, 2) \rightarrow n+1$ holds.

It is clear that many further problems can be posed.

We just state one more tirival result:

(10) $(m, n, i, r) \rightarrow u$ implies for every $t>0$ $(m+t, n, i+t, r) \rightarrow u$.

The simple proof of (10) we leave to the reader.

Let us now establish tho connection of our symbol with the RAMSEY numbers. Let $n = 2$ denote, say, by $g(i)$ the smallest integer for which

$$(11) \qquad (g(i), 2, i, 2) \not\rightarrow 3$$

holds. (11) means that there is a graph of $g(i)$ vertices which contains no triangle, and for which the complementary graph contains no $K_2(g(i) - i)$ and $g(i)$ is the smallest integer with this property in other words, $g(i)$ is the smallest integer for which

$$(12) \qquad g(i) < f(g(i) - i, 3).$$

It seems hopeless to determine $f(k,3)$, even to get an asymptotic formula is probably very difficult, thus the determination of $g(i)$ is no doubt very difficult.

It would be interesting to determine the largest integer m for which $(m, n, 1, 3) \rightarrow n+1$ holds.

One final remark. HAJNAL and I proved $(11, 3, 6, 2) \not\rightarrow 4$.

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