COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
4. COMBINATORIAL THEORY AND ITS APPLICATIONS, BALATONFÜRED (HUNGARY), 1969.

On a problem of Moser

by
P. Erdős and J. Komlós
Budapest, Hungary

Let $f(n)$ be the largest integer with the following property:

Every family F_n of n sets contains a subfamily F_n' of $f(n)$ sets so that the union of two sets of F_n' never equals a third*.

Moser asked for the determination or estimation of $f(n)$. A result of Kleitman [2] shows that $f(n) < cn/\sqrt{\log n}$. J. Riddell who communicated this problem to us pointed out that $f(n) > \sqrt{n}$.

The proof is easy. Let $A_1, \dots, A_n$ be any family of n sets, and order them by inclusion. By the theorem of Dilworth [1] either there is a chain of length $\geq \sqrt{n}$ or a family of $\geq \sqrt{n}$ incomparable sets. In any case we obtain $f(n) \geq \sqrt{n}$, which proves Riddell's result. Now we prove the following

THEOREM

$$\sqrt{n} \leq f(n) \leq 2\sqrt{2n} + 4.$$

*These three sets are assumed to be pairwise different.

The lower bound has just been proved. Thus we only have to prove the upper bound.

Define the positive integer $k \geq 3$ by the relation

$$(1) \qquad \binom{k-1}{2} \leq n \leq \binom{k}{2},$$

and consider k equidistant points on a circle. By an arc of length i, we shall mean a set of i neighbouring points. In the case $i < k$ the meaning of the notion "endpoints of an arc" is clear. It is easy to see that the total number of arcs of length i for $\frac{k}{2} \leq i < k$ is at least $\binom{k}{2}$, so we can take n arcs, whose lengths are between $\frac{k}{2}$ and k. These form our family F_n. Let us choose a subfamily F_n' of S sets from the family F_n, so that the union of two sets of F_n' never can be equal to a third one. We show $f_n \leq 2k$, which, together with (1), proves the theorem.

We say that an arc (in F_n') is minimal with respect to one of its endpoints if it does not contain any other arc (in F_n') with the same endpoint. For every point there are at most two arcs which are minimal with respect to this point (one "to the right" and one "to the left"), the number of minimal arcs is thus at most $2k$.

But all arcs in F_n' are minimal, since if one of them (say A) was not minimal with respect to any of its endpoints (say X and Y) then we should have the relation

$$(2) \qquad A = A_x \vee A_y$$

where A_x resp. A_y denote the shortest arcs contained in F_n' with endpoints x resp. y ((2) follous from $\frac{k}{2} \leq i < k$).

This contradiction proves our statement.

We remark that by the same argument we get that at most $2k$ arcs can be chosen from among the arcs of length i with $2^t \leq i < 2^{t+1}$ (t is arbitrary), if the requirement is the same as above. Thus, a subfamily F' of the family F of all arcs (F contains k^2-k+1 elements), having the property that the union of two of them is never equal to a third one, can contain at most $2k\log_2(k+1)$ elements. ($\log_2$ denotes the logarithm based on 2.)

The arcs of lengths 2^t-1, $t=2,\ldots$ show that a family of $k\log_2 k$ arcs can have this property.

Probably $f(n)=(c+o(1))\sqrt{n}$, but we cannot prove this and have no conjecture about the value of c.

REFERENCES

[1] R. P. DILWORTH, A decomposition theorem for partially ordered sets, Annals of Math. 51 (1950), 161-166.

[2] D. KLEITMAN, On a combinatorial problem of Erdős, Proc. Amer. Math. Soc. 17 (1966), 139-141.