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ON SETS OF DISTANCES OF n POINTS

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Let $f(n)$ be the largest integer so that n distinct points in the plane always determine at least $f(n)$ distinct distances. It is easy to see that $f(3)=1, f(4)=f(5)=2$, $f(6)=f(7)=3$. I proved [1]

$$\sqrt{n-1}-1<f(n)<\frac{c_1 n}{\sqrt{\log n}} \cdot \tag{1}$$

Moser proved [2]

$$f(n)>\frac{n^{2/3}}{2\,(9^{1/3})}-1 \tag{2}$$

which is the best-known lower bound for $f(n)$.

It seems likely that $c_2 n/(\log n)^{1/2}$ is the right order of magnitude for $f(n)$. In fact perhaps the following result holds: Let $x_1, \cdots, x_n$ be n distinct points in the plane. Then for at least one point x_i there are at least $c_3 n/(\log n)^{1/2}$ distinct numbers amongst the distances $d(x_i, x_j)$, where $1 \leqq j \leqq n$.

Assume next that the points $x_1, \cdots, x_n$ are vertices of a convex n-gon. I conjectured [1] and Altman [3] proved that the n points determine at least $[n/2]$ distinct distances. (The regular n-gon shows that this is best possible.) I made two further conjectures [1]. Let $x_1, \cdots, x_n$ be the vertices of a convex n-gon. Then there always is an x_i so that there are at least $[n/2]$ distinct distances among the $d(x_i, x_j)$, where $1 \leqq j \leqq n$ and $j \neq i$. This is probably true but has not yet been settled. The second conjecture asserts that there always is an x_i so that there are no three vertices equidistant from it.

The second conjecture would clearly imply the first, but Danzer disproved it (unpublished). In fact Danzer showed that for each k, there is a convex n-gon with $n > n_0(k)$ so that every vertex has at least k vertices which are equidistant from it.

Let $g(n)$ be the largest integer so that there are n points $x_1, \cdots, x_n$ in the plane for which there are $g(n)$ pairs x_i, x_j with $d(x_i, x_j)=1$. I proved [1]

$$n^{1+c/\log\log n} < g(n) < cn^{3/2}.$$

It seems likely that the lower bound gives the correct order of magnitude, but I could not even prove $g(n)=o(n^{3/2})$.

All these problems can be posed in the case the points are in k-dimensional Euclidean space E_k. Curiously some of them become more tractable for $k \geqq 4$ [4].

Let 7 points be given in E_2. L. M. Kelly and I proved [5] that there are always three of them which determine a nonisosceles triangle. The regular pentagon and its center shows that 7 is best possible. Croft [6] proved that 9 points in E_3 gives the best possible answer and believes that $2k+3$ points in E_k always determine the vertices of a nonisosceles triangle.

More generally one can ask the following question. Let $f(n, k)$ be the smallest integer so that if $x_1, \cdots, x_l$ are $l=f(n, k)$ points in E_k, one can always select k of them so that all the $C_{k,2}$ distances are distinct. A good estimation for $f(n, k)$ seems difficult even for $k=1$. I conjecture $f(n, 1)=(1+o(1))n^2$. A result of Turán and myself [7] shows that $f(n, 1) \geqq (1+o(1))n^2$.

For some of these and other geometric problems, see my Hungarian paper in Mat. Lapok, 8 (1957) 86–92.

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