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Set mappings and polarized partition relations

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1. INTRODUCTION

A set mapping on a set S is a function f from S into the set of subsets of S such that $x \notin f(x)$ $(x \in S)$; $A \subset S$ is called a free set (for the set mapping) if $y \notin f(x)$ for all $x, y \in A$, i.e. $A \cap f(A) = \phi$. It was an old conjecture of Ruziewicz [1] that, if $|S| = m \geq \aleph_0$, and if $|f(x)| < n$ $(x \in S)$, where n is a fixed cardinal less than m, then there is a free set of cordinal m. D. Lázár [2] proved this in the case when m is a regular cardinal and Sophie Piccard [3] proved the conjecture for those cardinals m which are the sums of $\aleph_0$ smaller cardinals. Erdős [4] gave a solution of the complete conjecture using the generalized continuum hypothesis, and finally Hajnal [5] proved the result without this hypothesis.

It is very easy to see that the result is no longer true if the hypothesis $|f(x)| < n < m$ is weakened to simply $|f(x)| < m$ $(x \in S)$. For, let $x_0 < x_1 < \dots < x_\nu < \dots$ $(\nu < \lambda)$ be a well ordering of S, where λ is the initial ordinal of cardinal m. If we put $f(x_\mu) = \{x_\nu : \nu < \mu\}$ $(\mu < \lambda)$, then f is set mapping on S such that $|f(x)| < |S|$ $(x \in S)$ and there is no free subset of S containing more than one element. It will be noted that in this counter example, the order types of the image sets $f(x)$ are not bounded below λ. This suggests the following strengthening of Ruziewicz's conjecture proved by Erdős and Specker [6]. If λ is an initial ordinal number and f is any set mapping of order $\alpha (< \lambda)$ on a set S of type λ, then there is a free set of the full type λ. The set mapping f has order α if the order type of $f(x)$ is less than α for all $x \in S$.

In this paper we shall consider set mappings on a well ordered set S in the case when the order type of S is not necessarily an initial ordinal. In particular, we examine the truth status of the following statement $SM(\alpha,\lambda)$. If f is any set mapping of order α on a set of type λ, then there is a free subset having the same order type λ. The Erdős-Specker generalization of the Ruziewicz conjecture asserts that $SM(\alpha,\lambda)$ holds if λ is an infinite initial ordinal and $\alpha < \lambda$. We only examine the problem for the case when $|\lambda| = \aleph_1$ although some of our results hold more generally.* We will prove (Theorems 4, 5 & 6) that $SM(\alpha,\lambda)$ holds in the following cases: (i) $\alpha < \omega_1$ and $\lambda = \omega_1^{\sigma_1+1} + \dots + \omega_1^{\sigma_k+1} < \omega_1^{\omega+2}$ (k finite); (ii) $\alpha = \omega_0$ and $\lambda = \omega_1\gamma < \omega_1^{\omega+2}$; (iii) $\alpha < \omega_0$; $\lambda = \omega\Theta$, where Θ is arbitrary. Note that the form given for λ in (i) is the most general for which $SM(\alpha,\lambda)$ is true with any $\alpha < \omega_1$. For example, $SM(\alpha,\omega_1^{\omega})$ is false if $\omega < \alpha < \omega_1$. The condition $\lambda < \omega_1^{\omega+2}$ is (i) and (ii) is also essential for we show (see Theorem 3) that $SM(\omega,\lambda)$ is false if $\omega_1^{\omega+2} \le \lambda < \omega_2$.

There is a connection between set mappings and polarized partition relations. The symbol

*This is the first of a sequence of forthcoming papers by the three of us. In these we shall consider similar problems for types with higher cardinals. Many new phenomena and new difficulties appear already for types of power $\aleph_2$ which is why they will be treated separately.

$$(1.1) \qquad \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \to \begin{pmatrix}\alpha_0 & \alpha_1\\ \beta_0 & \beta_1\end{pmatrix}$$

first used in [7] means, by definition, that the following is true: If A, B are ordered sets with types α, β respectively, and if $A \times B = K_0 \cup K_1$, then there are $i < 2$ and sets $A_i \subset A$ and $B_i \subset B$ such that $\operatorname{tp} A_i = \alpha_i$, $\operatorname{tp} B_i = \beta_i$ and $A_i \times B_i \subset K_i$. The negation of (1.1) is expressed by replacing $\to$ by $\not\to$ in the symbol.

We prove (Lemma 2) that $SM(\alpha, \gamma)$ implies the relation

$$\begin{pmatrix}\gamma\\ \gamma\end{pmatrix} \to \begin{pmatrix}\alpha & \gamma\\ 1 & \gamma\end{pmatrix}$$

but we do not know if the converse implication is also true. Using the continuum hypothesis $2^{\aleph_0} = \aleph_1$, we will show (Theorem 2), if $\omega_1 \le \gamma$ and $\gamma = \sum(n < \omega)\gamma_n$, where $2 \le |\gamma_n| \le \aleph_1$, then

$$\begin{pmatrix}\gamma\\ \omega_1\end{pmatrix} \not\to \begin{pmatrix}\omega+1 & \gamma\\ 1 & \omega_1\end{pmatrix}$$

This easily implies that

$$\begin{pmatrix}\gamma\\ \gamma\end{pmatrix} \not\to \begin{pmatrix}\omega+1 & \gamma\\ 1 & \gamma\end{pmatrix}$$

and hence that $SM(\omega+1, \gamma)$ is false. This confirms our remark about (i) above. We will also prove (Theorem 3) that,

$$(1.2) \qquad \begin{pmatrix}\omega_1\\ \beta\end{pmatrix} \not\to \begin{pmatrix}\omega & 1\\ 1 & \omega_1^{\omega+2}\end{pmatrix} \quad (\beta < \omega_2).$$

From this, it follows that

$$\begin{pmatrix}\beta\\ \beta\end{pmatrix} \not\to \begin{pmatrix}\omega & \beta\\ 1 & \beta\end{pmatrix} \quad (\omega_1^{\omega+2} \le \beta < \omega_2),$$

and hence that $SM(\omega, \beta)$ is false. The relation (1.2) is a little surprising for it is equivalent to the following seemingly paradoxical statement: If S is an ordered set of type β $(\beta < \omega_2)$, then there are $\aleph_1$ subsets of type less than $\omega_1^{\omega+2}$ (i.e. their order type is small compared with the order type of S) such that the union of any $\aleph_0$ of these subsets is the whole set S. This is closely related to the negative partition relation $\beta \nrightarrow (1, \omega_1, \omega_1^2, \dots)^1_\omega$ proved by Milner and Radó [8].

In contrast to (1.2) we prove (Theorem 1) that

$$\begin{pmatrix} \gamma \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \gamma \\ 1 & \beta \end{pmatrix}$$

holds if $\alpha < \omega_1$, $\beta < \omega_1^{\omega+2}$ and γ is a finite sum of order types $\gamma_0 + \dots + \gamma_k$ which are expressible as an ω_1-sum of increasing ordinals, i.e. $\gamma_i = \sum(\nu < \omega_1)\delta_{i\nu}$ with $\delta_{i0} \le \delta_{i1} \le \dots$ $(i \le k)$.

As an application of the set mapping theorems we shall prove the following result about transfinite graphs (Theorem 7). If S is an ordered set of order type $\omega\Theta < \omega_1^{\omega+2}$, and if G is any graph on S, thein either there is an infinite path in G or there is an independent set (i.e. a set containing no edges of G) of the same type $\omega\Theta$. To prove this we make use of (ii) above. We know by (1.2) that (ii) is false for order types greater than or equal to $\omega_1^{\omega+2}$, but it is possible that Theorem 7 is true for arbitrary Θ.

2. ADDITIONAL NOTATION

Greek letters denote ordinal numbers and capital letters denote sets. The obliterator sign ^ written above a symbol means that that symbol is to be disregarded, e.g. $\{x_0, \dots, \hat{x}_\lambda\} = \{x_\mu : \mu < \lambda\}$. We write $S = \{x_0, \dots, \hat{x}_\lambda\}_<$ to indicate that the elements of S are ordered so that $x_0 < x_1 < \dots < \hat{x}_\lambda$; similarly, $\{x_0, \dots, \hat{x}_\lambda\}_{\neq}$ means that $x_\mu \neq x_\nu$ $(\mu < \nu < \lambda)$. If S is an ordered set, then $\operatorname{tp} S$ denotes the order type of S. If X, Y are subsets of S, then $X < Y$ means that $x < y$ holds for all $x \in X$ and $y \in Y$. We write

$$S = S_0 \cup S_1 \cup \dots \cup \hat{S}_\lambda \; (<)$$

if S is the disjoint union of the sets $S_\mu \; (\mu < \lambda)$ and $S_0 < S_1 < \dots < \hat{S}_\lambda$.

If $S = A \cup B \; (<)$, then A, B are respecitvely called initial and final sections of S; they are proper sections if non-empty. If $x \in S$, the section $\{ y \in S : x < y \}$ is denoted by $R(x)$. More generally, if $X \subset S$, then $R(X) = \cap (x \in X) R(x)$. An interval of S is a set I such that $S = A \cup I \cup B \; (<)$. The interval of ordinal numbers $\{ \nu : \alpha \leq \nu < \beta \}$ is denoted by $[\alpha, \beta)$.

A subset X of the ordered set S is cofinal with S if $X < \{a\}$ is false for all $a \in S$. If X is not cofinal with S we write $X \overset{N}{\subset} S$. If $tp\, S = \alpha$, then $co(\alpha)$ denotes the least ordinal β such that $tp\, B = \beta$ for some cofinal subset B of S. Thus, if $\alpha > 0$, then $co(\alpha)$ is either 1 or an infinite initial ordinal. The ordinal α is indecomposable if the equation $\alpha = \beta + \gamma$ implies that either $\beta = \alpha$ (and $\gamma = 0$) or $\gamma = \alpha$. It is well known that the indecomposable ordinals are 0 and the powers of ω and that every ordinal $\alpha > 0$ has a unique representation as a sum (the Cantor standard form)

$$\alpha = \alpha_0 + \dots + \alpha_n ,$$

with $n < \omega, \alpha_i$ indecomposable $(\underline{i} \leq \underline{n})$ and $\alpha_0 \geq \alpha_1 \geq \dots \geq \alpha_n > 0$.

The cardinal of S is $|S|$. If m is any cardinal, we write $[S]^m = \{ X \subset S : |\underline{X}| = m \}$ and $[S]^{\leq m} = \{ X \subset S : |X| \leq m \}$. A graph is an ordered pair of sets $G = (S, E)$ with $E \subset [S]^2$. The elements of E are called the edges of the graph. $X \subset S$ is called in independent set if $[X]^2 \cap E \neq \phi$. An infinite path in G is a set $\{ x_0, \dots, \hat{x}_\omega \}_{\neq} \subset S$ such that $\{ x_n, x_{n+1} \} \in E \; (n < \omega)$.

The ordinary partition symbol

$$\alpha \to (\alpha_\nu)^r \tag{2.1}$$

means that the following is true: If $tp\, S = \alpha$ and $[S]^r = K_0 \cup \dots \cup \hat{K}_\lambda$, then there are $\mu < \lambda$ and $A \subset S$ such that $tp\, A = \alpha_\mu$ and $[A]^r \subset K_\mu$. If $\alpha_\nu = \beta \; (\beta < \lambda)$, we write (2.1) in the alternative form $\alpha \to (\beta)^r_\lambda$. In this paper we only require some special relations of the form (2.1) when $r = 1$. If $n < \omega$ and α is indecomposable, then

$$(2.2) \qquad \alpha \to (\alpha)^1_n ;$$

if $n < \omega$ and $\lambda < \omega_{\beta+1}$, then

$$(2.3) \qquad \omega^n_{\beta+1} \to (\omega^n_{\beta+1})^1_\lambda .$$

Also, we need the negative relation

$$(2.4) \qquad \alpha \not\to (\omega^n_\beta)^1_{n<\omega} \quad \text{if} \quad \alpha < \omega_{\beta+1} .$$

The above results are all easy to prove, but details can be found in [7].

The cartesian product of two sets A, B is denoted by $A \times B$. If $A \times B = K_0 \cup K_1$ in any partition, then we write

$$F_i(a) = \{ b \in B : (a,b) \in K_i \} \quad (a \in A; i < 2)$$

$F_i(b)$ is similarly defined for $b \in B$ and $i < 2$. If $D \subset A$ or $D \subset B$ we define $F_i(D) = \bigcup_{x \in D} F_i(x)$.

3. CONSTRUCTION OF SETS WITH PRESCRIBED ORDER TYPE.

We now describe a systematic procedure, which we follow in later parts of the paper, for constructing a subset of a well ordered set so that this subset has prescribed order type $\omega_\alpha \Theta$, where Θ is a fixed ordinal less than $\omega_{\alpha+1}$. In the applications we are concerned only with the special case $\alpha = 1$, but it seems worthwhile formulating the procedure in more general terms. For a set S of type $\omega_\alpha \Theta$ we shall describe a standard sequence $I(S) = (I_0, I_1, \ldots, \hat{I}_{\omega_\alpha})$ of intervals of S whose essential features are that (i) every one-element subset of S appears as a term and (ii) if two intervals I_μ, I_ν $(\mu < \nu < \omega_\alpha)$ of the sequence overlap, then $I_\nu \subset I_\mu$. In the applications the set Z of type $\omega_\alpha \Theta$ which we want to construct will have certain special properties peculiar to the particular problem. What we do is to construct by transfinite induction a sequence $(Z_0, Z_1, \ldots, \hat{Z}_{\omega_\alpha})$ of subsets of the given set so that the terms have certain properties relevant to the problem and at the same time imitate precisely the order structure of the standard sequence $I(S)$. That is to say, the sets Z_ν are constructed so that $\mathrm{tp}\, Z_\nu = \mathrm{tp}\, I_\nu$ $(\nu < \omega_\alpha)$ and

$$Z_\nu \lhd Z_\mu \Longleftrightarrow I_\nu \lhd I_\mu$$

hold for $\mu < \nu < \omega_\alpha$, where $\lhd$ denotes any of the binary relations $<, >, \subset$ or $\overset{N}{\subset}$. This will ensure that, in addition to certain other properties, the set $Z = \bigcup(|Z_\nu| = 1)\, Z_\nu$ will have the required order type $\omega_\alpha \theta$.

We first make the trivial observation that, if $1 < \Theta < \omega_{\alpha+1}$, then there is $\chi = \chi(\Theta) \leq \omega_\alpha$ such that Θ has a representation as a sum of powers of ω_α,

$$\Theta = \sum(\xi < \chi)\, \omega_\alpha^{\rho_\xi}, \tag{3.1}$$

in which the terms are all strictly less than Θ. We assume that $\chi(\Theta)$ is the minimal value of χ for which there is such a representation (3.1) for Θ. Note that $\chi(\omega_\alpha^\rho) = \mathrm{co}(\omega_\alpha^\rho)$ if $1 < \rho < \omega_{\alpha+1}$. In general, however, $\chi(\Theta)$ differs from $\mathrm{co}(\Theta)$, e.g. $\chi(\omega_\alpha 2) = 2$.

Let $\mathrm{tp}\, S = \omega_\alpha \Theta$, where $1 \leq \Theta < \omega_{\alpha+1}$. We assert that there is a sequence of intervals of S, $I(S) = (I_0, \dots, \hat{I}_{\omega_\alpha})$, and a regressive function $\phi = \phi_S : [1, \omega_\alpha) \to [0, \omega_\alpha)$ such that the conditions (3.2) - (3.8) are satisfied:

$$I_0 = S; \tag{3.2}$$

$$\text{if } x \in S \text{ then } \{x\} = I_\nu \text{ for some } \nu < \omega_\alpha; \tag{3.3}$$

$$\mathrm{tp}\, I_\nu = \omega_\alpha^{\sigma_\nu} \quad (1 \leq \nu < \omega_\alpha); \tag{3.4}$$

$$I_\nu \subset I_{\phi(\nu)} \text{ and } \mathrm{tp}\, I_\nu < \mathrm{tp}\, I_{\phi(\nu)} \quad (1 \leq \nu < \omega_\alpha); \tag{3.5}$$

$$I_\mu \cap I_\nu = \emptyset \quad (\phi(\nu) < \mu < \nu < \omega_\alpha); \tag{3.6}$$

$$I_\mu < I_\nu \quad \text{if } \mu < \nu \text{ and } \phi(\mu) = \phi(\nu); \tag{3.7}$$

(3.8) $$\operatorname{tp}\{\mu:\ \phi(\nu)<\mu<\nu;\ \phi(\mu)=\phi(\nu)\}<\chi(\operatorname{tp} I_{\phi(\nu)}) \quad (1\le\nu<\omega_\alpha).$$

For $\Theta=1$ this is obvious. We simply put $I_0=S=\{x_0, x_1, \dots, \hat{x}_{\omega_\alpha}\}$, $I_{1+\nu}=\{x_\nu\}$ $(\nu<\omega_\alpha)$ and $\phi(\nu)=0$ $(1\le\nu<\omega_\alpha)$. In this case $\chi(\operatorname{tp} I_0)=\omega_\alpha$ and (3.2) - (3.8) all hold. We now suppose that $\Theta>1$ and use induction.

By (3.1), if $\operatorname{tp} S=\omega_\alpha\Theta$, then

$$S=S_0\cup S_1\cup\dots\cup\hat{S}_\chi\ (<),$$

where $\chi=\chi(\Theta)$ and $\operatorname{tp} S_\xi=\omega_\alpha^{1+\rho_\xi}<\omega_\alpha\Theta$ $(\xi<\chi)$. By the induction hypothesis there are sequences $I(S_\xi)=(J_{\xi 0}, \dots, \hat{J}_{\xi\omega_\alpha})$ and regressive functions $\phi_\xi=\phi_{S_\xi}$ $(\xi<\chi)$ such that the stated conditions are satisfied. Let

$$f:\ \{(\xi,\nu):\ \xi<\chi,\ \nu<\omega_\alpha\}\to[1,\omega_\alpha)$$

be any bijection which satisfies

(3.9) $$f(\xi,0)<f(\xi',0) \qquad (\xi<\xi'<\chi),$$

(3.10) $$f(\xi,\nu)<f(\xi,\nu') \qquad (\xi<\chi;\ \nu<\nu'<\omega_\alpha).$$

(For example, such a mapping is defined by putting $f(\xi,\nu)=1+(\xi\dotplus\nu)$, where $\xi\dotplus\nu$ denotes the natural sum.) Now put $I_0=S$, $I_\nu=J_{f^{-1}(\nu)}$ $(0<\nu<\omega_\alpha)$. Also, if $0<\mu<\omega_\alpha$ and $f^{-1}(\mu)=(\xi,\nu)$, then we define $\phi(\mu)=0$ if $\nu=0$ and $\phi(\mu)=f(\xi,\phi_\xi(\nu))$ if $\nu>0$. This defines the sequence $I(S)=(I_0, \dots, \hat{I}_{\omega_\alpha})$ and the regressive function $\phi_S=\phi$.

From (3.10) we see that $I(S)$ contains each $I(S_\xi)$ $(\xi<\chi)$ as a proper subsequence. Therefore, since the sets S_ξ are mutually disjoint, the conditions (3.2) - (3.6) follow from the corresponding statements for $I(S_\xi)$ $(\xi<\chi)$. Similarly, (3.7) and (3.8) hold when $\phi(\nu)>0$. Since (3.9) holds and $S_0 < S_1 < \dots < \hat{S}_\chi$, it follows that (3.7) and (3.8) also hold when $\phi(\nu) = 0$. This proves our assertion.

Now suppose that $I(S)$ and $\phi = \phi_S$ have been defined in the manner just described. If $\nu < \omega_\alpha$, put $\phi_0(\nu) = \nu$. Also, if $k < \omega$ and $\phi_k(\nu)$ has been defined and is positive, put $\phi_{k+1}(\nu) = \phi(\phi_k(\nu))$; if $\phi_k(\nu) = 0$, then $\phi_{k+1}(\nu)$ is not defined. Then, since ϕ is regressive, for $\nu < \omega_\alpha$ there is a non-negative integer $\iota = \iota(\nu)$ such that $\nu = \phi_0(\nu) > \phi_1(\nu) > \dots > \phi_\iota(\nu) = 0$. By (3.5) it follows that:

$$I_\nu \subset I_{\phi(\nu)} \subset \dots \subset I_{\phi_\iota(\nu)} = I_0 .$$

Now suppose that $i < \iota$ and $\phi_{i-1}(\nu) > \mu > \phi_i(\nu)$. Then, by (3.6) $I_\mu \cap I_{\phi_{i-1}(\nu)} = \phi$ and hence $I_\mu \cap I_\nu = \phi$. Therefore, if $\mu < \nu < \omega_\alpha$, then either $I_\nu < I_\mu$ or $I_\mu < I_\nu$ or $I_\nu \subset I_\mu$. In fact, apart from one possible exception, if $I_\nu \subset I_\mu$ holds we can make the stronger assertion that

(3.11) $$I_\nu \overset{N}{\subset} I_\mu .$$

This follows from the fact that, by (3.5), I_ν is a sub-interval of I_μ of smaller type and so it cannot be cofinal with I_μ unless $tp(I_\mu)$ is decomposable. By (3.2) and (3.4) the only possible exception to the above remark is when $\chi(\omega_\alpha\Theta) = \delta + 1$, in which case $I_{\delta+1}$ is cofinal with I_0 .

We make one additional minor remark. From the inductive manner in which we defined $I(S)$ and $\phi = \phi_S$, it is apparent that, if $tp\, S = \omega_\alpha\Theta > \omega_\alpha$, then

(3.12) $$|I_\nu| = \aleph_\alpha \quad \text{if } \phi(\nu) = 0 \text{ and } 1 \le \nu < \omega_\alpha .$$

In the special case when $tp\, S = \omega_\alpha$, (3.12) does not hold.

4. POLARIZED PARTITION RELATIONS

In this section we establish some positive and negative polarized partition relations. The positive result (Theorem 1) will be used to establish the set mapping theorems in the next section. The negative relations (Theorems 2 & 3) show that Theorem 1 is best possible in certain senses and they also show that the set mapping results (Theorems 4 & 5) cannot be improved.

We say that γ is an increasing ω_1-sum if $\gamma = \gamma_0 + \gamma_1 + \cdots + \hat{\gamma}_{\omega_1}$ and $\gamma_0 \leq \gamma_1 \leq \cdots \leq \hat{\gamma}_{\omega_1}$.

THEOREM 1. If $\alpha < \omega_1$; $\beta < \omega_1^{\omega+2}$; $\gamma = \gamma_0 + \cdots + \gamma_k$, where $k < \omega$ and each γ_i is an incereasing ω_1-sum, then

$$\begin{pmatrix}\gamma\\ \beta\end{pmatrix} \rightarrow \begin{pmatrix}\alpha & \gamma\\ 1 & \beta\end{pmatrix}. \tag{4.1}$$

Theorems 2 and 3 show that the conditions placed upon β and γ in Theorem 1 cannot be relaxed. We use the continuum hypothesis to establish Theorem 2, but this is not needed in Theorem 3.

THEOREM 2. If $2^{\aleph_0} = \aleph_1$ and $\gamma = \gamma_0 + \cdots + \hat{\gamma}_{\omega}$, where $2 \leq |\gamma_n| \leq \aleph_1$ $(n < \omega)$, then

$$\begin{pmatrix}\gamma\\ \omega_1\end{pmatrix} \not\rightarrow \begin{pmatrix}\omega+1 & \gamma\\ 1 & \omega_1\end{pmatrix}. \tag{4.2}$$

THEOREM 3. If $\beta < \omega_2$, then

$$\begin{pmatrix}\omega_1\\ \beta\end{pmatrix} \not\rightarrow \begin{pmatrix}\omega & 1\\ 1 & \omega_1^{\omega+2}\end{pmatrix}. \tag{4.3}$$

As we remarked in the introduction, Theorem 3 is equivalent to the following statement: If $\operatorname{tp} S = \beta < \omega_2$, then there are $\aleph_1$ sets

$F_\mu \subset S$ $(\mu < \omega_1)$ such that $\operatorname{tp} F_\mu < \omega_1^{\omega+2}$ and such that the union of any $\aleph_0$ of these $\aleph_1$ sets is the whole set S.

PROOF OF THEOREM 1. We first prove that

$$\begin{pmatrix}\omega_1\\ \omega_1^\lambda\end{pmatrix} \to \begin{pmatrix}\alpha & 1\\ 1 & \omega_1^\lambda\end{pmatrix} \qquad (\alpha < \omega_1;\ \lambda \leq \omega+1) \tag{4.4}$$

which is weaker than (4.1).

Let A, B be ordered sets with types ω_1 and ω_1^λ respectively. Let $A \times B = K_0 \cup K_1$. In order to prove (4.4) we shall assume that $\operatorname{tp} F_1(a) < \omega_1^\lambda$ for all $a \in A$ and deduce that $\operatorname{tp} F_0(b) \geq \alpha$ for some $b \in B$.

Case 1. $\lambda < \omega$. Let N be any subset of A of type α. Since $\operatorname{tp} F_1(a) < \omega_1^\lambda$ for $a \in N$, it follows from the partition relation (2.3) that

$$\operatorname{tp}(\bigcup(a \in N)\, F_1(a)) < \omega_1^\lambda .$$

Therefore, there is $b \in B - \bigcup(a \in N)\, F_1(a)$ and $\operatorname{tp} F_0(b) \geq \alpha$ since $N \subset F_0(b)$.

Case 2. $\lambda = \omega$. In this case, there are $A_1 \in [A]^{\aleph_1}$ and $n < \omega$ such that $\operatorname{tp} F_1(a) < \omega_1^n$ $(a \in A_1)$ and the result follows from Case 1.

Case 3. $\lambda = \omega + 1$. We may write $B = B_0 \cup B_1 \cup \dots \cup \hat{B}_{\omega_1}$ $(<)$, where $\operatorname{tp} B_\nu = \omega_1^\omega$ $(\nu < \omega_1)$. Since $\operatorname{tp} F_1(\alpha) < \omega_1^{\omega+1}$ for $a \in A$, it follows that there are $\mu(a) < \omega_1$ and $n(a) < \omega$ such that

$$\operatorname{tp}(F_1(a) \cap B_\nu) < \omega_1^{n(a)} \quad (\mu(a) \leq \nu < \omega_1)$$

There is $A_1 \in [A]^{\aleph_1}$ such that $n(a) = n$ for all $a \in A_1$. Let N be any subset of A_1 of type α. There is $\mu < \omega_1$ such that $\mu(a) < \mu$ for $a \in N$ and, as in Case 1, there is

$$b \in B_\mu - \bigcup(a \in N)\, F_1(a) .$$

This implies that $N \subset F_0(b)$ and hence that $tp\, F_0(b) \geq \alpha$. This proves (4.4).

We now extend (4.4) slightly and prove

$$\begin{pmatrix} \omega_1 \\ \omega^\sigma \end{pmatrix} \to \begin{pmatrix} \alpha & 1 \\ 1 & \omega^\sigma \end{pmatrix} \qquad (\alpha < \omega_1;\ \omega^\sigma < \omega_1^{\omega+2}). \tag{4.5}$$

We may write $\sigma = \omega_1 \lambda + \rho$, where $\lambda \leq \omega+1$ and $\rho < \omega_1$. If $\rho = 0$, then (4.5) is the same as (4.4). Now assume $\rho > 0$ and use induction on ρ.

There are $\rho_n < \rho$ $(n < \omega)$ such that $\omega^\rho = \sum(n<\omega)\, \omega^{\rho_n}$. Then if $tp\, B = \omega^\sigma$, $B = B_0 \cup \dots \cup \hat{B}_\omega$, where $tp\, B_n = \omega_1^\lambda \omega^{\rho_n}$ $(n < \omega)$. Let $tp\, A = \omega_1$ and let $A \times B = K_0 \cup K_1$. Suppose that $tp\, F_1(a) < \omega^\sigma$ $(a \in A)$.

Then for each $a \in A$ there is $n(a) < \omega$ such that $tp\,(F_1(a) \cap B_{n(a)}) < tp\, B_{n(a)}$. There are $A_1 \in [A]^{\aleph_1}$ and $n < \omega$ such that $n(a) = n$ $(a \in A_1)$.

Applying the induction hypothesis to the partition induced on $A_1 \times B_n$, it follows that there is $b \in B_n$ such that $tp\, F_0(b) \geq \alpha$ and (4.5) follows.

The main step in our proof of Theorem 1 will be to strengthen (4.5) to

$$\begin{pmatrix} \omega_1 \\ \omega^\sigma \end{pmatrix} \to \begin{pmatrix} \alpha & \omega_1 \\ 1 & \omega^\sigma \end{pmatrix} \qquad (\alpha < \omega_1;\ \omega^\sigma < \omega_1^{\omega+2}). \tag{4.6}$$

Let $tp\, A = \omega_1$, $tp\, B = \omega^\sigma$ and consider any partition $A \times B = K_0 \cup K_1$. We will assume that $tp\, F_0(b) < \alpha$ for all $b \in B$ and deduce that there are $A_1 \subset A$ and $B_1 \subset B$ such that $tp\, A_1 = \omega_1$, $tp\, B_1 = \omega^\sigma$ and $A_1 \times B_1 \subset K_1$.

If $\omega^\sigma < \omega_1$ the result is obvious. Simply put $B_1 = B$ and

$A_1 = A - \bigcup(b \in B) F_0(b)$. Therefore, we may assume that $\omega^\sigma = \omega_1 \Theta$, where Θ is indecomposable and less than $\omega_1^{\omega+2}$.

Let C be any subset of B whose order type is a power of ω, say $C = \omega^\mu$. Then there is a countable set $D(C) \subset A$ such that

$$(4.7) \qquad \mathrm{tp}\,(C - F_0(D)) = \mathrm{tp}\,C \quad \text{for} \quad D \in [A - D(C)]^{\leq \aleph_0} .$$

For, if there were no such set $D(C)$, then there would be an uncountable sequence of countable sets $D_0 < D_1 < \ldots < \hat{D}_{\omega_1}$ such that the order type of each of the sets

$$E_\nu = C - F_0(D_\nu) \quad (\nu < \omega_1)$$

is strictly less than $\mathrm{tp}\,C$. Applying (4.5) (with $\sigma = \mu$) , it follows that there are $b \in C$ and $N \subset [0, \omega_1)$ such that $\mathrm{tp}\,N = \alpha$ and $b \notin E_\nu$ $(\nu \in N)$. This implies that $F_0(b) \cap D_\nu \neq \emptyset$ $(\nu \in N)$ and hence that $F_0(b) \geq \mathrm{tp}\,N = \alpha$. This contradiction proves the existence of a countable set $D(C) \subset A$ such that (4.7) holds.

Now let $I(B) = (I_0, \ldots, \hat{I}_{\omega_1})$ and $\phi = \phi_B$ be as defined in §3. We are going to define sets $Z_\mu \subset B$ and elements $a_\mu \in A$ $(\mu \in \omega_1)$ such that the conditions (4.8) - (4.11) hold:

$$(4.8) \qquad \mathrm{tp}\,Z_\mu = \mathrm{tp}\,I_\mu ;$$

$$(4.9) \qquad Z_\mu \triangleleft Z_\varrho \iff I_\mu \triangleleft I_\varrho \quad (\varrho < \mu),$$

where $\triangleleft$ denotes $<$, $>$ or $\overset{N}{\subset}$;

$$(4.10) \qquad Z_\mu \cap F_0(a_\varrho) = \emptyset \quad (\varrho < \mu);$$

$$(4.11) \qquad a_\mu \in A - D(Z_\mu) \cup \bigcup(\varrho < \mu)(\{a_\varrho\} \cup D(Z_\varrho)) .$$

Put $Z_0 = B$ and choose $a_0 \in A - D(Z_0)$. Now let $0 < \nu < \omega_1$ and suppose that Z_μ, a_μ have already been chosen for $\mu < \nu$ so that (4.8) - (4.11) hold. We want to define Z_ν, a_ν so that these relations remain valid with

Since $\phi(\nu) < \nu$, it follows from (4.10) that

$$Z_{\phi(\nu)} \cap F_0(\{a_0, \ldots, \hat{a}_{\phi(\nu)}\}) = \emptyset .$$

Also, by (4.11), $\{a_{\phi(\nu)}, \ldots, \hat{a}_\nu\}$ is a countable subset of $A - D(Z_{\phi(\nu)})$ and therefore

$$\text{(4.12)} \qquad tp(Z_{\phi(\nu)} - F_0(\{a_0, \ldots, \hat{a}_\nu\})) = tp\, Z_{\phi(\nu)} .$$

If $Q = \{\mu : \phi(\nu) = \phi(\mu) < \mu < \nu\}$, then

$$\bigcup(\mu \in Q)\, I_\mu < I_\nu$$

by (3.7). By (3.5) and (3.11) it follows that $I_\mu \overset{N}{\subset} I_{\phi(\nu)}$ $(\mu \in Q)$ and therefore, by (4.9),

$$Z_\mu \overset{N}{\subset} Z_{\phi(\nu)} \quad (\mu \in Q).$$

By (3.8), $tp\, Q < \chi(tp\, I_{\phi(\nu)}) = co(tp\, Z_{\phi(\nu)})$, and hence

$$\text{(4.13)} \qquad Z' = \bigcup(\mu \in Q)\, Z_\nu \overset{N}{\subset} Z_{\phi(\nu)} .$$

From (4.12) and (4.13) we see that there is Z_ν such that $Z' < Z_\nu \overset{N}{\subset} Z_{\phi(\nu)} - F_0(\{a_n, \ldots, \hat{a}_\nu\})$ and $tp\, Z_\nu = tp\, I_\nu (< tp\, I_{\phi(\nu)} = tp\, Z_{\phi(\nu)})$.

It is obvious that (4.8) and (4.10) hold for $\mu = \nu$ with this definition of Z_ν. We now verify that (4.9) also holds.

Let $\rho < \nu$. Case 1. $I_\rho \not\subset I_{\phi(\nu)}$. Then $I_{\phi(\nu)} \lhd I_\rho$, where $\lhd$ denotes $<$, $>$ or $\overset{N}{\subset}$. Therefore, with the same meaning for $\lhd$, we have both $I_\nu \lhd I_\rho$ and $Z_\nu \lhd Z_\rho$ since I_ν and Z_ν are respectively subsets of $I_{\phi(\nu)}$ and $Z_{\phi(\nu)}$. Case 2. $I_\rho \subset I_{\phi(\nu)}$. Then either (i) $\rho = \phi(\nu)$ or (ii) $I_\rho \subset I_\sigma$ for some $\sigma \in Q$ (see the remark preceeding (3.11)). If (i) holds, then $I_\nu \overset{N}{\subset} I_\varrho$ and, by the construction, $Z_\nu \overset{N}{\subset} Z_{\phi(\nu)} = Z_\varrho$. If (ii) holds, then $I_\varrho \subset I_\sigma < I_\nu$ by (3.7), and by (4.9) with $\mu = \sigma$ and the definition of Z_ν, we also have $Z_\varrho \subset Z_\sigma < Z_\nu$. This shows that (4.9) holds with $\mu = \nu$.

Finally, we choose $a_\nu \in A - D(Z_\nu) \cup U(\varrho < \nu)(\{a_\varrho\} \cup D(Z_\varrho))$ so that (4.11) also holds with $\mu = \nu$.

Since (4.8) and (4.9) hold, it follows that the set $B_1 = U(|Z_\nu| = 1)\, Z_\nu$ has the same order type as $B = U(|I_\nu| = 1)\, I_\nu$, i.e. $tp\, B_1 = \omega^\sigma$. Also, by (4.11), $A_1 = \{a_0, \dots, \hat{a}_{\omega_1}\}_{\neq}$ has type ω_1.

The proof of (4.6) will be complete if we show that $A_1 \times B_1 \subset K_1$. Let $\mu, \nu < \omega_1$, and suppose that $|Z_\nu| = 1$. If $\mu < \nu$, then (4.10) shows that $Z_\nu \cap F_0(a_\mu) = \emptyset$. If $\nu \leq \mu$, then by (4.11), $a_\mu \in A - D(Z_\nu)$ and so

$$tp(Z_\nu - F_0(a_\mu)) = tp\, Z_\nu = 1 ,$$

i.e. $Z_\nu \cap F_0(a_\mu) = \emptyset$. This implies that $A_1 \times B_1 \subset K_1$.

The generalization from (4.6) to (4.1) is straightforward. First we extend (4.6) to

$$(4.14) \qquad \begin{pmatrix} \omega_1 \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \omega_1 \\ 1 & \beta \end{pmatrix} \qquad (\alpha < \omega_1 ;\ \beta < \omega_1^{\omega+2}) .$$

To see this, let $tp\, A = \omega_1$, $tp\, B = \beta$, $A \times B = K_0 \cup K_1$ and assume that $tp\, F_0(b) < \alpha$ for $b \in B$. We may write $B = B_0 \cup \dots \cup B_{n-1}(<)$, where $n < \omega$, $tp\, B_i = \beta_i$ and

$$\beta = \beta_0 + \beta_1 + \dots + \beta_{n-1}$$

in the standard decomposition of β as a finite sum of non-increasing indecomposable ordinals. Applying (4.6) we find successively sets $A_i, B_i' \ (i < n)$ such that $A_0 \supset A_1 \supset \dots \supset A_{n-1}$, $B_i' \subset B_i$. $tp\, A_i = \omega_1$,

$tp\, B_i' = \beta_i$ and $A_i \times B_i \subset K_1$. The set $B' = B_0' \cup \ldots \cup B_{n-1}'$ has type β and $A_{n-1} \times B' \subset K_1$ This proves (4.14).

We now show that if $\alpha < \omega_1$; $\beta < \omega_1^{\omega+2}$ and γ is an increasing ω_1-sum, then

$$(4.15) \qquad \begin{pmatrix} \gamma \\ \beta \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \gamma \\ 1 & \beta \end{pmatrix} .$$

Let $tp\, A = \gamma$, $tp\, B = \beta$, $A \times B = K_0 \cup K_1$ and suppose that $tp\, F_0(b) < \alpha \quad (b \in B)$. We may write $A = A_0 \cup \ldots \cup \hat{A}_{\omega_1} (<)$, where $tp\, A_\nu = \gamma_\nu$ and $0 < \gamma_0 \leq \gamma_1 \leq \ldots \leq \hat{\gamma}_{\omega_1} < \gamma$. Comsider the partition

$$[0, \omega_1) \times B = K_0' \cup K_1' .$$

where $(\nu, b) \in K_0'$ if and only if $F_0(b) \cap A_\nu \neq \phi$. Then, for $b \in B$,

$$F_0'(b) = \{\nu < \omega_1 : (\nu, b) \in K_0'\}$$

has type less than or equal to $tp\, F_0(b)$, i.e. $tp\, F_0'(b) < \alpha$. It follows from (4.14) that there are $N \subset [0, \omega_1)$ and $B' \subset B$ such that $tp\, N = \omega_1$, $tp\, B' = \beta$ and $N \times B' \subset K_1'$. This implies that $A' \times B' \subset K_1$, where $A' = \bigcup (\nu \in N) A_\nu$. This proves (4.15) since

Finally, (4.1) follows by a finite number of applications of (4.15). This completes the proof.

PROOF OF THEOREM 2. Let $tp\, A = \gamma$, $B = [0, \omega_1)$. By the hypothesis, we may write $A = A_0 \cup \cdots \cup \hat{A}_\omega \; (<)$, where $tp\, A_n = \gamma_n \; (n < \omega)$. Since $2 \leq |\gamma_n| \leq \aleph_1$, it follows from the continuum hypothesis that there are $2^{\aleph_0} = \aleph_1$ sets $C \subset A$ such that $tp\, C = \omega$ and $|C \cap A_n| \leq 1 \; (n < \omega)$. Let $C_0, C_1, \ldots, \hat{C}_{\omega_1}$ be a well ordering of all these sets C.

Let $\xi \in B$ and let $f = f_\xi$ be any mapping (not necessarily 1-1) from $[0, \omega)$ onto $[0, \xi+1)$. Since each set $C_{f(\varrho)} \; (\varrho < \omega)$ has a non-empty intersection with infinitely many of the A_n $(n<\omega)$, there is an increasing sequence of integers $n_0 < n_1 < \dots$ such that $C_{f(\varrho)} \cap A_{n_\varrho} \neq \emptyset$ $(\varrho < \omega)$. Put

$$F_0(\xi) = \bigcup(\varrho<\omega) C_{f(\varrho)} \cap A_{n_\varrho}.$$

Then $F_0(\xi)$ is a subset of A of type ω and

$$F_0(\xi) \cap C_\eta \neq \emptyset \quad (\eta \leq \xi). \tag{4.16}$$

Now consider the partition $A \times B = K_0 \cup K_1$ in which $(\mu, \xi) \in K_0$ if and only if $\mu \in A$, $\xi \in B$ and $\mu \in F_0(\xi)$. Suppose that A_1 is a subset of A of type γ and that $B_1 \in [B]^{\aleph_1}$. Then there is $\eta < \omega_1$ such that $C_\eta \subset A_1$. Also there is $\xi \in B_1$ such that $\eta \leq \xi$ and (4.16) shows that $A_1 \times B_1 \not\subset K_1$. Since $tp\, F_0(\xi) < \omega+1$ for $\xi \in B$, this proves (4.2).

PROOF OF THEOREM 3. In order to prove (4.3) it is enough to show that if $tp\, B = \beta < \omega_2$, then there are $\aleph_1$ sets $F_1(\xi) \subset B$ $(\xi < \omega_1)$ such that $tp\, F_1(\xi) < \omega_1^{\omega+2}$ $(\xi < \omega_1)$ and the union of any $\aleph_0$ of these sets is the whole set B. (4.3) follows from this result by considering the partition $A \times B = K_0 \cup K_1$, where $A = [0, \omega_1)$ and $(\xi, \mu) \in K_1$ if and only if $\mu \in F_1(\xi)$.

There is no loss of generality if we prove the result stated in the last paragraph only for the case when $\beta = \omega_1^\gamma < \omega_2$. If $\gamma < \omega+2$, the result is obvious, we just put $F_1(\xi) = B$ $(\xi < \omega_1)$. Now assume that $\omega+2 \leq \gamma < \omega_2$ and use induction on γ.

Case 1. $co(\omega_1^\gamma) = \omega$. Then $B = B_0 \cup \dots \cup \hat{B}_\omega (<)$, where $tp\, B_n = \omega_1^{\gamma_n} < \omega_1^\gamma$ $(n < \omega)$. By the induction hypothesis, there are sets $F_1(n, \xi) \subset B_n$ for $n < \omega$ and $\xi \in A$ such that $tp\, F_1(n, \xi) < \omega_1^{\omega+2}$ and $\bigcup(\xi \in N) F_1(n, \xi) = B_n$ for $n < \omega$ and $N \in [A]^{\aleph_0}$. The sets

$$F_1(\xi) = \bigcup_{n<\omega} F_1(n, \xi) \quad (\xi \in A)$$

clearly have the required properites.

Case 2. $co(\omega_1^\gamma) = \omega_1$. In this case we may write

$B = B_0 \cup \ldots \cup \hat{B}_{\omega_1}\ (<)$, where $tp\ B_\mu = \omega_1^{\gamma_\mu} < \omega_1^{\gamma}\ (\mu < \omega_1)$. By the induction hypothesis, there are sets $F_1(\mu, \xi) \subset B_\mu\ (\mu < \omega_1;\ \xi \in A)$ such that $tp\ F_1(\mu, \xi) < \omega_1^{\omega+2}$ and

$$\bigcup(\xi \in N)\, F_1(\mu, \xi) = B_\mu \qquad N \in [A]^{\aleph_0} .$$

By the partition relation (2.4) of Milner and Rado, there is a partition of $B_\mu\ (\mu < \omega_1)$,

$$B_\mu = \bigcup_{n<\omega} B_{\mu n} ,$$

in which $tp\ B_{\mu n} < \omega_1^n\ (\mu < \omega_1;\ n < \omega)$.

For $0 < \mu < \omega_1$, let $[0, \mu) = \{\nu_{\mu 0}, \nu_{\mu 1}, \ldots, \hat{\nu}_{\mu\omega}\}$ (the $\nu_{\mu n}$ are not necessarily different). Then, if $\xi < \mu$, there is some integer $n = n(\mu, \xi)$ such that $\xi = \nu_{\mu n}$. Now define

$$F_1(\xi) = \bigcup_{\mu \le \xi} F_1(\mu, \xi) \cup \bigcup_{\xi < \mu < \omega_1} \bigcup_{n < n(\mu, \xi)} B_{\mu n}$$

for $\xi \in A$. Clearly,

$$tp\ F_1(\xi) \le \sum_{\mu \le \xi} tp\ F_1(\mu, \xi) + \omega_1^{\omega}\, \omega_1 < \omega_1^{\omega+2} \qquad (\xi \in A) .$$

Also, if N is any infinite subset of A, then N contains an increasing sequence $\xi_0, \xi_1, \ldots, \hat{\xi}_\omega$ of ordinals with limit $\xi^* = \lim \xi_n < \omega_1$.

If $\mu < \xi^*$, then $\mu < \xi_m$ for some $m < \omega$ and

$$\bigcup(\xi \in N)\, F_1(\xi) \supset \bigcup(m \le i < \omega)\, F_1(\mu, \xi_i) = B_\mu .$$

If $\mu \ge \xi^*$, then the integers $n(\mu, \xi_i)\ (i < \omega)$ are all defined and distinct and therefore

$$\bigcup(\xi \in N)\, F_1(\xi) \supset \bigcup_{i<\omega} \bigcup_{n < n(\mu, \xi_i)} B_{\mu n} = B_\mu .$$

This shows that $\bigcup(\xi \in N)\, F_1(\xi) = B$ and the proof of Theorem 3 is complete.

5. SET MAPPINGS

Let $SM(\alpha,\beta)$ denote the following statement: if $tp S = \beta$ and f is any set mapping of order α on S, then there is a free subset of S of the full type β. Lemma 1 establishes a simple connection between $SM(\alpha,\beta)$ and the polarized partition symbol. Note that, if $\alpha > 1$, then $SM(\alpha,\beta+1)$ is trivially false and so it is only necessary to consider the statement $SM(\alpha,\beta)$ in the case when β is a limit number.

LEMMA 1. $SM(\alpha,\beta)$ implies

(5.1)
$$\begin{pmatrix}\beta\\ \beta\end{pmatrix} \to \begin{pmatrix}\alpha & \beta\\ 1 & \beta\end{pmatrix}$$

PROOF. If $\alpha = 1$, then (5.1) certainly holds. Therefore, we may assume that $\alpha > 1$ and that β is a limit number.

Let $tp B = \beta$ and let $B \times B = K_0 \cup K_1$. We will assume that $tp\{x \in B: (x,b) \in K_0\} < \alpha$ for all $b \in B$ and deduce that there are sets $B_1, B_2 \subset B$ which both have type β and are such that $B_1 \times B_2 \subset K_1$.

Consider the set mapping f defined on B by putting

$$f(b) = \{x \in B: x \neq b, (x,b) \in K_0\} \quad (b \in B).$$

By assumption f is a set mapping of order α and the hypothesis $SM(\alpha,\beta)$ implies that there is a free set B' of type β. Since β is a limit number, $2\beta = \beta$ and therefore B' is the union of two disjoint sets B_1, B_2 each of type β. If $b_1 \in B_1$ and $b_2 \in B_2$, then $b_1 \neq b_2$ and $b_1 \notin f(b_2)$. Hence $(b_1, b_2) \in K_1$. This proves the lemma.

We do not know if $SM(\alpha,\beta)$ and (5.1) are actually equivalent. If $\gamma = \gamma_0 + \dots + \gamma_k$, where k is finite and $\gamma_i = \omega_1^{\sigma_{i+1}} < \omega_1^{\omega+2}$ $(i \leq k)$, then it follows from Theorem 1 that

$$\begin{pmatrix}\gamma\\ \gamma\end{pmatrix} \to \begin{pmatrix}\alpha & \gamma\\ 1 & \gamma\end{pmatrix} \quad (\alpha < \omega_1).$$

In Theorem 4 we show that the corresponding set mapping statement $SM(\alpha,\gamma)$ is true.

THEOREM 4. If $\alpha < \omega_1$; $\gamma < \omega_1^{\omega+2}$ and γ is a finite sum of ordinals of the form $\omega_1^{\sigma+1}$, then $SM(\alpha,\gamma)$ holds.

In particular, Theorem 4 implies that

(*) $SM(\alpha,\omega_1^{\sigma+1})$ holds if $\alpha < \omega_1$ and $\sigma \le \omega$.

If $1 < \gamma < \omega_2$, then the conditions on γ stated in Theorem 4 are necessary for $SM(\alpha,\gamma)$ to hold for any $\alpha < \omega_1$. From Theorem 2 we see that if γ is an ordinal of the form $\gamma = \gamma_0 + \gamma_1 + \cdots + \hat{\gamma}_\omega$ with $0 < \gamma_i < \omega_2$, then

$$\begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \nrightarrow \begin{pmatrix} \omega+1 & \gamma \\ 1 & \gamma \end{pmatrix}$$

and this implies that $SM(\omega+1,\gamma)$ is false by Lemma 1. For example, in contrast with (*) the last remark shows that $SM(\omega+1,\omega_1^\omega)$ is false. As a special case of Theorem 5 we know that $SM(\omega,\omega_1^\omega)$ holds and this result is best possible in the sense that ω cannot be increased.

THEOREM 5. $SM(\omega,\omega_1\gamma)$ holds for any $\gamma < \omega_1^{\omega+2}$.

If $\omega_1^{\omega+2} \le \gamma < \omega_2$, then

$$\begin{pmatrix} \gamma \\ \gamma \end{pmatrix} \nrightarrow \begin{pmatrix} \omega & \gamma \\ 1 & \gamma \end{pmatrix}$$

by Theorem 3. Therefore, $SM(\omega,\gamma)$ is false if $\omega_1^{\omega+2} \le \gamma < \omega_2$. This shows that the condition $\gamma < \omega_1^{\omega+2}$ in Theorems 4 and 5 is necessary. For set mappings of finite order n, we have a very general positive result.

THEOREM 6. If $n < \omega$, then $SM(n,\omega\Theta)$ holds for arbitrary Θ.

PROOF OF THEOREM 4. Let $tp\,S = \gamma$ and let f be any set mapping of order α on S. We want to show that there is a free set $S_1^* \subset S$ having the same type γ.

Case 1. $\gamma = \omega_1^n < \omega_1^\omega$

The set mapping f induces two auxiliary mappings g and h defined by putting

$$g(x) = \{y \in f(x):\ y < x\}, \qquad h(x) = \{y \in f(x):\ x < y\}.$$

Thus, for any $x \in S$, we have

$$g(x) < \{x\} < h(x) \tag{5.2}$$

The set mappings g and h are also of order α and it is convenient to consider these separately. We will show that there are sets $S_0 \subset S$ and $S_1 \subset S_0$ such that $tp\, S_0 = tp\, S_1 = \omega_1^n$, S_0 is free in the set mapping g and S_1 is free in the set mapping h. This will give the result since we then have $f(S_1) \cap S_1 = \emptyset$.

We first show that there is $S_0 \subset S$ such that $tp\, S_0 = \omega_1^n$ and

$$S_0 \cap g(S_0) = \emptyset. \tag{5.3}$$

If $n = 0$ then (5.3) holds with $S_0 = S$, for in this case S has a single element. We therefore assume that $n > 0$ and use induction on n.

We begin by showing that there is a set $S' \subset S$ of type ω_1^n such that

$$tp\,(S' - g^{-1}(X)) = \omega_1^n \quad \text{whenever} \quad X \overset{N}{\subset} S'. \tag{5.4}$$

Suppose there is no such set S'. Then we define sets $X_\nu \overset{N}{\subset} S$ and $Y_\nu \subset S$ for $\nu < \omega_1$ in the following way. Let $\mu < \omega_1$ and suppose we have already defined X_ν, Y_ν for $\nu < \mu$. Since the sets X_ν $(\nu < \mu)$ are non-cofinal with S and $\mu < \omega_1$, there is a proper final section T of S such that $X_\nu < T$ $(\nu < \mu)$. By our assumption, (5.4) is false with $S' = T$ and hence there is $X_\mu \overset{N}{\subset} T$ such that $tp\,(T - g^{-1}(X_\mu)) < \omega_1^n$. Since T is a proper final section of S, this implies that

$$Y_\mu = S - g^{-1}(X_\mu) \tag{5.5}$$

also has type less than ω_1^n. This defines the sets $X_\mu, Y_\mu \subset S$ for $\mu < \omega_1$.

Note that, by the construction, $X_0 < X_1 < \dots < \hat{X}_{\omega_1}$, $tp\, Y_\mu < \omega_1^n$ and (5.5) holds.

By (4.4) we have

$$(5.6) \qquad \begin{pmatrix} \omega_1 \\ \omega_1^k \end{pmatrix} \to \begin{pmatrix} \alpha & 1 \\ 1 & \omega_1^k \end{pmatrix} \qquad (k < \omega)$$

Therefore, since $Y_\mu < \omega_1^n$ $(\mu < \omega_1)$, the polarized partition relation (5.6) implies that there are $x \in S$ and $N \subset [0, \omega_1)$ such that $tp\, N = \alpha$ and

$$x \notin Y_\mu \quad (\mu \in N).$$

According to (5.4) this means that $g(x) \cap X_\mu \neq \emptyset$ $(\mu \in N)$ and hence $tp\, g(x) \geq tp\, N = \alpha$. This contradiction shows that there is a set $S' \subset S$ of type ω_1^n such that (5.4) holds.

We now define non-cofinal subsets C_ν of S'. Let $\mu < \omega_1$ and suppose that we have already defined $C_\nu \overset{N}{\subset} S$ for $\nu < \mu$. By (5.4)

$$tp(S' - g^{-1}(\cup(\nu < \mu) C_\nu)) = \omega_1^n$$

and therefore there is $C'_\mu \overset{N}{\subset} S' - g^{-1}(\cup(\nu < \mu) C_\nu))$ such that $C_\nu < C'_\mu$ $(\nu < \mu)$ and $tp\, C'_\mu = \omega_1^{n-1}$. By the induction hypothesis, there is a g-free set $C_\mu \subset C'_\mu$ having the same type ω_1^{n-1}. This defines the sets C_ν for $\nu < \omega_1$. Clearly

$$S_0 = C_0 \cup C_1 \cup \dots \cup \hat{C}_{\omega_1} \, (<)$$

has order type ω_1^n. If $\nu \leq \mu < \omega_1$, then $C_\nu \cap g(C_\mu) = \emptyset$ by the construction. Also, by (5.2), $g(C_\nu) < C_\mu$. This shows that (5.3) holds.

We now consider the set mapping h restricted to S_0. First we observe that, if T is any non-cofinal subset of S_0 such that $tp\, T = \omega_1^m$ with $m < n$, then there is a proper final section $F(T)$ of S_0 such that $T < F(T)$ and

(5.7) $$tp(T-h^{-1}(D)) = tp\,T \quad \text{whenever} \quad D \overset{N}{\subset} F(T).$$

Suppose this is not the case. Then since* $R(T)$ is a proper final section of S_0, there is $D_0 \overset{N}{\subset} R(T)$ such that $tp(T-h^{-1}(D_0)) < \omega_1^m$. More generally, if $\mu < \omega_1$ and $D_0, \ldots, \hat{D}_\mu$ have been chosen so that $D_\nu \overset{N}{\subset} S_0$ $(\nu < \mu)$, then $R' = R(T \cup \bigcup(\nu<\mu) D_\nu)$ is a proper final section of S_0 and, by the assumption that (5.7) is false, there is $D_\mu \overset{N}{\subset} R'$ such that

(5.8) $$tp(T-h^{-1}(D_\mu)) < \omega_1^m .$$

In this way, we define sets D_μ $(\mu < \omega_1)$ so that $D_0 < D_1 < \ldots < \hat{D}_{\omega_1}$ and (5.8) holds. The relation (5.6) applied to the sets $T-h^{-1}(D_\mu)$ $(\mu < \omega_1)$ shows that there are $N \subset [0, \omega_1)$ and $x \in T$ such that $tp\,N = \alpha$ and $x \notin T-h^{-1}(D_\mu)$ $(\mu \in N)$. Therefore, $h(x) \cap D_\mu \neq \phi$ $(\mu \in N)$ and $tp\,h(x) \geq \alpha$. This contradiction proves that there is a final section $F(T)$ of S_0 such that (5.7) holds.

We want to prove that there is a set $S_1 \subset S_0$ of type ω_1^n such that

(5.9) $$S_1 \cap h(S_1) = \phi .$$

If $n = 0$, (5.9) holds with $S_1 = S_0$. We shall therefore assume that $n > 0$ and use induction on n.

Let $I(S_0)$ and $\phi = \phi_{S_0}$ be as described in §3. We are going to define sets $Z_\nu \subset S_0$ $(\nu < \omega_1)$ such that (5.10) - (5.14) hold for $\nu < \omega_1$.

(5.10) $$tp\,Z_\nu = tp\,I_\nu ;$$

(5.11) $$Z_\nu \triangleleft Z_\varrho \Leftrightarrow I_\nu \triangleleft I_\varrho \quad (\varrho < \nu),$$

where $\triangleleft$ denotes $<$, $>$ or $\overset{N}{\subset}$;

*Here $R(T) = \{y \in S_0 : T < \{y\}\}$.

(5.12) $$Z_\nu \cap h(Z_\nu) = \phi \qquad (\nu \neq 0);$$

(5.13) $$Z_\nu \overset{N}{\subset} \cap(0<\varrho<\nu) F(Z_\varrho) \quad \text{if} \quad \phi(\nu) = 0;$$

(5.14) $$g(Z_\nu) \cap Z_\varrho = \phi \qquad \text{if } \varrho < \nu \text{ and } \quad I_\nu < I_\varrho.$$

Put $Z_0 = S_0$. Now let $0 < \mu < \omega_1$ and suppose that Z_ν has been defined for $\nu < \mu$ so that (5.10) - (5.14) hold. We want to define Z_μ so that these relations remain valid with $\nu = \mu$.

Let $K = \{\sigma : \phi(\mu) < \sigma < \mu ; \phi(\sigma) = 0\}$.

Case (i) $\phi(\mu) = 0$. If $0 < \nu < \mu$, then by (3.5) there is some $\sigma \in K$ such that $I_\nu \subset I_\sigma$. Therefore, by (3.7), $I_\nu \subset I_\sigma < I_\mu$. Since (3.11) holds, $I_\nu \overset{N}{\subset} I_0$. Therefore $Z_\nu \overset{N}{\subset} Z_0$ by (5.11) and $F(Z_\nu)$ is a proper final section of $Z_0 = S_0$. By (3.5) we have that $tp\, I_\mu < tp\, I_0 = \omega_1^n$ and therefore Z' can be chosen so that

$$Z' \overset{N}{\subset} \cap(0<\nu<\mu) F(Z_\nu)$$

and $tp\, Z' = tp\, I_\mu$. Since $tp\, Z' = tp\, I_\mu = \omega_1^k < \omega_1^n$, it follows from the induction hypothesis that there is an h-free subset Z_μ of Z' having the same order type. With this choice for Z_μ it is clear that (5.10) - (5.13) hold with $\nu = \mu$. In this case (5.14) holds vacuously for $\nu = \mu$, since $I_\varrho < I_\mu$ $(\varrho < \mu)$.

Case (ii) $\phi(\mu) > 0$. Let $A = \cup(\varrho \in K) Z_\varrho$. Since K is countable and, by (5.13), $Z_\varrho \overset{N}{\subset} F(Z_{\phi(\mu)})$ $(\varrho \in K)$, it follows that $A \overset{N}{\subset} F(Z_{\phi(\mu)})$.

Therefore, by (5.7),

(5.15) $$tp\,(Z_{\phi(\mu)} - h^{-1}(A)) = tp\, Z_{\phi(\mu)}.$$

Let $L = \{\nu : \phi(\nu) = \phi(\mu) < \nu < \mu\}$. By (3.5) and (5.11) we have $I_\nu \overset{N}{\subset} I_{\phi(\mu)}$ and $Z_\nu \overset{N}{\subset} Z_{\phi(\mu)}$ for $\nu \in L$ and therefore

(5.16) $$B = \cup(\nu \in L) Z_\nu \overset{N}{\subset} Z_{\phi(\mu)} .$$

It follows from (5.15) and (5.16) that Z_μ can be chosen so that

$$B < Z_\mu \overset{N}{\subset} Z_{\phi(\mu)} - h^{-1}(A)$$

and $tp\, Z_\mu = tp\, I_\mu \; (< tp\, I_{\phi(\mu)} = tp\, Z_{\phi(\mu)})$.

It is clear that (5.10) and (5.11) both hold for $\nu = \mu$ with this definition of Z_μ. Also (5.12) holds since $Z_\mu \subset Z_{\phi(\mu)}$ and $Z_{\phi(\mu)}$ is h-free. In this case (5.13) is satisfied vacuously when $\nu = \mu$. It remains for us to verify that (5.14) also holds for $\nu = \mu$, i.e. we have to show that

(5.17) $$h(Z_\mu) \cap Z_\varrho = \phi \quad \text{if} \quad \varrho < \mu \quad \text{and} \quad I_\mu < I_\varrho .$$

If $0 < \nu < \omega_1$ then, as we observed in §3, there is an integer $\iota(\nu)$ such that $\nu = \phi_0(\nu) > \phi_1(\nu) > \phi_2(\nu) > \cdots > \phi_\iota(\nu) = 0$. Put $\bar{\phi}(\nu) = \phi_{\iota-1}(\nu)$. Then $0 < \bar{\phi}(\nu) \leq \nu$, $I_\nu \subset I_{\bar{\phi}(\nu)}$ and $\phi(\bar{\phi}(\nu)) = 0$. Note that, since $\phi(\mu) > 0$, we have $\bar{\phi}(\mu) < \mu$.

Let $\varrho < \mu$ and suppose that $I_\mu < I_\varrho$. If $\bar{\phi}(\varrho) = \bar{\phi}(\mu)$, then Z_ϱ and Z_μ are subsets of $Z_{\bar{\phi}(\mu)}$ and (5.17) holds since $Z_{\bar{\phi}(\mu)}$ is free by (5.12). Therefore, we can assume that $\bar{\phi}(\varrho) \neq \bar{\phi}(\mu)$. If $\bar{\phi}(\varrho) < \bar{\phi}(\mu)$, then $I_{\bar{\phi}(\varrho)} < I_{\bar{\phi}(\mu)}$ by (3.7) and this contradicts the assumption that $I_\mu < I_\varrho$ (for $I_\mu \subset I_{\bar{\phi}(\mu)}$ and $I_\varrho \subset I_{\bar{\phi}(\varrho)}$. Therefore, $\bar{\phi}(\phi(\mu)) = \bar{\phi}(\mu) < \bar{\phi}(\varrho)$ and $I_{\phi(\mu)} \subset I_{\bar{\phi}(\mu)} < I_{\bar{\phi}(\varrho)}$. If $\bar{\phi}(\varrho) < \phi(\mu)$, then

$$g(Z_{\phi(\mu)}) \cap Z_{\bar{\phi}(\varrho)} = \phi$$

since (5.14) holds with $\nu = \phi(\mu)$, and this implies (5.17) since $Z_\mu \subset Z_{\phi(\mu)}$ and $Z_\varrho \subset Z_{\bar{\phi}(\varrho)}$. If, on the other hand, $\phi(\mu) < \bar{\phi}(\varrho)$, then $\bar{\phi}(\varrho) \in K$ and $Z_\varrho \subset Z_{\bar{\phi}(\varrho)} \subset A$ and (5.17) holds since $Z_\mu \cap h^{-1}(A) = \phi$ by the definition of Z_μ.

This shows that there are sets Z_ν $(\nu < \omega_1)$ satisfying all the conditions (5.10) - (5.14).

By (5.10) and (5.11), it follows that

$$S_1 = \bigcup(|Z_\nu| = 1;\ \nu < \omega_1)\, Z_\nu$$

has the same type as $S_0 = \bigcup(|I_\nu| = 1)\, I_\nu$. To complete the proof of Theorem 4 for Case 1 it remains to show that S_1 is h-free.

Let $\mu, \nu < \omega_1$ and suppose that $|Z_\mu| = |Z_\nu| = 1$ and $Z_\mu < Z_\nu$. In view of (5.2) it is enough to show that

$$h(Z_\mu) \cap Z_\nu = \phi . \tag{5.18}$$

If $Z_\mu \subset Z_{\bar\phi(\nu)}$, then (5.18) holds since $\bar\phi(\nu) \neq 0$ and $Z_{\bar\phi(\nu)}$ is h-free by (5.12). Therefore, we may suppose that $\bar\phi(\nu) \neq \mu$ and $Z_\mu < Z_{\bar\phi(\nu)}$. If $\bar\phi(\nu) < \mu$, then $h(Z_\mu) \cap Z_{\bar\phi(\nu)} = \phi$ by (5.14) and this implies (5.18) since $Z_\nu \subset Z_{\bar\phi(\nu)}$. If $\mu < \bar\phi(\nu)$, then by (5.13), $Z_\nu \subset Z_{\bar\phi(\nu)} \overset{N}{\subset} F(Z_\mu)$. Therefore, by (5.7),

$$tp(Z_\mu - h^{-1}(Z_{\bar\phi(\nu)})) = tp\, Z_\mu .$$

But since $tp\, Z_\mu = 1$, this simply means that $h^{-1}(Z_{\bar\phi(\nu)})$ is disjoint from Z_μ and (5.18) follows. This completes the proof of (5.9) and Case 1 of the Theorem.

Case 2. $\gamma = \omega_1^{\omega+1}$.

To prove the Theorem in this case we require the following lemma.

LEMMA. Let $tp\, S = \omega_1^{\omega+1}$ and let $T_0, T_1, \ldots$ be cofinal subsets of S such that $tp\, T_n = \omega_1^{n+1}$ $(n < \omega)$. Then $tp \bigcup(n<\omega)\, T_n = \omega_1^{\omega+1}$.

PROOF. For $\mu < \omega_1$ we may write $\mu = \omega\lambda(\mu) + r(\mu)$, where $\lambda(\mu) \leq \mu$ and $r(\mu) < \omega$. We define non-cofinal subsets S_μ of S for $\mu < \omega_1$ in the following way. Let $\mu < \omega_1$ and suppose that $S_\nu \overset{N}{\subset} S$ has been defined for $\nu < \mu$. Since $T_{r(\mu)}$ is a cofinal subset of S of order type $\omega_1^{r(\mu)+1}$, there is a non-cofinal subset S_μ of $T_{r(\mu)}$ with type $\omega_1^{r(\mu)}$ such that $S_\nu < S_\mu$ holds for $\nu < \mu$. Then

$$tp(\bigcup(n<\omega)T_n) \geq tp(\bigcup(\mu<\omega_1)S_\mu) = \sum(\mu<\omega_1)\omega_1^{r(\mu)} = \omega_1^{\omega+1}.$$

This proves the lemma.

Let S be an ordered set of type $\omega_1^{\omega+1}$ and let f be a set mapping on S of order $\alpha(<\omega_1)$. We are going to define sets $T_n, S_n \subset S$ for $n < \omega$ such that

(5.19) $\qquad T_n$ is cofinal with S and $tp\, T_n = \omega_1^{n+1}$,

(5.20) $$S_{n+1}, T_n \subset S_n,$$

(5.21) $$T_n \cap f(S_{n+1}) = \emptyset,$$

(5.22) $$f(T_n) \cap (T_n \cup S_{n+1}) = \emptyset.$$

This will prove the theorem for Case 2 since (5.20), (5.21) and (5.22) imply that $T = \bigcup(n<\omega)T_n$ is a free set for the mapping f and (5.19) and the lemma imply that $tp\, T = \omega_1^{\omega+1}$.

Put $S_0 = S$. Now let $n < \omega$ and suppose that a subset S_n of S has already been defined so that $tp\, S_n = \omega_1^{\omega+1}$. We want to show that there are sets T_n and S_{n+1} such that $tp\, S_{n+1} = \omega_1^{\omega+1}$ and (5.19) - (5.22) are satisfied.

Since $tp\, S_n = \omega_1^{\omega+1}$, we may write $S_n = P_0 \cup \ldots \cup \hat{P}_{\omega_1} (<)$, where $P_\nu = P_{\nu 0} \cup \ldots \cup \hat{P}_{\nu\omega} (<)$ and $tp\, P_{\nu i} = \omega_1^{i+1}$ $(\nu < \omega_1;\ i < \omega)$. Put $A = \bigcup(\nu < \omega_1;\ i \leq n) P_{\nu i}$ and let $B = S_n - A$. Then $tp\, A = \omega_1^{n+1}$ and

$\text{tp}\, B = \omega_1^{\omega+1}$. Consider the partition $A \times B = K_0 \cup K_1$, where $(a,b) \in K_0$ if and only if $a \in f(b)$. By Theorem 1 we have that

$$\begin{pmatrix} \omega_1^{n+1} \\ \omega_1^{\omega+1} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \omega_1^{n+1} \\ 1 & \omega_1^{\omega+1} \end{pmatrix}$$

Therefore, since $\text{tp}\, F_0(b) \leq \text{tp}\, f(b) < \alpha \; (b \in B)$, it follows that there are sets $A' \subset A$ and $B' \subset B$ such that $\text{tp}\, A' = \omega_1^{n+1}$, $\text{tp}\, B' = \omega_1^{\omega+1}$ and $A' \times B' \subset K_1$, i.e. $A' \cap f(B') = \phi$.

Now consider the partition $A' \times B' = K_0' \cup K_1'$, where $(a,b) \in K_0'$ if and only if $b \in f(a)$. Again by Theorem 1, we have

$$\begin{pmatrix} \omega_1^{\omega+1} \\ \omega_1^{n+1} \end{pmatrix} \rightarrow \begin{pmatrix} \alpha & \omega_1^{\omega+1} \\ 1 & \omega_1^{n+1} \end{pmatrix}$$

and this implies that there are sets $A'' \subset A'$, $B'' \subset B'$ such that $\text{tp}\, A'' = \omega_1^{n+1}$, $\text{tp}\, B'' = \omega_1^{\omega+1}$ and $f(A'') \cap B'' = \phi$. By Case 1 of the present theorem, there is a free set $T_n \subset A''$ of type ω_1^{n+1}. Now (5.20) - (5.22) hold with this choice for T_n and $S_{n+1} = B''$. T_n is cofinal with A since it is a subset with the same ordinal number and similarly S_n is cofinal with S. Therefore, (5.19) also holds since, by definition, A is cofinal with S_n. It follows by induction that there are sets S_n, T_n satisfying (5.19) - (5.22) and the proof is complete.

Case 3. $\gamma = \omega_1^{\sigma_0+1} + \ldots + \omega_1^{\sigma_k+1}$, where $k < \omega$ and $\sigma_i \leq \omega$ $(i \leq k)$.

Let $\text{tp}\, S = \gamma$ and let f be a set mapping on S of order α. We want to show there is a free subset of S of type γ. If $k = 0$, this follows from Cases 1 and 2. Now assume that $k > 0$ and use induction on k.

We have $S = S_0 \cup S_1 \; (<)$, where $\text{tp}\, S_0 = \sum (i < k)\, \omega_1^{\sigma_i+1} = \gamma_0$

and $\operatorname{tp} S_1 = \omega_1^{\sigma_k+1} = \gamma_1$. By the induction hypothesis, there are f-free sets $S_0' \subset S_0$ and $S_1' \subset S_1$ such that $\operatorname{tp} S_i' = \operatorname{tp} S_i$ $(i < 2)$. By Theorem 1, the relations

$$\begin{pmatrix}\gamma_0\\ \gamma_1\end{pmatrix} \to \begin{pmatrix}\alpha & \gamma_0\\ 1 & \gamma_1\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\gamma_1\\ \gamma_0\end{pmatrix} \to \begin{pmatrix}\alpha & \gamma_1\\ 1 & \gamma_0\end{pmatrix}$$

hold and by succesively applying these in an obvious way we conclude that there are sets $S_i'' \subset S_i'$ $(i < 2)$ such that $\operatorname{tp} S_i'' = \gamma_i$ and $S_0'' \cup S_1''$ is f-free. This concludes the proof of Theorem 4.

PROOF OF THEOREM 5. Let $\operatorname{tp} S = \omega_1 \gamma < \omega_1^{\omega+2}$ and let f be any set mapping of order ω on S. Then $f(x)$ is finite for $x \in S$. We want to show that there is a free set which also has type $\omega_1\gamma$. If $\gamma = 1$ this is a consequence of Theorem 4 and so we assume $\gamma > 1$.

We observe first that whenever $A \subset S$ and the order type of A is a power of ω_1, then there is a countable set $C(A) \subset S$ such that

$$(5.23) \qquad \operatorname{tp}(A - f^{-1}(D)) = \operatorname{tp} A \quad \text{whenever} \quad D \in [S - C(A)]^{\leq \aleph_0}.$$

If this were not so there would be disjoint countable sets D_ν $(\nu < \omega_1)$ such that $\operatorname{tp}(A - f^{-1}(D_\nu)) < \operatorname{tp} A$ $(\nu < \omega_1)$. This leads to a contradiction since, by (4.4), there are $x \in A$ and an infinite set $N \subset [0, \omega_1)$ such that $x \notin (A - f^{-1}(D_\nu))$ $(\nu \in N)$, i.e. $f(x)$ is infinite.

This shows that there is a countable set $C(A) \subset S$ such that (5.23) holds.

Let $I(S) = (I_0, \ldots, \hat{I}_{\omega_1})$ and $\phi = \phi_S$ be as described in §3. We shall define by transfinite induction sets $Z_\nu \subset S$ $(\nu < \omega_1)$ such that the following conditions hold for $\nu < \omega_1$:

$$(5.24) \qquad \operatorname{tp} Z_\nu = \operatorname{tp} I_\nu .$$

(5.25) $$Z_\nu \lhd Z_\varrho \Leftrightarrow I_\nu \lhd I_\varrho \quad (\varrho < \nu)$$

where $\lhd$ denotes $<$, $>$, or $\overset{N}{\subset}$.

(5.26) $$Z_\nu \cap C(Z_\varrho) = \emptyset \quad (0 < \varrho < \nu).$$

(5.27) $$Z_\nu \cap f(Z_\varrho) = \emptyset \quad \text{if } \varrho < \nu \text{ and } |Z_\varrho| = 1.$$

(5.28) $$Z_\nu \subset I_\nu \quad \text{if} \quad \phi(\nu) = 0.$$

Put $Z_0 = S$. Let $\mu > 0$ and suppose that Z_ν has been defined fo $\nu < \mu$ so that the above relations hold. Put $U = \bigcup(\nu < \mu ; |Z_\nu| = 1) Z_\nu$, $V = \bigcup(0 < \nu < \mu) C(Z_\nu)$ and let $K = \{\nu : \phi(\nu) = \phi(\mu) < \nu < \mu\}$.

Case 1. $\phi(\mu) = 0$. By (5.28) and (3.7) we have in this case that

$$Z_\nu \subset I_\nu < I_\mu \quad (\nu \in K).$$

Since $tp\, S > \omega_1$, the remark (3.11) applies and therefore

$$Z_\mu = I_\mu - f(U) - V$$

has the same type as I_μ because U and V are both denumerable sets. It is easy to see that (5.24) - (5.28) hold for $\nu = \mu$ with this choice for Z_μ.

Case 2. $\phi(\mu) > 0$. In this case, $I_\nu < I_\mu \subset I_{\phi(\mu)}$ $(\nu \in K)$ by (3.5) and (3.7). Also, $tp\, K < \chi(tp\, I_{\phi(\mu)}) = co(tp\, I_{\phi(\mu)})$ by (3.8). By (5.25) we have that $Z_\nu \overset{N}{\subset} Z_{\phi(\mu)}$ $(\nu \in K)$ and hence $Z' = \bigcup(\nu \in K) Z_\nu \overset{N}{\subset} Z_{\phi(\mu)}$.

Since U and V are denumerable and $tp\, Z_{\phi(\mu)} = tp\, I_{\phi(\mu)}$ is a power of ω_1, there is a set Z_μ such that

$$Z' < Z_\mu \overset{N}{\subset} Z_{\phi(\mu)} - f(U) - V$$

and $\mathrm{tp}\, Z_\mu = \mathrm{tp}\, I_\mu\, (< \mathrm{tp}\, I_{\phi(\mu)})$. It is obvious that (5, 24), (5. 26), (5. 27) and (5. 28) now hold with $\nu = \mu$ and routine to verify that (5. 25) also holds. This defines the sets Z_ν for $\nu < \omega_1$.

By (5. 24) and (5. 25) we see that $Z^* = \bigcup(\nu < \omega_1;\ |Z_\nu| = 1)\, Z_\nu$ has the same type as $S = \bigcup(|I_\nu| = 1)\, I_\nu$. Also Z^* is a free set. For, if $0 < \rho < \nu < \omega$, and $|Z_\rho| = |Z_\nu| = 1$, then

$$f(Z_\rho) \cap Z_\nu = \phi$$

by (5. 27) and

$$f(Z_\nu) \cap Z_\rho = \phi$$

since $\mathrm{tp}(Z_\nu - f^{-1}(Z_\rho)) = \mathrm{tp}\, Z_\nu = 1$ by 5. 26. This completes the proof of Theorem 5.

PROOF OF THEOREM 6.

Let $\mathrm{tp}\, S = \omega\alpha$ and let f be a set mapping of order $n\, (< \omega)$ on S. We have to show that there is a free set of type $\omega\alpha$.

Case 1. $\omega\alpha$ is indecomposable. By a theorem of Erdős and de Bruijn [9], S is the union of $2n-1$ free sets. Since $\omega\alpha$ is indecomposable it follows by (2. 2) that one of these free sets has type $\omega\alpha$.

Case 2. $\omega\alpha$ is decomposable. Let $\omega\alpha = \alpha_0 + \dots + \alpha_k$ be the Cantor standard representation for $\omega\alpha$ as a finite sum of non-increasing indecomposable ordinals. Then $k \geq 1$ and α_i is indecomposable. We shall give details of the proof only for the case $k = 1$. The general result follows by an obvious extension of the argument.

Then $S = S_0 \cup S_1\, (<)$, where $\mathrm{tp}\, S_i = \alpha_i$ is indecomposable and infinite $(i < 2)$. By case 1 we can assume that each S_i is a free set. Since α_0 is infinite, S_0 is the union of n disjoint sets $A_i\ (i < n)$ each having the same type α_0. Then for $x \in S_1$ there is an index $i(x) < n$ such that $f(n) \cap S_{i(x)} = \phi$. Put $B_i = \{x \in S_1 :\ i(x) = i\}$.

Then there is $i_0 < n$ such that B_{i_0} has type α_1 and

$$A_{i_0} \cap f(B_{i_0}) = \phi .$$

Applying a similar argument to the sets A_{i_0} and B_{i_0} we find that there are sets $A' \subset A_{i_0}$ and $B' \subset B_{i_0}$ such that $tp\, A' = \alpha_0$, $tp\, B' = \alpha_1$ and

$$f(A') \cap B' = \phi .$$

Then $A' \cup B'$ is a free subset of type $\alpha_0 + \alpha_1$.

6. GRAPHS WITHOUT INFINITE PATHS.

In this final section we apply the results of this paper to prove the following theorem.

THEOREM 7. Let S be an ordered set of type $\omega\Theta < \omega_1^{\omega+2}$ and let $G = (S, E)$ be any graph on S *which does not contain an infinite path.* Then there is an independent set $S' \subset S$ with the same type $\omega\Theta$.

REMARK. In our proof of Theorem 7 we employ Theorem 5 and this explains the restriction on the size of S (i.e. $tp\, S < \omega_1^{\omega+2}$. While Theorem 5 is false for larger order types (of cardinal $\aleph_1$), we suspect that Theorem 7 holds for arbitrary Θ but we are unable to prove this.

PROOF OF THEOREM 7.

We shall prove the theorem in three stages.

Case 1. $\Theta = \omega_1 \gamma$.

We claim that if T is any subset of S, then there is an element $x = x(T) \in T$ with finite relative valence, i.e. such that

(6.1) $\qquad E(x) \cap T$ is finite.

For suppose this is false for some $T \subset S$. Then we construct an infinite path in T as follows. Choose $x_0 \in T$. If $n < \omega$ and x_n has been chosen then, since $E(x_n) \cap T$ is infinite, we can chose $x_{n+1} \in E(x_n) \cap T - \{x_0, \ldots, x_n\}$.

Then $x_0, x_1, \dots$ is an infinite path contrary to the hypothesis.

Hence there is $x = x(T) \in T$ such that (6.1) holds.

Now define a well ordering of the elements of S in the following way. Put $a_0 = x(S)$, $a_1 = x(S-\{a_0\})$, etc. This process must terminate after λ steps for some ordinal $\lambda < \omega_2$. We then have $S = \{a_0, a_1, \dots, \hat{a}_\lambda\}_{\neq}$. Now define a set mapping on S by putting $f(a_\mu) = E(a_\mu) \cap \{a_{\mu+1}, \dots, \hat{a}_\lambda\}$. Then, by (6.1), f is a set mapping of order ω and, by Theorem 5, there is a free set $S' \subset S$ of the same type $\omega_1 \gamma$. The set S' is also independent in the graph G. For, if $a_\mu, a_\nu \in S'$ and $\mu < \nu$, then $\{a_\mu, a_\nu\}$ is not an edge of G since $a_\nu \notin f(a_\mu)$.

Case 2. $\Theta < \omega_1$.

In this case we shall apply the construction described in §3. Let $I(S) = (I_0, \dots, \hat{I}_\omega)$ and $\phi = \phi_S$ be as defined earlier.

If T is any subset of S of type $\omega^\lambda \geq \omega$, then there is a finite set $F(T) \subset S$ such that

$$tp(T - E(X)) = tp\, T \quad \text{for all} \quad X \in [S - F(T)]^{<\aleph_0}$$

For, if this were not so, there would be infinitely many disjoint finite sets $X_0, X_1, \dots$ such that $tp(T - E(X_n)) < tp\, T$ $(n < \omega)$. Since $\omega^\lambda \to (\omega^\lambda)^1_k$ holds for any finite k, it follows that there are elements $x_n \in X_n$ $(n < \omega)$ such that $tp(T - E(x_n)) < \omega^\lambda$. Each x_n is joined by edges of G to almost all the points of T (all but a set of type less than ω^λ) and hence $tp\, E(\{x_i, x_j\}) \cap T = \omega^\lambda$ for $i, j < \omega$. Now define integers n_i and elements y_i $(i < \omega)$ as follows. Put $n_0 = 0$, $n_1 = 1$ and choose $y_0 \in E(\{x_0, x_1\})$. If $1 \leq k < \omega$ and n_i $(i \leq k)$, y_i $(i < k)$ have been defined, choose n_{k+1} and y_k so that

$$x_{n_{k+1}} \notin \{x_{n_i} : i \leq k\} \cup \{y_i : i < k\} = Z_k$$

and $y_k \in E(x_{n_k}, x_{n_{k+1}}) - Z_k$. Then the graph G contains the infinite path $x_{n_0}, y_0, x_{n_1}, y_1, \ldots$, a contradiction.

Put $Z_0 = S$. We are going to define sets $Z_i \subset S$ $(i < \omega)$ such that the following conditions hold:

(6.2) $$\operatorname{tp} Z_i = \operatorname{tp} I_i ,$$

(6.3) $$Z_i \triangleleft Z_j \iff I_i \triangleleft I_j \quad (j < i),$$

where $\triangleleft$, as usual, denotes either $<$, $>$, $\subset$ or $\overset{N}{\subset}$.

(6.4) $Z_i \cap F(Z_j) = \phi$ if $j < i$ and $\operatorname{tp} Z_j = \omega^{\lambda_j} \geq \omega$,

(6.5) $Z_i \cap E(Z_j) = \phi$ if $j < i$ and $|Z_j| = 1$,

(6.6) $Z_i \subset I_i$ if $\phi(i) = 0$ and $\omega\Theta$ is decomposable.

Condition (6.6) is rather special and is introduced only to take into account the case when $\omega\Theta$ is decomposable. But, in this case, by the definition of $I(S)$, we have

$$S = I_1 \cup I_2 \cup \ldots \cup I_\chi \ (<) ,$$

where $\chi = \chi(\omega\Theta)$ is finite, and we define

$$Z_i = I_i - F(I_1) \cup \ldots \cup \hat{F}(I_i)$$

fir $i < \chi$. With these definitions, it is clear that (6.2) - (6.6) hold for $i \leq \chi$ We can now assume that $n > 0$ (and $n > \chi$ if χ is finite) and that Z_i has been defined for $i < n$ so that the above conditions are satisfied. From our assumption we have that $\operatorname{tp} I_{\phi(n)} = \operatorname{tp} Z_{\phi(n)}$ is indecomposable. We want to define Z_n so that (6.2) - (6.6) remain valid.

Let $K = \{i : \phi(i) = \phi(n) < i < n\}$, $Z' = \bigcup (i \in K) Z_i$. Then

$I_i \overset{N}{\subset} I_{\phi(n)}$, $Z_i \overset{N}{\subset} Z_{\phi(n)}$ $(i \in K)$ and, since $tp\, Z_{\phi(n)}$ is indecomposable and K is finite, $Z' \overset{N}{\subset} Z_{\phi(\mu)}$. Put $J = \{ j < n : \ tp\, Z_j = \omega^{\lambda_j} \geq \omega \}$, $L = \bigcup (j < \phi(n), |Z_j| = 1) Z_j$, $M = \bigcup (\phi(n) < j < n, |Z_j| = 1) Z_j$. By (6.5), $Z_{\phi(n)} \cap E(L) = \phi$. By (6.4), M is a finite subset of $S - F(Z_{\phi(n)})$ and therefore

$$tp(Z_{\phi(n)} - E(L \cup M)) = tp\, Z_{\phi(n)} .$$

It follows that there is a set Z_n having the same type as I_n $(< tp\, Z_{\phi(n)})$ such that

$$Z' < Z_n \overset{N}{\subset} Z_{\phi(n)} - E(L \cup M) - \bigcup (j \in J) F(Z_j) .$$

With this choice for Z_n it is obvious that (6.2), (6.4) and (6.5) hold, and routine to verify that (6.3) also holds for $i = n$. (6.6) holds vacuously for $i = n$ (from our assumption about n).

From (6.2) and (6.3) it follows that $S' = \bigcup (|Z_i| = 1) Z_i$ has the same order type as S. If $|Z_i| = |Z_j| = 1$ and $j < i$, then $Z_i \cap E(Z_j) = \phi$ by (6.5). Therefore, S' is an independent set of type $\omega\Theta$.

Case 3. $\omega\Theta = \omega_1\gamma + \omega\beta$, where $0 < \gamma < \omega_1^{\omega+2}$ and $0 < \beta < \omega_1$.

Let $S = A \cup B \ (<)$, where $tp\, A = \omega_1\gamma$ and $tp\, B = \omega\beta$. In view of cases 1 and 2 we can assume that A, B are actually independent and the edges of G join points of A to points of B. We want to show that there are sets $A' \subset A$ and $B' \subset B$ such that $tp\, A' = \omega_1\gamma$ and $tp\, B' = \omega\beta$ and $A' \cup B'$ is independent.

We shall assume first that γ is indecomposable. Consider a new graph $G' = (B, E')$ in which two points $b, b' \in B$ are joined by an edge if and only if $E(b) \cap E(b')$ is infinite. If G' contained an infinite path $b_0, b_1, \ldots$ then we should be able to find distinct points $a_0, a_1, \ldots$ in A such that $a_n \in E(b_n) \cap E(b_{n+1})$. Then $b_0, a_0, b_1, a_1, \ldots$ is an infinite path in G contrary to the hypothesis of the theorem. Therefore, G' contains no infinite path and, by Case 2, it follows that there is a set $B_1 \subset B$ of type $\omega\beta$ which contains no edge of G'. This implies that $E(b) \cap E(b')$ is finite for every distinct pair of points $b, b' \in B_1$. Therefore, since $|B_1| = \aleph_0$, the set $A_1 = \{a \in A : |E(a) \cap B_1| \geq 2\}$ is countable and hence $A'_1 = A - A_1$ has type $\omega_1\gamma$ and each point of A'_1 is joined to at most one point of B_1.

Since $tp\, B_1 = \omega\beta$ and $2 \cdot \omega = \omega$, it follows that B_1 is the union of two disjoint sets B'_1, B'_2 having the same type $\omega\beta$. Let $A''_i = \{a \in A'_1 : E(a) \cap B'_i = \phi\}$ $(i = 1, 2)$. Then $A'_1 = A''_1 \cup A''_2$ and, since γ is indecomposable, there is $i \in \{1, 2\}$ such that $tp\, A''_i = \omega_1\gamma$. Then $A''_i \cup B'_i$ is an independent set of type $\omega_1\gamma + \omega\beta$.

Now suppose that γ is decomposable. In this case we have that $A = A_0 \cup A_1 \cup \ldots \cup A_k\ (<)$, where $k < \omega$, $tp\, A_i = \omega_1\gamma_i$ and γ_i is indecomposable. Applying the previous argument k times we find sets $A'_i \subset A_i$ and $B' \subset B$ such that $tp\, A'_i = \omega_1\gamma_i$ $(i \leq k)$, $tp\, B' = \omega\beta$ and $A'_1 \cup \ldots \cup A'_k \cup B'$ is independent. This completes the proof.

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