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Some extremal problems in graph theory

by

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We consider only graphs without loops and multiple edges. G^n denotes a graph of n vertices, $v(G)$, $e(G)$ and $\chi(G)$ denote the number of vertices, edges, and the chromatic number of the graph G respectively. The star of a vertex x will be denoted by $\mathrm{st}\,x$ (that is the set of vertices joined to x), the valency of x will be denoted by $\sigma(x)$. $K(m,n)$ denotes the complete bichromatic graph with m and n vertices in its classes. $\{K(m,n)-r\}$ is the graph obtained from $K(m,n)$ omitting r $(r \le \min(m,n))$ independent edges. Thus $\{K(4,4)-4\} = C$ is the graph formed by the vertices and edges of a cube.

Let us denote by $f(n; L_1, \dots, L_\lambda)$ the maximum number of edges a graph G^n can have if it does not contain any L_i as a subgraph.

If it does not cause any confusion, $f(n; L_1, \dots, L_\lambda)$ will be abbreviated by $f(n)$.

According to [1]

(1) $$f(n; K(\ell, m)) = O_m(n^{2-\frac{1}{\ell}}) \quad (\ell \le m)$$

This result is sharp if $\ell = 2,3$ [1] [2]

(2) $$f(n; K(2,2)) = (1+o(1))\frac{n^{3/2}}{2}$$

(3) $$c_{3,\ell}\, n^{5/3} \le f(n; K(3,\ell)) \le c'_{3,\ell}\, n^{5/3}$$

Hereinafter $c, c_i, c_{ij}, \dots$ will denote positive absolute constants, and if c_i is used in different formulas, it can have different values.)

TURÁN asked for the determination or estimation of $f(n;C)$. A very special case of a result of ERDŐS gives [3]

$$f(n;C) < cn^{5/3} \quad .$$

Further ERDŐS showed that [4]

$$c_1 n^{3/2} < f(n; \{C\} - \{x\}) < c_2 n^{3/2}$$

where $\{C\} - \{x\}$ is the graph, obtained by omitting a vertex from the cube. We shall prove a stronger assertion, namely

(4) $$c_3 n^{3/2} < f(n; \{\, C-1 \,\}) < c_4 n^{3/2}$$

where $\{C-1\}$ is the graph, obtained by omitting an edge from the cube. ERDŐS conjectured that $n^{5/3}$ is also the lower bound for C, but this conjecture is false. In fact,

(5) $$f(n;C) \le O(n^{8/5})$$

ERDŐS conjectured [5] that for any graph L, if $\chi(L) = 2$, then there exists an $\alpha = \alpha(L)$ such that

(6) $$\lim_{n\to\infty} f(n;L)/n^{\alpha}$$

exists, and perhaps

$$(7) \qquad \alpha = 1 + \frac{1}{k} \qquad \text{or} \qquad \alpha = 2 - \frac{1}{k}$$

would also hold in all cases (where k is an integer). This conjecture is disproved by the following results of this paper:

Let $D(k,\ell)$ be the graph, obtained by joining two given vertices x and y by k independent paths of length ℓ and let $E(t,k,\ell)$ be the graph obtained by joining each vertex of the first colour of a $K(t,t)$ to each one of the first colour in $D(k,\ell)$ and joining the vertices of the second colour of $K(t,t)$ to the vertices of the second colour in $D(k,\ell)$. ($D(k,\ell)$ can be coloured by 2 colours in exactly one way, thus $E(t,k,\ell)$ is well determined). Then

$$(8) \quad c_{k,t}\, n^{2-\frac{2k+2t}{3k+t^2+2t(k+1)-1}} \le f(n; E(t,k,3)) \le c'_{k,t}\, n^{2-\frac{2}{2t+3}}$$

where the exponent in the lower estimation tends to the exponent in the upper one, therefore (7) does not always hold.

Besides, we shall also obtain that

$$(9) \qquad f(n; E(t,2,\ell)) = O\left(n^{2-\frac{\ell-1}{t(\ell-1)+\ell}}\right)$$

Further, we obtain that

$$(10) \qquad f(n; \{K(r,r)-3\}) = O\left(n^{2-\frac{2}{2r-3}}\right)$$

which is a generalization of (5). (10) is trivial from $\{K(r,r)-3\} = E(r-3,2,3)$ and from (9).

All these results are obtained from two main theorems of the paper. In order to formulate Theorem 1, we need

DEFINITION 1. G_n is d-regular, if

$$d \min_{x \in G^n} \sigma(x) \geq \max_{x \in G^n} \sigma(x) \qquad (d \geq 1).$$

THEOREM 1. If $e(G^n) \geq n^{1+\alpha}$ and $d = 10 \cdot 2^{\frac{1}{\alpha^2}+1}$ then $G^{(n)}$ contains a d-regular subgraph G^m such that

$$e(G^m) \geq \frac{2}{5} m^{1+\alpha}$$

and $m \geq n^{\alpha \frac{1-\alpha}{1+\alpha}}$ unless n is too small.

COROLLARY. If $f_d(n)$ denotes the maximum number of edges a d-regular graph can have if it does not contain any L_i, $\left(d = 10 \cdot 2^{\frac{1}{\alpha^2}+1}\right)$ and if

$$f_d(n) = O(n^{1+\alpha})$$

then

$$f(n; L_1, \dots, L_\lambda) = O(n^{1+\alpha})$$

THEOREM 2. Let $L(t)$ be a graph obtained from L where $\chi(L) = 2$ by joining each vertex of its i-th class to each vertex of the i-th class of $K(t,t)$ $(i = 1,2)$.

If

(10) $$f(n; L) = O(n^{2-\alpha}) \qquad (\alpha \in (0,1])$$

and

(11) $$\frac{1}{\beta} - \frac{1}{\alpha} = t$$

then

(12) $$f(n; L(t)) = O(n^{2-\beta})$$

The latter theorem is a recursive one. It can be used for many estimations if $\chi(L) = 2$.

PROOFS.

PROOF of THEOREM 1. Let A be a large number and let us divide the vertices of G^n into $2A$ classes of equal size. (Hereinafter no difference will be made between x and $[x]$. This is allowed here.) The i-th class will be denoted by C_i and we may suppose that $x \in C_1$, $y \notin C_1$ implies that $\sigma(x) \geq \sigma(y)$ There are two cases.

a) If C_1 represents less than $\frac{1}{2} n^{1+\alpha}$ edges, we consider $G^{m_1} = G^n - C_1$. $e(G^{m_1}) \geq \frac{1}{2} n^{1+\alpha}$. If G^{m_1} contains a vertex x_1 of valence $\leq \frac{1}{10} n^{\alpha}$, we omit it, $G^{m_2} = G^{m_1} - \{x_1\}$.

If G^{m_2} contains a vertex x_2 of valence $\leq \frac{1}{10} n^{\alpha}$ we omit it, and so on. At last we cannot omit any x_j from G^{m_j} and since we omitted less than $\frac{1}{10} n^{1+\alpha}$ edges, $e(G^{m_j}) \geq \frac{2}{5} n^{1+\alpha}$. Further

$$2e(G) \leq v(G) \cdot \max_{x \in G} \sigma(x)$$

implies that

$$v(G^{m_j}) = m_j \geq \frac{4}{5A} n \tag{13}$$

Here we used that

$$\max_{x \in G^{m_j}} \sigma(x) \leq A n^{\alpha} \tag{14}$$

but (14) follows from the fact that C_1 contains a vertex x_0 of valency

$$\leq \frac{1}{2} n^{1+\alpha} \Big/ \frac{n}{2A} = A n^{\alpha}$$

and if $y \notin C_1$,

$$\sigma(y) \leq \sigma(x_0) = A n^{\alpha}.$$

Therefore, the maximal valence in G^{m_j} is less than An^{α} and the minimal one is greater than $\frac{1}{10} n^{\alpha}$. Thus G^{m_j} is $10A$-regular and

$$e(G^{m_j}) \geq \frac{2}{5} m_j^{1+\alpha}, \qquad m_j \geq \frac{4}{5A} \cdot n$$

In this case the theorem is proved.

b) If G represents more than $\frac{1}{2} n^{1+\alpha}$ edges, we consider the graphs G_j $(j = 2, \dots, 2A)$ spanned by $C_j \cup C_1$. Since

$$\sum e(G_j) \geq \frac{1}{2} n^{1+\alpha}$$

there exists a j_0 such that

$$e(G_{j_0}) \geq \frac{1}{2A} \cdot \frac{1}{2} n^{1+\alpha} = \frac{1}{4A} \cdot n^{1+\alpha}$$

Let us put $G_{j_0} = G^{m_1}$ and apply to it this splitting method again. Either we obtain a $10A$-regular graph by a) or a G^{m_2} by b) from it. In the latter case we apply the method to G^{m_2}, and so on.

We prove here that the iteration gives a G^{m_j} at last, which posseses the properties, described in Theorem 1. Clearly

$$e(G^{m_k}) \geq \frac{1}{(4A)^k} n^{1+\alpha}$$

and

$$m_k \approx \frac{1}{A^k} \cdot n$$

Since $v^2(G) > e(G)$ for every G,

$$(4A)^{-k} \cdot n^{1+\alpha} \leq n^2 \cdot A^{-2k}.$$

Thus

$$\left(\frac{A}{4}\right)^k \le n^{1-\alpha}$$

and consequently

$$k \le (1-\alpha)\log n / \log \frac{A}{4} .$$

On the other hand

$$\log m_k \approx \log \frac{n}{A^k} = \log n - k \log A .$$

Therefore

(15) $$\log m_k \ge \left(1-(1-\alpha)\frac{\log A}{\log \frac{A}{4}}\right) \log n .$$

Therefore, if $\alpha' < \alpha$ and A is large enough,

(16) $$m_k \ge n^{\alpha'}$$

This shows that the procedure stops at last, and the obtained graph G^m will have at least $n^{\alpha'}$ vertices, if A is large enough. Clearly G^m is $10A$ -regular and

$$e(G^m) \ge \frac{4}{5} m^{1+\alpha}$$

E.g. if $A = 2^{\alpha^{-2}+1}$, then a short computation shows that

(17) $$m \ge n^{\alpha \frac{1-\alpha}{1+\alpha}}$$

Here $d = 10 \cdot 2^{\alpha^{-2}+1}$, and this completes our proof.

PROOF of the COROLLARY. There exists a $c_1 > 1$ such that

(18) $$f_d(n) \le c_1 n^{1+\alpha}$$

Let now G^n be arbitrary, but having at least $3c_1 n^{1+\alpha}$ edges. According to Theorem 1, it contains a d-regular G^m such that $e(G^m) > c_1 m^{1+\alpha}$ and $m \geq n^{\varepsilon(\alpha)}$. According to (18) G^m contains at least one L_i, therefore G^n also contains an L_i. Thus

$$f(n) = O(n^{1+\alpha}).$$

PROOF of THEOREM 2. We use induction on t. Let $L_1 = L(t)$ and $L_2 = L_1(1)$. Then $L_2 = L(t+1)$. Therefore the induction is trivial, if we know the theorem for $t = 1$.

A triple (x, y, z) will be called a cap, if x, y are joined to z. First we estimate the number of caps in G^n by $e(G^n)$ then the number of $K(2,2)$ and finally the number of $K(2,2)$ containing a fixed edge. This will give the recursion.

Let G be a graph of E edges, $E > 1000\, n^{3/2}$ and let it be d-regular, where d is a large integer fixed later. The valency of x_i is $\sigma(x_i)$, therefore the number of caps is

$$N_c = \sum_{x_i \in G^n} \binom{\sigma(x_i)}{2} = \frac{1}{2} \sum (\sigma^2(x_i) - \sigma(x_i)) \geq$$

$$\geq \frac{1}{2} \frac{(\sum \sigma(x_i))^2}{n} - \frac{1}{2} \sum \sigma(x_i) = \frac{1}{2}\left(\frac{4E^2}{n} - E\right) \geq \frac{E^2}{n}$$

for sufficiently large n. Here we applied the Cauchy inequality.

Let us denote by $v(x, y)$ the number of caps (x, y, z) and by N_k the number of $K(2,2)$ in G^n. Clearly

$$N_k = \frac{1}{2}\sum_{x,y}\binom{v(x,y)}{2} = \frac{1}{2}\sum\left(\frac{v^2(x,y)}{2} - \frac{v(x,y)}{2}\right) \geq$$

(19)

$$\geq \frac{1}{4}\,\frac{\left(\sum v(x,y)\right)^2}{\binom{n}{2}} - \frac{1}{4}\sum v(x,y) = \frac{1}{4}\left(\frac{N_c^2}{\binom{n}{2}} - N_c\right) > \frac{1}{3}\,\frac{N_c^2}{n^2}$$

since $N_c \geq 10^6 \cdot n^2$. Thus

$$N_k > \frac{1}{3}\,\frac{E^4}{n^4} .$$

Therefore, there exists an edge (x,y) such that (x,y) is contained in at least

$$\frac{1}{3}\,\frac{E^3}{n^4}$$

$K(2,2)$ -s. Since the graph is d-regular, $\sigma(x), \sigma(y) \leq \frac{2dE}{n}$ and the even graph, obtained by considering only the edges joining $\mathrm{st}\,x$ to $\mathrm{st}\,y$, has at least

$$\frac{1}{3}\,\frac{E^3}{n^4}$$

edges. If G^n does not contain $L(1)$, then this graph does not contain L, thus

$$\frac{1}{3}\,\frac{E^3}{n^4} \leq f\left(\frac{4dE}{n}; L\right) \leq c\left(\frac{4dE}{n}\right)^{2-\alpha}$$

Let

(20) $$\frac{1}{\beta} - \frac{1}{\alpha} = 1$$

and $a = 10 \cdot 2^{\beta^{-2}+1}$. Then

$$c(4d)^{2-\alpha} \cdot E^{2-\alpha} \cdot n^{\alpha-2} \geq 3^{-1} \cdot E^3 \cdot n^{-4} .$$

Thus we obtain

$$c_1 n^{2+\alpha} \geq E^{1+\alpha} \tag{21}$$

where c_1 is a constant, depending on c, d and α. (20) and (21) imply that

$$E \leq c_2 n^{2-\beta}$$

i.e.

$$f_d(n; L(1)) = O(n^{2-\beta})$$

Applying Theorem 1 we obtain

$$f(n; L(1)) = O(n^{2-\beta})$$

which was to be proved

APPLICATIONS.

1) Let us denote by $D(k,\ell)$ the graph obtained by joining two given vertices x and y by k independent paths of length ℓ.

An unpublished result of Erdős states that

$$f(n; D(2,\ell)) = O(n^{1+\frac{1}{\ell}})$$

Let us denote by $E(t,k,\ell)$ the graph, obtained by joining t vertices of the first class of a $K(t,t)$ to the vertices of $D(k,\ell)$ having the same colour, the other vertices of $K(t,t)$ to the other vertices of $D(k,\ell)$. (I.e., if $L = D(k,\ell)$ then $E(t,k,\ell) = L(t)$). If L is connected and bichromatic, $L(t)$ is uniquely determined, thus $E(t,k,\ell)$ is also uniquely determined. According to Theorem 2

$$f(n; E(t,2,\ell)) = O(n^{2-\frac{\ell-1}{t(\ell-1)+\ell}})$$

On the other hand, the method used in [7] gives that

$$f(n;L) \geq n^{2-\frac{v-2}{e-1}} \tag{23}$$

where $v = v(L), \qquad e = e(L)$

Therefore

$$f(n; E(t,2,\ell)) \geq C_{t,2,\ell}\, n^{2-2\frac{t+\ell-1}{t^2+2t\ell+2\ell-1}}$$

If $\ell \to \infty$, t is fixed, the exponents tend to $2-\frac{1}{t}$ and $2-\frac{1}{t+1}$ respectively. In this sense the estimations are not too bad.

2) It can be proved that

$$f(n; D(k,3)) = O(n^{4/3})$$

Therefore

$$f(n; E(t,k,3)) = O(n^{2-\frac{2}{2t+3}}).$$

(23) gives

$$f(n; E(t,k,3)) \geq c_{t,k,3}\, n^{2-\frac{2k+2t}{3k+t^2+2t(k+1)-1}} . \tag{24}$$

If t is fixed, $k \to \infty$, the exponent in (24) tends to $2-\frac{2}{2t+3}$. This shows that the estimation is not too bad and at the same time it disproves the conjecture of Erdős, mentioned in the introduction. If $k=2$ we obtain estimations for $f(n;C)$, where C is the cube, moreover, for $f(n;\{K(4,4)-3\})$ (see in the introduction) but (24) is not better than the trivial $n^{3/2}$ obtained from (2). (Clearly, if $L_1 \subset L_2$, $f(n;L_1) \leq f(n;L_2)$ and here $C \supset K(2,2)$.)

3) Though it is not a new result, it is interesting to see that (1) is an immediate consequence of Theorem 2. Indeed, if $L = K(1,1)$ and $t = r-1$, then $L(t) = K(r,r)$ thus

$$\frac{1}{\alpha_r} - \frac{1}{\alpha_1} = r-1$$

defines an α_r such that

$$f(n; K(r,r)) = O(n^{2-\alpha_r})$$

Since $\alpha_1 = 1, \frac{1}{\alpha_r} = r$ and this gives (1). Since (1) is sharp if $r = 1,2,3$, thus Theorem 2 is the best possible in a certain sense.

4) Let T be any tree, then $f(n,T) = O(n)$. Therefore

$$f(n; T(t)) = O(n^{2-\frac{1}{t+1}}).$$

If e.g. T is a path of length 5, then $f(n; T(1)) = O(n^{3/2})$ But $T(1) = \{C-1\}$ and this proves (4).

Here we stop our investigations, though Theorem 2 has many further applications.

(The reader can easily check that all the results, stated in the introduction, were really proved.)

OPEN PROBLEMS

By the method of random graphs we can show that for every d and ε there is G^n, $e(G^n) = [n^{3/2}]$, which does not have a d-regular subgraph G^m such that $e(G^m) \geq \varepsilon\sqrt{n}\, m$.

Many open problems remain, we just state two of them: Is it true that for every ε and α if $n > n_0(\varepsilon,\alpha)$ and $d > d\cdot(\varepsilon,\alpha)$ every G^n $e(G^n) > n^{1+\alpha}$ contains a d-regular subgraph

$$G^m, \quad m > n^{1-\alpha}, \quad e(G^m) > \varepsilon m^{1+\alpha} \ ?$$

Is it true that every G^n, $e(G^n) = [n \log n]$ contains a d-regular subgraph G^m, $e(G^m) > \varepsilon m \log m$ where m tends to infinity together with n?

It would be interesting to determine the correct order of magnitude of $f(n, C)$.

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