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SOME RESULTS AND PROBLEMS ON CERTAIN POLARIZED PARTITIONS

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§ 1. Introduction. Notation

1. 1. A short list of general notations

$\alpha, \beta, \gamma, \delta$ denote cardinals. $\xi, \zeta, \eta, \mu, \nu, \sigma, \varrho$ denote ordinals. $|A|$ is the cardinality of the set A. α^+ is smallest cardinal greater than α. ω_ξ is the sequence of infinite cardinals $\omega_0 = \omega$. i, j, r, s, l, k denote integers (cardinals $<\omega$). α is a strong limit cardinal if $2^\beta < \alpha$ for every $\beta < \alpha$. For $\alpha \geqq \omega$ $\mathrm{cf}(\alpha)$ is the least cardinal cofinal with α

$$[A]^\alpha = \{X: X \subset A \wedge |X| = \alpha\}, \ [A]^{<\alpha} = \{X: X \subset A \wedge |X| < \alpha\}.$$

For the convenience of the reader we recall the definition of some of the partition symbols defined in earlier papers [1], [2], [3].

DEFINITION 1. 1. 1. *The ordinary partition symbol.* $\alpha \to (\beta_\nu)^\delta_{\nu<\gamma}$ denotes that the following statement is true.

Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma} I_\nu$ then there are $A \subset \alpha$, $\nu < \gamma$ such that $|A| = \beta_\nu$, $[A]^\delta \subset I_\nu$.

Here and for all other symbols to be defined $\alpha \nrightarrow (\beta_\nu)^\delta_{\nu<\gamma}$ denotes the negations of the corresponding statements. $\alpha \to (\beta)^\delta_\gamma$ denotes $\alpha \to (\beta_\nu)^\delta_{\nu<\gamma}$ where $\beta_\nu = \beta$ for $\nu < \gamma$.

We use some other self explanatory abbreviations which are defined in detail in [2].

Note that the ordinary partition symbol and some of the other symbols can be defined for types instead of ordinals in a natural way. If α; β_ν, $\nu < \gamma$ are types, $\alpha \to (\beta_\nu)^\delta_\gamma$ means the following:

Whenever $A, \prec$ is an ordered set, $\mathrm{tp}(\prec) = \alpha$ and $[A]^\delta = \bigcup_{\nu<\gamma} I_\nu$, then there are $A' \subset A$, $\nu < \gamma$ such that $\mathrm{tp}\, A'(\prec) = \beta_\nu$ and $[A']^\delta \subset I_\nu$.

Since we do not investigate these problems here, we will give all the definitions for cardinals.

DEFINITION 1. 1. 2. *The polarized partition symbol.* Let $r, s < \omega$; $r = r_0 + \dots + r_{s-1}$. Let α_i, $\beta_{i,\nu}$ be cardinals for $i < s$, $\nu < \gamma$.

$\begin{pmatrix} \alpha_0 \\ \cdots \\ \alpha_{s-1} \end{pmatrix} \to \begin{pmatrix} \beta_{0,\nu} \\ \cdots \\ \beta_{s-1,\nu} \end{pmatrix}^{r_0, \dots, r_{s-1}}_{\nu<\gamma}$ means that the following statement is true.

Whenever

$$[\alpha_0]^{r_0} \times \dots \times [\alpha_{s-1}]^{r_{s-1}} = \bigcup_{\nu<\gamma} I_\nu$$

then there exist sets $A_i \subset \alpha_i$, $i < s$ and $\nu < \gamma$ such that

$$[A_0]^{r_0} \times \dots \times [A_{s-1}]^{r_{s-1}} \subset I_\nu \quad \text{and} \quad |A_i| = \beta_{i,\nu} \quad \text{for} \quad i < s.$$

DEFINITION 1. 1. 3. $\alpha \to (\beta)_\gamma^{<\omega}$ means that the following statement is true: Whenever

$$[\alpha]^r = \bigcup_{\nu<\gamma} I_\nu^r \quad \text{for} \quad r < \omega$$

then there are $A \subset \alpha$ and $f \in {}^\omega\gamma$ such that $|A| = \beta$ and $[A]^r \subset I_{f(r)}$ for $r < \omega$.

DEFINITION 1. 1. 4. $\alpha \to [\beta_\nu]_{\nu<\gamma}^\delta$ means that the following statement is true. Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma} I_\nu$ then there are $A \subset \alpha$, $\nu_0 < \gamma$ such that

$$|A| = \beta_{\nu_0} \quad \text{and} \quad A \subset \bigcup_{\nu<\gamma, \nu \neq \nu_0} I_\nu.$$

If $\beta_\nu = \beta$ for $\nu < \gamma$, we write $\alpha \to [\beta]_\gamma^\delta$.

DEFINITION 1. 1. 5. $\alpha \to [\beta]_{\gamma_0, \gamma_1}^\delta$ means that the following statement is true: Whenever $[\alpha]^\delta = \bigcup_{\nu<\gamma_0} I_\nu$ then there are $A \subset \alpha$ and $C \subset \gamma_0$ such that $|A| = \beta$, $|C| \leqq \gamma_1$ and $A \subset \bigcup_{\nu \in C} I_\nu$. The symbols defined in 1. 1. 4, 1. 1. 5 are the "square bracket" symbols corresponding to the ordinary partition symbol. Quite similarly two square bracket symbols

$$\begin{pmatrix}\alpha_0 \\ \cdots \\ \alpha_{s-1}\end{pmatrix} \to \begin{bmatrix}\beta_{0,\nu} \\ \cdots \\ \beta_{s-1,\nu}\end{bmatrix}^{r_0, \ldots, r_{s-1}}, \quad \begin{pmatrix}\alpha_0 \\ \cdots \\ \alpha_{s-1}\end{pmatrix} \to \begin{bmatrix}\beta_0 \\ \cdots \\ \beta_{s-1}\end{bmatrix}^{r_0, \ldots, r_{s-1}}_{\gamma_0, \gamma_1}$$

can be defined corresponding to the polarized partition symbol.

The symbols defined in 1. 1, 1. 2, 1. 3 were defined in [2], where we gave a detailed discussion of the ordinary partition symbol and the special case. $s=2$, $r=2$, $r_0 = r_1 = 1$, $\gamma = 2$ of the polarized partition symbol.

The aim of the present is to consider the special cases

$$\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_{0,\nu} \\ \beta_{1,\nu}\end{pmatrix}^{1,r}, \quad 2 \leqq r < \omega$$

of the polarized partition symbol, mainly in case $r=2$, $\gamma=2$ and some related problems.

1. 2 A new notation for the main problems considered

1. 2. 1. $\begin{pmatrix}\alpha_0 \\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_{0,\nu} \\ \beta_{1,\nu}\end{pmatrix}^{1,r}_{\nu<\gamma}$ is obviously equivalent to the following statement:

Let $[\alpha_1]^r = \bigcup_{\nu<\gamma} I_\nu^\xi$ for $\xi < \alpha_0$. Then there are $A_0 \subset \alpha_0$, $A_1 \subset \alpha_1$, $\nu < \gamma$ such that $|A_0| = \beta_0, \nu$, $|A_1| = \beta_{1,\nu}$ and

$$[A_1]^r \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

If $\mathscr{B}_\nu = \{[A]^r : A \subset \alpha_1 \wedge |A| = \beta_{1,\nu}\}$ then 1. 2. 1 can be expressed as follows: There are $A_0 \subset \alpha$, $\nu < \gamma$ such that $|A_0| = \beta_{0,\nu}$ and there is an $X \in \mathscr{B}_\nu$ such that

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

We will consider the more general problem when $\mathscr{B}_\nu$ can be more general classes.

DEFINITION 1. 2. 2. Let $\alpha_0, \alpha_1; \beta_\nu, \nu<\gamma$ be cardinals $r<\omega$, and let $\mathscr{B}_\nu, \nu<\gamma$ be a sequence, where $\mathscr{B}_\nu \subset \mathscr{P}([\alpha_1]^r)$. Then $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_\nu\\ \mathscr{B}_\nu\end{pmatrix}^{1,r}_{\nu<\gamma}$ means that the following statement is true. Whenever

$$[\alpha_1]^r = \bigcup_{\nu<\gamma} I_\nu^\xi \quad \text{for} \quad \xi < \alpha_0$$

then there are $A_0 \subset \alpha_0$, $\nu<\gamma$ such that $|A_0| = \beta_\nu$ and there is $X \in \mathscr{B}_\nu$ with

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

We will write

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0 & \beta_1\\ \mathscr{B}_0 & \mathscr{B}_1\end{pmatrix}^{1,r} \quad \text{for} \quad \begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_\nu\\ \mathscr{B}_\nu\end{pmatrix}^{1,r}_{\nu<2}.$$

Note that an $X \subset [\alpha_1]^2$ can be considered as a graph $\langle \alpha, X\rangle$ whose vertices and edges are the elements of α and the elements of X, respectively. We will sometimes use graph terminology for expressing certain properties of such classes.

We will sometimes use the $\vee$ (or) sign in the symbol; e.g.

$$\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0' & \beta_0'' & \beta_1' & \beta_1''\\ \mathscr{B}_0' \vee \mathscr{B}_0'', & \mathscr{B}_1' \vee \mathscr{B}_1''\end{pmatrix}^{1,r}$$

has the following self explanatory meaning.

Whenever

$$[\alpha_1]^r = \bigcup_{\nu<2} I_\nu^\xi \quad \text{for} \quad \xi < \alpha_0$$

then there are $A_0 \subset \alpha$, $\nu<2$ such that either $|A_0| = \beta_\nu'$ and there is an $X \in \mathscr{B}_\nu'$ for which

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi,$$

or $|A_0| = \beta_\nu''$ and there is an $X \in \mathscr{B}_\nu''$ for which

$$X \subset \bigcap_{\xi \in A_0} I_\nu^\xi.$$

1. 3 About the results

Though we have defined above a general symbol which can be used to express the results and problems we are going to state, it will be clear to everyone familiar with the subject that a systematical discussion of all the problems involved is hopeless presently (and perhaps would not even be worth while). We came across the special cases of these problems when working on ordinary partition problems. Some of the results are ten years old, some are new and give the solution ot several porblems stated in our paper [3].

We will consider different instances in different chapters and we give short summaries there.

Though most of the results will only interest those who know the basic results un partition relations in detail, there will be some simple unsolved problems which seem to be fundamental. Trying to clear these problems up we will prove some obviously not final partial results too.

We mention that Theorems 4. 1, 4. 3, 6. 1, 6. 3 give solution of Problems 61, 59, 60 stated in [3], respectively.

§ 2. A positive result for measurable α

DEFINITION 2. 1. $\mathscr{B}_{\alpha,\gamma,\delta}$ will denote the class of complete γ, δ even graphs with set of vertices α i.e.

$$\mathscr{B}_{\alpha,\gamma,\delta}=\{X\subset[\alpha]^2:\exists C, D(C\subset\alpha\wedge D\subset\alpha\wedge C\cap D=\bigcirc\wedge|C|=$$
$$=\gamma\wedge|D|=\delta\wedge(\{\xi,\eta\}\in X\Leftrightarrow\zeta\in C\wedge\eta\in D))\}.$$

If $X\in\mathscr{B}_{\alpha,\gamma,\delta}$ we write $X=[C, D]$.

2. 2. *Let* $\alpha\geqq\omega$ *and* $\mathscr{B}^0_\alpha=\{X\subset[\alpha]^2:\langle\alpha, X\rangle$ *contains an odd circuit*$\}$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \alpha\\ \mathscr{B}^0_\alpha, & \mathscr{B}_{\alpha,1,\alpha}\end{pmatrix}^{1,2}.$$

PROOF. Put $I_0^\xi=\{\{\zeta,\eta\}\in[\alpha]^2:\zeta\leqq\xi<\eta\}$ $I_1^\xi=[\alpha]^2-I_0^\xi$ for $\xi<\alpha_0$. Then $I_0^\xi\notin\mathscr{B}^0_\alpha$. If $X\in\mathscr{B}_{\alpha,1,\alpha}$ i.e. $X=[\{\xi_0\}, D]$, $|D|=\alpha$ then $X\cap I_0^\xi\neq 0$ for $\xi\geqq\xi_0$.

Note that if $\mathscr{B}^1_\alpha=\{X\subset[\alpha]^2: X$ contains α edges$\}$ then one can prove e.g.

$$\text{2. 3.}\qquad\begin{pmatrix}\omega\\ \omega\end{pmatrix}\rightarrow\begin{pmatrix}\omega & \omega\\ k, & \mathscr{B}^1_\omega\end{pmatrix}^{1,2}\quad for\quad k<\omega,$$

which shows that in 2. 2 $\mathscr{B}_{\alpha,1,\alpha}(\subset\mathscr{B}^1_\alpha)$ can not be replaced by $\mathscr{B}^1_\alpha$. We omit the routine proof.

2. 4. *Let* $\alpha\geqq\omega$. *Put* $\mathscr{B}^2_\alpha=\{X\subset[\alpha]^2: X\neq\bigcirc\}$. *We have*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \alpha\\ \mathscr{B}^1_\alpha, & \mathscr{B}^2_\alpha\end{pmatrix}^{1,2}.$$

PROOF. For $\xi<\alpha$ put $I_0^\xi=[\xi]^2$, $I_1^\xi=[\alpha]^2-I_0^\xi$.

Then $|I_0^\xi|<\alpha$, hence $I^0_\xi\notin B^1_\alpha$ and if $X\neq 0$ then there is $\xi_0<\alpha$ such that $X\cap I_0^\xi\neq 0$ for $\xi\geqq\xi_0$.

The above negative results suggest the formulation of the following property:

2. 5. $\mathbf{P}(\alpha)\Leftrightarrow$ for every $\beta,\gamma<\alpha$, $\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}1 & 1 & \alpha\\ \beta\vee & \mathscr{B}_{\alpha,\gamma,\alpha}, & \alpha\end{pmatrix}^{1,2}$.

Our main result in this chapter is

THEOREM 1. 1. *Let* $\alpha\geqq 0$ *be* 0, 1-*measurable. Then for every* $\beta,\gamma<\alpha$

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}\alpha & 1 & \alpha\\ \beta\vee & \mathscr{B}_{\alpha,\gamma,\alpha}, & \alpha\end{pmatrix}^{1,2}.$$

As a corollary of this, $\mathbf{P}(\alpha)$ holds for measurable α. Before proving the theorem we mention

PROBLEM 1. a) Can one prove Theorem 1. 1 under some weaker hypothesis than the measurability of α? (E.g. $\alpha \to (\alpha)_2^2$?)

b) Is $\mathbf{P}(\alpha)$ true for the first strongly inaccessible cardinal?

Note that we will prove that $\mathbf{P}(\alpha)$ is false for cardinals not strongly inaccessible.

PROOF OF THEOREM 1. 1. By the definitions 1. 2. 1, 1. 2. 2 it is obviously sufficient to prove the following statement.

Let I_ξ, $\xi < \alpha$ be a sequence of type α, where $I_\xi \subset [\alpha]^2$. Assume that the following conditions (1), (2) hold:

(1) If $A, B \subset \alpha$, $A \cap B = 0$, $|A| = \gamma$, $|B| = \alpha$ then $[A, B] \not\subset I_\xi$ for $\xi < \alpha$.

(2) If $A, B \subset \alpha$, $|A| = \beta$, $|B| = \alpha$ then $[A]^2 \not\subset \bigcup_{\xi \in B} I_\xi$.

Then there are $C, D \subset \alpha$, $|C| = |D| = \alpha$ such that

$$[C]^2 \cap I_\xi = 0 \quad \text{for} \quad \xi < \alpha.$$

Let μ denote a non-trivial α-complete 0, 1-valued measure on α. For each $P \in [\alpha]^2$ put

(3) $$N(P) = \{\xi \in \alpha \colon P \in I_\xi\}.$$

Put

(4) $$I = \{P \in [\alpha]^2 \colon \mu(N(P)) = 1\}.$$

α being strongly inaccessible we have $\alpha \to (\beta, \alpha)^2$ (see [1]). Applying this for the partition $[\alpha]^2 = I \cup ([\alpha]^2 - I)$ and using the assumption (2) we obtain that

(5) There is an $A_0 \subset \alpha$, $|A_0| = \alpha$, $[A_0]^2 \cap \mathrm{I} = 0$ i.e.

$$\mu(N(P)) = 0 \quad \text{for each} \quad P \in [A_0]^2.$$

Let now μ' be a non trivial α-complete 0, 1-valued measure on A_0.

Put

(6) $U_\xi(x) = \{y \in A_0 \colon \{x, y\} \in I_\xi\}$ for every $x \in A_0$, $\xi < \alpha$

$M(x) = \{\xi < \alpha \colon \mu'(U_\xi(x)) = 1\}$ for every $x \in A_0$

$T = \{x \in A_0 \colon \mu(M(x)) = 1\}$

$T_\xi = \{x \in A_0 \colon \xi \in M(x)\}$ for every $\xi < \alpha$.

We prove

(7) $|T_\xi| < \gamma$ for every $\xi < \alpha$. In fact if $T' \subset T_\xi$ $|T'| = \gamma$ then $\mu'\left(\bigcap_{x \in T'} U_\xi(x)\right) = 1$ and $\left[T', \bigcap_{x \in T'} U_\xi(x) - T'\right] \subset I_\xi$ which contradicts the assumption (1).

It follows that

(8) $|T| < \gamma$, for if $T' \subset T$, $|T'| = \gamma$ then there is a $\xi \in \bigcap_{x \in T'} M(x)$ and $T' \subset T_\xi$ for this ξ.

Put $A_1 = A_0 - T$. By (5) and (6) and (8)

(9) $\mu'(A_1)=1$, $\mu(N(P))=0$ for every $P\in[A_1]^2$, $\mu(M(x))=0$ for every $x\in A_1$.
We define the sequences $\{x_\varrho\}_{\varrho<\alpha}\subset A_1$, $\{\xi_\varrho\}_{\varrho<\alpha}\subset\alpha$ by induction on ϱ as follows. Assume that $\varrho<\alpha$, X_σ and ξ_τ are already defined and $\xi_\tau\notin M(X_\sigma)$ for $\tau,\sigma<\varrho$. Then by (8) and (9)

$$\mu'\Big(\bigcup_{\tau<\varrho}\bigcup_{\sigma<\varrho}U_{\xi_\tau}(x_\sigma)\Big)=0$$

hence by (7) there is an $X_\varrho\in A_1$, $X_\varrho\neq X_\sigma$ for $\sigma<\varrho$ such that

$$x_\varrho\notin\bigcup_{\tau<\varrho}\bigcup_{\sigma<\varrho}U_{\xi_\tau}(x_\sigma)\cup\bigcup_{\tau<\varrho}T_{\xi_\tau}. \tag{10}$$

By (9) the set

$$\bigcup_{\tau<\sigma\leqq\varrho}N(\{x_\tau,x_\sigma\})\cup\bigcup_{\sigma\leqq\varrho}M(x_\sigma)$$

has μ measure 0, hence there is a $\xi_\varrho<\alpha$, $\xi_\varrho\neq\xi_\tau$ for $\tau<\varrho$ such that

$$\xi_\varrho\notin\bigcup_{\tau<\sigma\leqq\varrho}N(\{x_\tau,x_\sigma\})\cup\bigcup_{\sigma\leqq\varrho}M(x_\sigma). \tag{11}$$

By (10) and (11) $\xi_\tau\notin M(x_\sigma)$ holds for $\tau,\sigma,\leqq\varrho+1$ as well. Thus the sequences are defined. Put $C=\{x_\varrho\}_{\varrho<\alpha}$, $D=\{\xi_\varrho\}_{\varrho<\alpha}$. By the definition $|C|=|D|=\alpha$. Let $\tau<\sigma<\alpha$, $\varrho<\alpha$. We prove $\{x_\tau,x_\sigma\}\notin I_{\xi_\varrho}$. We distinguish two cases: (i) $\varrho<\sigma$, (ii) $\sigma\leqq\varrho$. By (10) $x_\sigma\notin U_{\xi_\varrho}(x_\tau)$ if (i) holds. By (11) $\xi_\varrho\notin N(\{x_\tau,x_\sigma\})$ if (ii) holds.

Hence by (3) and (6) $\{x_\tau,x_\sigma\}\notin I_{\xi_\varrho}$ in both cases. This proves that $[C]^2\cap I_\xi=0$ for every $\xi\in D$.

Now we prove a number of negative results which show that $\mathbf{P}(\alpha)$ is false for not strongly inaccessible cardinals.

2. 6. *Assume* $\mathrm{cf}(\alpha)<\alpha$ *and* $\alpha\geqq\omega$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha\\ \mathrm{cf}(\alpha) & \vee & \mathscr{B}^1_\alpha, & \alpha\end{pmatrix}^{1,2}.$$

We give the (trivial) proof in § 7 (Theorem 7. 1) where we discuss singular cardinals.

2. 7. $$\begin{pmatrix}\beta^+\\ \beta^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \beta^+\\ \omega & \vee & \mathscr{B}^1_{\beta^+}, & \beta^+\end{pmatrix}^{1,2}, \quad \textit{for}\quad \beta\geqq\omega.$$

PROOF. For each $\xi<\beta^+$ let $\prec_\xi$ be a well ordering of ξ such that $\mathrm{tp}\,\xi(\prec_\xi)\leqq\beta$. Put

$$I_0^\xi=\{\{\zeta,\eta\}\in[\xi]^2:\zeta<\eta\wedge\eta\prec_\xi\zeta\},\quad I_1^\xi=[\beta^+]^2-I_0^\xi.$$

Obviously I_0^ξ does not contain a complete ω graph, and $I_0^\xi\notin\mathscr{B}^1_{\beta^+}$. On the other hand if $X\subset\alpha$, $\mathrm{tp}X(<)=\beta+1$ then there is a $\xi_0<\beta$, $X\subset\xi_0$ and $X\cap I_0^\xi\neq\bigcirc$ for $\xi\geqq\xi_0$.

With a similar idea one gets

2. 8. *Assume* $\alpha\geqq\omega$, $\beta\nrightarrow(\gamma,\gamma)^2$ *for every* $\beta<\alpha$, $\mathrm{cf}(\alpha)\neq\mathrm{cf}(\gamma)$ *then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha\\ \gamma & \vee & \mathscr{B}^1_\alpha, & \gamma\end{pmatrix}^{1,2}.$$

PROOF. For each $\xi<\alpha$, let $I_0^\xi\subset[\xi]^2$ be such that the partition $[\xi]^2=I_0^\xi\vee([\xi]^2-I_0^\xi)$ establishes the negative partition relation $|\xi|\nrightarrow(\gamma,\gamma)^2$. Put $I_1^\xi=[\alpha]^2-I_0^\xi$. Obviously $I_0^\xi\notin B_\alpha^1$ and I_0^ξ does not contain a complete γ-graph. Assume $X\subset\alpha$, $|X|=\gamma$. Then by the assumption $\mathrm{cf}(\gamma)\neq\mathrm{cf}(\alpha)$, there are $Y\subset X$, $|Y|=\gamma$ and $\xi_0<\alpha$ such that $Y\subset\xi_0$. But then by the construction $I_0^\xi\cap[Y]^2$ for $\xi\geqq\xi_0$.

2. 9. COROLLARY. *Assume* $\alpha=(2^\beta)^+\geqq\omega$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha\\ \beta^+ & \vee & \mathscr{B}_\alpha^1, & \beta^+\end{pmatrix}^{1,2}.$$

PROOF by 2. 8 considering that $2^\beta\nrightarrow(\beta^+)_2^2$.

2. 10. COROLLARY. $\mathbf{P}(\alpha)$ *is false if* $\alpha\geqq\omega$ *is not strongly inaccessible.*

PROOF. By 2. 6 and 2. 7 we assume that α is regular, $2^\beta\geqq\alpha$ for some $\beta<\alpha$, and $\beta^+<\alpha$. Considering $2^\beta\nrightarrow(\beta^+)_2^2$ the statement follows from 2. 8.

Without using G.C.H. we could not prove stronger negative results. Assuming G.C.H., much stronger negative results will be proved in § 3.

It is obvious that many quantitative questions can be asked here; we point out one.

PROBLEM 2. Can one prove without assuming C.H. that

$$\begin{pmatrix}\omega_1\\ \omega_1\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \omega_1\\ 3 & \vee & \mathscr{B}_{\omega_1}^1, & \omega_1\end{pmatrix}^{1,2}?$$

This should be compared with 2. 7 and Theorems 3. 1 and 3. 2.

§ 3. Stronger counter examples for $P(\alpha^+)$, assuming G.C.H.

DEFINITON 3. 1. Let $\mathscr{B}_\alpha^3=\{X\subset[\alpha]^2;\langle\alpha,X\rangle$ contains a circuit$\}$,

$$\mathscr{B}_{\alpha^+}^4=\{X\subset[\alpha^+]^2:|X|=\alpha\}.$$

THEOREM 3. 1. *Assume G.C.H.,* $\alpha\geqq\omega$. *Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & & 1 & \alpha^+\\ \mathscr{B}_{\alpha^+}^1 & \vee & \mathscr{B}_{\alpha^+}^3, & \mathscr{B}_{\alpha^+}^4\end{pmatrix}^{1,2}.$$

PROOF. First we prove

(1) Assume $\{Y_\mu\}_{\mu<\alpha}$ is a sequence of type α of elements of $\mathscr{B}_{\alpha^+}^4$. Then there is a set $I\subset[\alpha^+]^2$, $|I|=\alpha$, such that the graph $\langle\alpha^+,I\rangle$ does not contain a circuit, and $I\cap Y_\mu\neq 0$ for $\mu<\alpha$.

First we define a sequence $\{P_\mu\}_{\mu<\alpha}$ of elements of $[\alpha^+]^2$ by transfinite induction on μ. Assume P_ν is defined for every $\nu<\mu$ for some $\mu<\alpha$. Then $\left|\bigcup_{\nu<\mu}P_\nu\right|<\alpha$, hence $Y_\mu-[\bigcup_{\nu<\mu}P_\nu]^2\neq 0$ and let P_μ be an element of it. Put $I=\{P_\mu\}_{\mu<\alpha}$. Then I obviously satisfies the requirements ot (1).

By G.C.H., there exists a well-ordering $\{X_\xi\}_{\xi<\alpha^+}=\mathscr{B}^4_{\alpha^+}$ of type α^+ of $\mathscr{B}^4_{\alpha^+}$. By (1) for each $\xi<\alpha^+$ there exists an $I^\xi_0\notin\mathscr{B}^1_{\alpha^+}\cup\mathscr{B}^3_{\alpha^+}$ such that $I^\xi_0\cap X_\zeta\neq 0$ for each $\zeta<\xi$. Put $I^\xi_1=[\alpha^+]^2-I^\xi_0$ for $\xi<\alpha^+$.

If $X\in\mathscr{B}^4_{\alpha^+}$, then $X=X_\zeta$ for some $\zeta<\alpha^+$ and $X\cap I^\xi_0\neq 0$ for $\zeta<\xi<\alpha^+$.

DEFINITION 3. 2. Let $\mathscr{B}^5_\alpha=\{X\subset[\alpha]^2: P\cap Q\neq$ for some $P\neq Q\in X\}$. $\mathscr{B}^6_\alpha=\{X\subset[\alpha]^2: X$ consists of α disjoint edges$\}$. Our next theorem is incomparable with Theorem 3. 1 since $\mathscr{B}^1_{\alpha^+}\cup\mathscr{B}^3_{\alpha^+}\subset\mathscr{B}^1_{\alpha^+}\cup\mathscr{B}^5_{\alpha^+}$ but $\mathscr{B}^6_{\alpha^+}\subset\mathscr{B}^4_{\alpha^+}$.

THEOREM 3. 2. *Assume G.C.H., $\alpha\geqq\omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\nrightarrow\begin{pmatrix}1 & \vee & 1 & \alpha^+\\ \mathscr{B}^1_{\alpha^+} & & \mathscr{B}^5_{\alpha^+}, & \mathscr{B}^6_{\alpha^+}\end{pmatrix}^{1,2}.$$

PROOF. First we prove

(1) If $\{Y_\mu\}_{\mu<\alpha}$ is a sequence of type α of elements of $\mathscr{B}^6_\alpha$ then there is an $I\subset[\alpha^+]^2$, $|I|\leqq\alpha$, which consists of disjoint pairs such that $Y_\mu\cap I\neq 0$ for every $\mu<\alpha$.

To prove this we define a sequence $\{P_\mu\}_{\mu<\alpha}\subset[\alpha^+]^2$ by transfinite induction on μ. Assume P_ν is defined for every $\nu<\mu$ for some $\mu<\alpha$. Then $\left|\bigcup_{\nu<\mu}P_\nu\right|<\alpha$. Considering $Y_\mu\in\mathscr{B}^6_{\alpha^+}$ there is a $P_\mu\in Y_\mu$ such that $P_\mu\cap\bigcup_{\nu<\mu}P_\nu=0$, $I=\{P_\mu\}_{\mu<\alpha}$ satisfies the requirements of (1). By G.C.H., there exists a well-ordering $\{X_\xi\}_{\xi<\alpha^+}=$ $=\mathscr{B}^6_{\alpha^+}$ of type α^+ of $\mathscr{B}^6_{\alpha^+}$. Applying (1) for $\{X_\zeta\}_{\zeta<\xi}$ for each $\xi<\alpha^+$ we obtain that there exists an $I^\xi_0\subset[\alpha^+]^2$, $|I^\xi_0|=\alpha$ $I^\xi_0\notin\mathscr{B}^1_{\alpha^+}\cup\mathscr{B}^5_{\alpha^+}$ such that

$$I^\xi_0\cap X_\zeta\neq 0 \quad \text{for every} \quad \zeta<\xi.$$

Put $I^\xi_1=[\alpha^+]^2-I^\xi_0$ to $\xi<\alpha^+$. If $X\in\mathscr{B}^6_{\alpha^+}$ then $X=X_\zeta$ for some $\zeta<\alpha^+$ and

$$I^\xi_0\cap X\neq 0 \quad \text{for every} \quad \xi>\zeta.$$

Our next theorem shows that Theorems 3. 1 and 3. 2 do not have a common generalisation.

THEOREM 3. 3. *Assume $\alpha\geqq\omega$, $\gamma^+<\alpha$. Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\rightarrow\begin{pmatrix}1 & \alpha\\ \mathscr{B}^5_\alpha, & \mathscr{B}_{\alpha,\gamma,\alpha}\end{pmatrix}^{1,2}.$$

PROOF. Let $[\alpha]^2=I^\xi_0\cup I^\xi_1$ for $\xi<\alpha$.

Assume $I^\xi_0\notin\mathscr{B}^5_\alpha$ for every $\xi<\alpha$. Let $C\subset\alpha$, $|C|=\gamma$.

Put $B=\alpha-C$. For each $\xi<\alpha$ let

(1) $B_\xi=\{y\in B$: there is an $x\in C$ such that $\{x,y\}\in I^\xi_0\}$.

Considering $I^\xi_0\notin\mathscr{B}^5_\alpha$, we have $|B_\xi|\leqq\gamma$ for every $\xi<\alpha$.

By a theorem of G. FODOR [7] then there exist $D\subset B$, $A\subset\alpha$ such that $|D|=|A|=\alpha$ and $D\cap B_\xi=0$ for $\xi\in A$.

Then $[C,D]\in\mathscr{B}_{\alpha,\gamma,\alpha}$ and by (1) $[C,D]\subset I^\xi_1$ for every $\xi\in A$.

As to the counterexamples in Theorem 3. 1, 3. 2, it is obvious that assuming G.C.H., neither of the classes $\mathscr{B}^4_{\alpha^+}$, $\mathscr{B}^6_{\alpha^+}$ can be replaced by a class containing graphs with fewer than α edges since then α^+ graphs coincide on a set of power γ (where $\gamma<\alpha$) and the problems are reduced to ordinary partition problems.

§ 4. Further counterexamples (assuming G.C.H).

In this § we consider problems of the type

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha\\ \mathscr{B}_0, & \mathscr{B}_1\end{pmatrix}^{1,2}$$

where the complement of B_0 consists of very small graphs and the graphs in $\mathscr{B}_1$ have α^+ edges.

Though most of the results are negative and technically complicated to prove, they are surprisingly sharp. That is why we think it is worth to give them in detail.

DEFINITION 4. 1. $\mathscr{B}^7_\alpha = \{X \subset [\alpha]^2 : \langle \alpha, X\rangle$ contains an infinite path$\}$.

THEOREM 4. 1. *Assume G.C.H.,* $\alpha \geqq \omega$. *Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & \alpha\\ \mathscr{B}^3_{\alpha+} & \vee & \mathscr{B}^7_{\alpha+}, & \mathscr{B}^1_{\alpha+}\end{pmatrix}^{1,2}$$

i.e. in a set of power α^+ *we can define* α^+ *forests not containing infinite paths so that given* α^+ *edges all but less than* α *of the forests have an edge among the given edges.*

Note that $\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha\\ \mathscr{B}^3_{\alpha+}, & \mathscr{B}^1_{\alpha+}\end{pmatrix}^{1,2}$ could technically much simpler be proved.

PROOF. We will define a sequence $I_\xi \subset [\alpha^+]^2$, $\xi < \alpha^+$ with the intention that the partitions

(1) $$I_\xi = I^\xi_0,\ I^\xi_1 = [\alpha^+]^2 - I^\xi_0,\ [\alpha^+]^2 = I^\xi_0 \cup I^\xi_1$$

should establish the required counterexample.

For each $\varrho < \alpha^+$ we will define a function β_ϱ and its domain $D_\varrho \subset \varrho$, $\beta_\varrho \in {}^{D_\varrho}\varrho$ and we will put

(2) $$I_\xi = \{\{\beta_\varrho(\xi), \varrho\} : \xi \in D_\varrho \wedge \varrho < \alpha^+\} \quad \text{for every} \quad \xi < \alpha^+.$$

We will define β_ϱ, D_ϱ and a one-to-one mapping φ_ϱ of D_ϱ onto an ordinal $\leqq \alpha$ by transfinite induction on ϱ.

By G.C.H., there exists a well-ordering $\{R_\nu\}_{\nu<\alpha^+} = [\alpha^+]^\alpha$ of type α^+ of $[\alpha^+]^\alpha$.
Assume $\varrho < \alpha^+$ and β_σ, D_σ, φ_σ are already defined for every $\sigma < \varrho$.
We want β_ϱ, D_ϱ, and φ_ϱ to satisfy the following conditions (3) and (4)

(3) For every $\sigma < \varrho$ and for every $\nu < \varrho$ $R_\nu \subset \varrho$ there is a $\xi \in R_\nu$ such that

$$\beta_\varrho(\xi) = \sigma.$$

(4) If $\xi \in D_\sigma \cap D_\varrho$, $\sigma < \varrho$ and $\beta_\varrho(\xi) = \sigma$ then

$$\varphi_\sigma(\xi) > \varphi_\varrho(\xi).$$

We define $\varphi^{-1}_\varrho(\mu)$ by transfinite induction on μ. Let $\mathscr{H} = \{R_\nu : \nu < \varrho \wedge R_\nu \subset \varrho\}$.
If $\mathscr{H} = 0$ put $D_\varrho = \varphi_\varrho = \beta_\varrho = 0$. If $\mathscr{H} \neq 0$, $\varphi^{-1}_\varrho(\mu)$ will be defined for every $\mu < \alpha$.

Let $\{\langle P_\mu, \sigma_\mu\rangle\}_{\mu<\alpha}$ be a sequence containing all elements of $\mathscr{H}\times\varrho$ (may be with repetitions) and suppose that $\varphi_\varrho^{-1}(\tau)$ is defined for every $\tau<\mu$ for some $\mu<\alpha$.

(5) If $|P_\mu\cap D_{\sigma_\mu}|<\alpha$ then let $\varphi_\varrho^{-1}(\mu)$ be an element of $P_\mu - D_{\sigma_\mu} - \{\varphi_\varrho^{-1}(\tau)\colon \tau<\mu\}$.

(6) If $|P_\mu\cap D_{\sigma_\mu}|=\alpha$ then φ_{σ_μ} being one-to-one $P_\mu\cap D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\colon\tau<\mu\}$ has an element ξ such that $\varphi_{\sigma_\mu}(\xi)>\mu$. Put $\varphi_\varrho^{-1}(\mu)=\xi$ for this ξ. Thus $\varphi_\varrho^{-1}(\mu)$ is defined for every $\mu<\alpha$. Put

(7) $\mathscr{R}(\varphi_\varrho^{-1}(\mu))=D_\varrho$.

Then if $\mathscr{H}\neq 0$, φ_ϱ is a one-to-one mapping of D_ϱ onto α.
Put

(8) $\beta_\varrho(\xi)=\sigma$ if $\xi\in D_\varrho$, $\xi=\varphi_\varrho^{-1}(\mu)$, $\sigma_\mu=\sigma$.

This defines $\beta_\varrho(\xi)$ for $\xi\in D_\varrho$.

Assume $\sigma<\varrho$, $\nu<\varrho$, $R_\nu\subset\varrho$. Then $R_\nu\in\mathscr{H}$, hence there is $\mu<\alpha$ such that $P_\mu=R_\nu$, $\sigma_\mu=\sigma$. Put $\xi=\varphi_\varrho^{-1}(\mu)$. Then by (7) and (8) $\xi\in D_\varrho$, $\beta_\varrho(\xi)=\sigma$. By (5) and (6) $\xi\in R_\nu=P_\mu$. Thus (3) is satisfied. Assume $\xi\in D_\sigma\cap D_\varrho$, $\beta_\varrho(\xi)=\sigma$. Then by (5) and (6) $\xi=\varphi_\varrho^{-1}(\mu)$, $\sigma=\sigma_\mu$, $\varphi_{\sigma_\mu}(\xi)>\mu=\varphi_\varrho(\xi)$. Thus β_ϱ, D_ϱ and φ_ϱ satisfy (4) as well.

It remains to show that the I_ξ defined by (2) and the I_0^ξ, I_0^ξ, defined by (1) satisfy the requirements of our theorem.

By (2) for every pair $\xi<\alpha^+$, $\varrho<\alpha^+$ there is at most one $\sigma<\varrho$ for which $\{\sigma,\varrho\}\in I_\xi$. This means that the I_ξ are forests, i.e. $I_0^\xi\notin\mathscr{B}^3_{\alpha^+}$ for $\xi<\alpha^+$. Using the above property of the I_ξ, if it contains an infinite path, then there is an increasing sequence $\{\varrho_n\}_{n<\omega}$ of type ω of ordinals $<\alpha^+$ such that $\{\varrho_n,\varrho_{n+1}\}\in I_\xi$ for every $n<\omega$. Then by (2) $\xi\in D_{\varrho_{n+1}}$, $\beta_{\varrho_{n+1}}(\xi)=\varrho_n$ for $n<\omega$. Hence by (4) $\varphi_{\varrho_n}(\xi)>\varphi_{\varrho_{n+1}}(\xi)$ for $n<\omega$, a contradiction. It follows by (1) that $I_0^\xi\notin\mathscr{B}^7_{\alpha^+}$.

Let now $X\in B^1_{\alpha^+}$ i.e. $X\subset[\alpha^+]^2$, $|X|=\alpha^+$. Put $T=\{\varrho<\alpha^+\colon\exists\sigma(\sigma<\varrho\wedge\{\sigma,\varrho\}\in X)\}$. Then $|T|=\alpha^+$. Let $C\subset\alpha^+$, $|C|=\alpha$. Then there is a $\nu<\alpha^+$ such that $C=R_\nu$. There is a $\varrho<\alpha^+$ such that $\nu<\varrho$, $R_\nu\subset\varrho$ and $\varrho\in T$. There is $\sigma<\varrho$ such that $\{\sigma,\varrho\}\in X$. By (3) there is a $\xi\in D_\varrho$ such that $\xi\in C$, $\beta_\varrho(\xi)=\sigma$. By (2) that means $\{\sigma,\varrho\}\in I_\xi$, hence

$$X\cap I_\xi\neq 0 \quad\text{for}\quad \xi\in C.$$

By (1) that means $C\subset\alpha^+$, $|C|=\alpha^+$, $X\in B^1_{\alpha^+}$ imply

$$X\not\subset\bigcap_{\xi\in C} I_1^\xi$$

This proves the theorem.

Our next theorem shows that the forests defined in Theorem 4. 1 can not be edge disjoint.

THEOREM 4. 2. *Let $\alpha\geqq\omega$, $\beta<\alpha$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}2 & \alpha^+\\ \mathscr{B}^2_{\alpha^+}, & \mathscr{B}_{\alpha^+,\beta,\alpha^+}\end{pmatrix}^{1,2}.$$

PROOF. Let $[\alpha^+]^2=I_0^\xi\cup I_1^\xi$ for $\xi<\alpha^+$ be arbitrary. Put briefly $I_0^\xi=I_\xi$. We assume that the I_ξ are disjoint.

Let $B \subset \alpha^+$, $|B| = \beta$ be arbitrary. For each $\zeta \in \alpha^+ - B$ put $V_\zeta = \{\xi < \alpha^+ : I_\xi \cap [B, \{\zeta\}] \neq 0\}$. By the assumption $|V_\zeta| \leqq \beta$ for each $\zeta \in \alpha^+ - B$. By the result of G. FODOR [7] already mentioned there are $C \subset \alpha^+$, $D \subset \alpha^+ - B$, $|C| = |D| = \alpha^+$ such that $V_\zeta \cap C = 0$ for every $\zeta \in D$. Put $X = [B, D]$. Then $X \in \mathscr{B}_{\alpha^+, \beta, \alpha^+}$ and $X \cap I_\xi = 0$ for $\xi \in C$ i.e.

$$X \subset \bigcap_{\xi \in C} I_1^\xi.$$

However one can prove a theorem corresponding to Theorem 4. 1 for edge disjoint forests as well.

THEOREM 4. 3. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & 2 & \alpha^+ \\ \mathscr{B}^3_{\alpha^+} & \vee \mathscr{B}^2_{\alpha^+}, & \mathscr{B}_{\alpha^+, \alpha, \alpha^+} \end{pmatrix}^{1,2}.$$

We postpone the proof to p. 384, where we are going to state two more general Theorems.

DEFINITION 4. 2. Let $\mathscr{B}^8_{\alpha,k} = \{X \subset [\alpha]^2 : \langle \alpha, X \rangle$ contains a path of length $k\}$ for $1 \leqq k < \omega$.

Note that $\mathscr{B}^8_{\alpha,1} = \mathscr{B}^2_\alpha$, $\mathscr{B}^8_{\alpha,2} = \mathscr{B}^5_\alpha$. Forests not contained in $\mathscr{B}^8_{\alpha,3}$, are usually called stars.

We will briefly write $\mathscr{B}^8_\alpha$ for $\mathscr{B}^8_{\alpha,3}$. Obviously $\mathscr{B}^8_\alpha \subset \mathscr{B}^3_\alpha$.

We turn back to the problem considered in Theorem 4. 1.

A very strong negative result holds still if we assume that the forests defined in Theorem 4. 1 are even smaller.

THEOREM 4. 4. *Assume G.C.H., $\alpha \geqq \omega$. Then*

a) $\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \omega, \alpha^+} \end{pmatrix}^{1,2}$ *if* $\mathrm{cf}(\alpha) > \omega$,

b) $\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \delta, \alpha^+} \end{pmatrix}^{1,2}$ *where* $\delta = \min[\alpha^+, \omega_2]$ *if* $\mathrm{cf}(\alpha) = \omega$.

We mention

PROBLEM 3. Assume G.C.H. Let $\alpha = \omega_\omega$. Does

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} 1 & \alpha \\ \mathscr{B}^8_{\alpha^+}, & \mathscr{B}_{\alpha^+, \omega_1, \alpha^+} \end{pmatrix}^{1,2} \text{ hold?}$$

PROOF OF THEOREM 4. 4. Theorems of [2] say that if G.C.H. is assumed then the following relations hold:

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} \alpha^+ & \omega & 1 \\ \alpha & , \alpha & \vee \alpha^+ \end{pmatrix}^{1,1} \quad \text{for} \quad \mathrm{cf}(\alpha) > \omega.$$

$$\begin{pmatrix} \alpha^+ \\ \alpha^+ \end{pmatrix} \nrightarrow \begin{pmatrix} \alpha^+ & \omega_2 & 1 \\ \alpha & , \alpha & \vee \alpha^+ \end{pmatrix}^{1,1} \quad \text{for} \quad \mathrm{cf}(\alpha) = \omega_1, \quad \alpha > \omega_1.$$

$$\begin{pmatrix} \omega_1 \\ \omega_1 \end{pmatrix} \nrightarrow \begin{pmatrix} \omega_1 & \omega_1 & 1 \\ \omega & , \omega & \vee \omega_1 \end{pmatrix}^{1,1}.$$

Note that we do not know if ω_2 can be replaced by ω_1 in the second negative relation (see [2]). This explains why we have Problem 3 unsolved.

By definitions the above results mean the following:

(1) There exists a sequence $\{S_\varrho\}_{\varrho<\alpha^+}$ of subsets of α^+, $|S_\varrho|=\alpha$ for $\varrho<\alpha$ satisfying the following conditions:

(a) If $E\subset\alpha^+$, $|E|=\alpha^+$ then $|\alpha^+ - \bigcup_{\varrho\in E} S_\varrho| < \alpha$

(b) If $F\subset\alpha$ then $|\bigcap_{\varrho\in F} S_\varrho| < \alpha$ provided one of the following conditions holds.

(i) $|F|=\omega \wedge \mathrm{cf}(\alpha)>\omega$, (ii) $|F|=\omega_2 \wedge \mathrm{cf}(\alpha)=\omega_1 \wedge \alpha>\omega$,
(iii) $|F|=\omega_1 \wedge \alpha=\omega$.

Let $\{R_\nu\}_{\nu<\alpha^+}=[\alpha^+]^\alpha$ be a well-ordering of type α^+ of $[\alpha^+]^\alpha$.

We will define a sequence I_ξ, $\xi<\alpha^+$, $I_\xi\subset[\alpha^+]^2$ with the intention that

(2) $$I_0^\xi = I_\xi, \quad I_1^\xi = [\alpha^+]^2 - I_\xi$$

should establish the required counterexample.

Similarly as in the proof of Theorem 4. 1 we will define a function β_ϱ for every $\varrho<\alpha^+$ $(\beta_\varrho\in{}^{D_\varrho}\varrho,\ D_\varrho\subset S_\varrho)$ and we will put

(3) $$I_\xi = \{\{\beta_\varrho(\xi), \varrho\}\colon \text{ for } \varrho<\alpha^+,\ \xi\in D_\varrho\} \quad \text{for every} \quad \xi<\alpha^+.$$

We want β_ϱ to satisfy the following condition:

(4) For each $\sigma<\varrho$ and for each $\nu<\varrho$ for which $|R_\nu\cap S_\varrho - S_\sigma| = \alpha$ there is a $\xi\in(R_\nu\cap S_\varrho)-S_\sigma$ such that $\sigma=\beta_\sigma(\xi)$. To do this we need the following lemma. Let H be a set, H_σ $\sigma<\varrho$, $|\varrho|\leqq\alpha$ be a sequence of subsets of H, and let $\mathscr{H}_\sigma\subset[H_\sigma]^\alpha$, $|\mathscr{H}_\sigma|\leqq\alpha$ for $\sigma<\varrho$. Then there exists a sequence $T_\sigma\subset H_\sigma$ $\sigma<\varrho$ such that the T_σ are disjoint and each T_σ meets each element of $\mathscr{H}_\sigma$. This is an easy generalization of a well-known theorem of F. BERNSTEIN. The proof is left to the reader.

Put $H=S_\varrho$, $H_\sigma = S_\varrho - S_\sigma$

$$\mathscr{H}_\sigma = \{(R_\nu\cap S_\varrho)-S_\sigma\colon \nu<\varrho \wedge |(R_\nu\cap S_\varrho)-S_\sigma| = \alpha\} \qquad \text{for} \quad \sigma<\varrho.$$

We obtain the existence of T_σ and we put $\beta_\varrho(\xi)=\sigma$ for $\xi\in T_\sigma$ (hence $D_\varrho=D(\beta_\varrho)=\bigcup_{\varrho<\sigma} T_\sigma$). Then the β_ϱ satisfy (4). The $I_\xi\notin\mathscr{B}^3_{\alpha^+}$ since by (3) for each ϱ there is at most one $\sigma<\varrho$ for $\{\sigma,\varrho\}\in I_\xi$. Using this property it is easy to see that if an I_ξ contained a path of length 3 then there were $\tau<\sigma<\varrho$ such that both $\{\tau,\sigma\}$ and $\{\sigma,\varrho\}$ would belong to I_ξ. By (3) then $\beta_\varrho(\xi)=\sigma$, $\beta_\sigma(\xi)=\tau$ hence by (4) $\xi\in(S_\sigma-S_\tau)\cap(S_\varrho-S_\sigma)$ a contradiction. Thus $I_\xi\notin\mathscr{B}^8_{\alpha^+}$. Hence the I_ξ are stars.

Let now $X=[F,E]\in\mathscr{B}_{\alpha^+,\delta,\alpha^+}$ where $\delta=\omega$, $\delta=\omega_2$, $\delta=\omega_1$, if $\mathrm{cf}(\alpha)>\omega$ $\mathrm{cf}(\alpha)=\omega\wedge\alpha>\omega_1$, $\alpha=\omega_1$ respectively, and let $C\subset\alpha^+$, $|C|=\alpha$. Then there is a $\nu<\alpha^+$ such that $C=R_\nu$. By the assumption (1),b we have

(5) $$|R_\nu\cap\bigcap_{\sigma\in F'} S_\sigma| < \alpha$$

for every $F'\subset F$, $|F'|=\delta$. Using $\mathrm{cf}(\delta)\neq\mathrm{cf}(\alpha)$ and the theorem: $\binom{\delta}{\alpha}\to\binom{\delta\ \ 1}{\alpha,\alpha}^{1,1}$

for cf $(\delta)=$ cf (α) of [2] it results from (5) that there is a $\sigma\in F$ for which $|R_\nu - S_\sigma|=\alpha$. Applying $\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix}\to\begin{pmatrix}\alpha^+ & 1\\ \alpha & ,\alpha\end{pmatrix}^{1,1}$ again it follows that there is a $\varrho\in E$, $\varrho>\max[\nu,\sigma]$ such that

$$|(R_\nu\cap S_\varrho)-S_\sigma| = \alpha.$$

By (4) there is a $\xi\in R_\nu$ such that $\sigma=\beta_\varrho(\xi)$. By (3) that means $\{\sigma,\varrho\}\in[F,E]\cap I_\xi$. By (2) this means

$$X\not\subset \bigcap_{\xi\in C} I_\xi^1.$$

Our next theorem shows that except for the case stated in Problem 3, Theorem 4. 4 is best possible of its kind.

DEFINITION 4. 3. Let $\mathscr{B}^9_{\alpha,k}=\{X\subset[\alpha]^2$: there are $\varrho_0<\dots<\varrho_k<\alpha^+$ such that $\{\varrho_i,\varrho_{i+1}\}\in X$ for $i<k\}$ for $1\leqq k<\omega$, i.e $\langle\alpha, X\rangle$ contains an increasing path of length k. Obviously $\mathscr{B}^9_{\alpha,k}\subset\mathscr{B}^8_{\alpha,k}$ and we have

THEOREM 4. 5. *Assume G.C.H., $\alpha\geqq\omega$. Then*

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}1 & \alpha\\ \mathscr{B}^9_{\alpha,k}, & \mathscr{B}_{\alpha^+,\delta,\alpha^+}\end{pmatrix}^{1,2} \quad \textit{for every}\quad k<\omega$$

and for $\delta<\omega$ or $\delta=\omega$ if cf $(\alpha)>\omega$ *or* cf $(\alpha)=\omega$ *respectively.*

PROOF. We prove the statement by induction on k. For $k=1$ it is trivial Assume $k>1$ and the statement is true for $k-1$.

Let $[\alpha^+]^2=I_0^\xi\cup I_1^\xi$ for $\xi<\alpha$. Put briefly $I_0^\xi=I_\xi$ for $\xi<\alpha$. Put

(1) $T_\xi=\{\zeta<\alpha^+:\zeta$ is the greatest point of an increasing path of length $k-1$ contained in $I_\xi\}$. Theorems of [2] say that

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}\alpha & \alpha\\ \alpha^+, & \delta\end{pmatrix}^{1,1}\quad\text{for}\quad \delta<\omega$$

and

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}\alpha & \alpha\\ \alpha^+, & \delta\end{pmatrix}^{1,1}\quad\text{for}\quad \delta\leqq\omega\quad\text{if}\quad \text{cf}(\alpha)=\omega.$$

It results that one of the following conditions hold:

(2) There are $C\subset\alpha$, $D\subset\alpha^+$, $|C|=\alpha$, $|D|=\alpha^+$ such that $D\cap T_\xi=0$ for every $\xi\in C$.

(3) There are $C\subset\alpha$, $D\subset\alpha^+$, $|C|=\alpha$ such that

$$D\subset\bigcap_{\xi\in D} T_\xi$$

and $|D|=\delta$ or $|D|=\omega$ if $\text{cf}(\alpha)\neq\omega$ or $\text{cf}(\alpha)=\omega$ respectively. If (2) holds then $\bar I_0^\xi=[D]^2\cap I_\xi$, $I_1^\xi=[D]^2-\bar I_0^\xi$, $\xi\in C$ are α 2-partitions of a set of power α^+, and by (1) $\bar I_0^\xi\notin\mathscr{B}^9_{\alpha,k-1}$; hence the result follows from the induction hypothesis.

Assume (3) holds. Let E be an arbitrary subset of α^+ such that $D<E$, $|E|=\alpha^+$. Then again by (1) $[D,E]\cap I_\xi=0$ for every $\xi\in C$; hence $[D,E]\subset\bigcap_{\xi\in C} I'_\xi$. Note

that in case $\alpha=\omega$ G.C.H. is not used since $\begin{pmatrix}\omega\\ \omega_1\end{pmatrix}\to\begin{pmatrix}\omega & \omega\\ \omega_1, & \omega\end{pmatrix}^{1,1}$ can be proved without any hypothesis.

DEFINITION 4. 4. $\mathscr{B}_\alpha^{10}=\bigcap_{1\leq k<\omega}\mathscr{B}_{\alpha,k}^8$.

Note that $\mathscr{B}_\alpha^7$ is a proper subset of $\mathscr{B}_\alpha^{10}$. Assuming G.C.H. Theorem 4. 5 implies trivially for $\alpha\geqq\omega$ that

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}1 & \alpha\\ \mathscr{B}_\alpha^{10}, & \mathscr{B}_{\alpha^+,\delta,\alpha^+}\end{pmatrix}^{1,2}\quad\text{for}\quad\delta<\omega$$

and

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\to\begin{pmatrix}1 & \alpha\\ \mathscr{B}_\alpha^{10}, & \mathscr{B}_{\alpha^+,\delta,\alpha^+}\end{pmatrix}^{1,2}\quad\text{for}\quad\delta<\omega\quad\text{if}\quad\mathrm{cf}(\alpha)\neq\omega.$$

However the following improvement of Theorem 4. 1 is still possible.

THEOREM 4. 6. *Assume G.C.H.* $\mathrm{cf}(\alpha)=\omega$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix}\not\to\begin{pmatrix}1 & 1 & \alpha\\ \mathscr{B}_{\alpha^+}^3 & \vee\ \mathscr{B}_\alpha^{10}, & \mathscr{B}_{\alpha^+}^1\end{pmatrix}^{1,2}.$$

PROOF. We will define a sequence $I_\zeta=I_0^\zeta\subset[\alpha^+]^2$, $I_1^\zeta=[\alpha^+]^2-I_0^\zeta$, $\zeta<\alpha$ with the intention that the partitions

(1) $[\alpha^+]^2=I_0^\zeta\cup I_1^\zeta,\ \zeta<\alpha$

should establish the required counterexample. For each $\zeta<\alpha^+$ we will define a function β_ϱ and its domain $D_\varrho\subset\alpha$, $\beta_\varrho\in{}^{D_\varrho}\varrho$ and we will put

(2) $I_\zeta=\{\{\beta_\varrho(\zeta),\varrho\}:\zeta\in D_\varrho,\ \varrho<\alpha^+\}$ for every $\zeta<\alpha$.

Similarly as in the proof of Theorem 4. 1, we will define β_ϱ, D_ϱ and a one-to-one mapping φ_ϱ of D_ϱ onto an ordinal $\leqq\alpha$, by transfinite induction on ϱ.

By G.C.H. there exists a well-ordering $\{R_\nu\}_{\nu<\alpha^+}=[\alpha]^\alpha$ of type α^+ of $[\alpha]^\alpha$. Considering that $\mathrm{cf}(\alpha)=\omega$ we may assume that

(3) $\alpha=\bigcup_{k<\omega}A_k$ where $|A_k|<\alpha$ and the A_k are disjoint, $|A_0|<\ldots<|A_k|<\ldots$.

Put $k(\zeta)=k$ if $\zeta\in A_k$.

Assume $\varrho<\alpha^+$ and β_σ, D_σ, φ_σ are defined for every $\sigma<\varrho$. We want β_ϱ, D_ϱ and φ_ϱ to satisfy the following conditions (4) and (5):

(4) For every $\sigma<\varrho$ and for every $\nu<\varrho$, there is a $\zeta\in R_\nu$ such that $\beta_\varrho(\zeta)=\sigma$.

(5) If $\zeta\in D_\sigma\cap D_\varrho$, $\beta_\varrho(\zeta)=\sigma$ for $\sigma<\varrho$ then $k(\varphi_\sigma(\zeta))>k(\varphi_\varrho(\zeta)$, and $k(\varphi_\varrho(\zeta))<k(\zeta)$ for $\zeta<\alpha$.

We define $\varphi_\varrho^{-1}(\mu)$ by transfinite induction on μ. Let $\mathscr{H}=\{R_\nu:\nu<\varrho\}$. If $\mathscr{H}=0$ or $\varrho=0$ put $D_\varrho=\beta_\varrho=\varphi_\varrho=0$. If $\mathscr{H}\neq0$, $\varrho>0$, $\varphi_\varrho^{-1}(\mu)$ will be defined for every $\mu<\alpha$.

Let $\{\langle P_\mu, \sigma_\mu\rangle\}_{\mu<\alpha}$ be a sequence containing all the elements of $\mathscr{H}\times\varrho$ (may be with repetitions) and suppose that $\varphi_\varrho^{-1}(\tau)$ is defined for every $\tau<\mu$ for some $\mu<\alpha$.

(6) If $|P_\mu\cap D_{\sigma_\mu}|<\alpha$, then $|P_\mu-D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}|=\alpha$. It follows from (3) that there exists a $\zeta\in P_\mu-D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}$ such that $k(\zeta)>k(\mu)$. Put $\zeta=\varphi_\varrho^{-1}(\mu)$ for this ζ.

(7) If $|P_\mu\cap D_{\sigma_\mu}|=\alpha$, then φ_{σ_μ} being one-to-one by (3) $P_\mu\cap D_{\sigma_\mu}-\{\varphi_\varrho^{-1}(\tau)\}_{\tau<\mu}$ has an element ζ such that $k(\varphi_{\sigma_\mu}(\zeta))>k(\mu)$. Put $\varphi_\varrho^{-1}(\mu)=\zeta$ for this ζ. Thus $\varphi_\varrho^{-1}(\mu)$ is defined for every $\mu<\alpha$. Put

(8) $\mathscr{R}(\varphi_\varrho^{-1})=D_\varrho$. Then if $\mathscr{H}\neq 0$, $\varrho>0$ φ_ϱ is a one-to-one mapping of D_ϱ onto α. Put

(9) $\beta_\varrho(\zeta)=\sigma$ if $\zeta\in D_\varrho, \zeta=\varphi_\varrho^{-1}(\mu), \sigma_\mu=\sigma$. This defines $\beta_\varrho(\zeta)$ for $\zeta\in D_\varrho$. Assume $\sigma<\varrho$, $\nu<\varrho$. Then $R_\nu\in\mathscr{H}$ hence there is a $\mu<\alpha$ such that $P_\mu=R_\nu$, $\sigma_\mu=\sigma$. Put $\zeta=\varphi_\varrho^{-1}(\mu)$. Then by (8) and (9) $\zeta\in D_\varrho$, $\beta_\varrho(\zeta)=\sigma$. By (6) and (7) $\zeta\in R_\nu=P_\mu$.

Thus (4) is satisfied.

Assume $\zeta\in D_\sigma\cap D_\varrho$ for some $\sigma<\varrho$, and $\beta_\varrho(\zeta)=\sigma$. Then by (8) and (9) $\zeta=\varphi_\varrho(\mu)$ for a $\mu<\alpha$, $\sigma=\sigma_\mu$ for this μ and by (6) $|P_\mu\cap D_\sigma|=\alpha$, hence by (7) $k(\varphi_\sigma(\zeta))>$ $>k(\varphi_\varrho(\zeta)=k(\mu)$. This proves the first statement of (5). To prove the second statement we use transfinite induction on ϱ. Assume that $k(\varphi_\sigma(\zeta))<k(\zeta)$ for every $\zeta\in D_\sigma$ for every $\sigma<\varrho$. Let $\zeta\in D_\varrho$. Then by (8) and (9) there is a $\mu<\alpha$ such that $\zeta=\varphi_\varrho^{-1}(\mu)$ and $\beta_\varrho(\xi)=\sigma_\mu$. If $|P_\mu\cap D_{\sigma_\mu}|<\alpha$ then by (6) $k(\zeta)>k(\mu)=k(\varphi_\varrho(\zeta))$. If $|P_\mu\cap D_{\sigma_\mu}|=\alpha$ then by (7) $\zeta\in D_{\sigma_\mu}$ and by the first statement of (5) already proved we have

$$k(\varphi_{\sigma_\mu}(\zeta))>k(\varphi_\varrho(\zeta));$$

hence the statement follows from the induction hypothesis.

It remains to show that the I_ζ defined by (2) and the I_0^ζ, I_1^ζ, defined by (1) satisfy the requirements of our theorem.

By (2) for every pair $\zeta<\alpha$, $\varrho<\alpha^+$ there is at most one $\sigma<\varrho$ for which $\{\sigma,\varrho\}\in I_\zeta$; this means that the I_ζ are forests, i.e. $I_0^\zeta\notin B_{\alpha^+}^3$ for $\zeta<\alpha$.

Using the above property of the I_ζ if it contains a path of length $2k-1$ then it contains an increasing path of length k. We will show that if $k(\zeta)=k$, then I_ζ does not contain an increasing path of length k. Assume $\varrho_0<\dots<\varrho_k<\alpha^+$ and $\{\varrho_i,\varrho_{i+1}\}\in I_\zeta$ for $i<k$. Then by (8) and (9) we have $\varrho_i=\beta_{\varrho_{i+1}}(\zeta)$ for $i<k$ and $\zeta\in D_{\varrho_i}\cap D_{\varrho_{i+1}}$ for $0<i<k$. Then by (5) $k>k(\varphi_{\varrho_0}(\zeta))>\dots>k(\varphi_{\varrho_k}(\zeta))\geqq 0$ a contradiction. Hence $I_0^\zeta\notin\mathscr{B}_{\alpha^+,k}^9$ and then by the above remark $I_0^\zeta\in\mathscr{B}_{\alpha^+,2k-1}^8$ for $k=k(\zeta)$, hence $I_0^\zeta\notin\mathscr{B}_{\alpha^+}^{10}$ for every $\zeta<\alpha$.

Let now $X\in\mathscr{B}_\alpha^1$, $C\subset\alpha$, $|C|=\alpha$ the statement that $X\not\subset\bigcup_{\xi\in C}I'_\xi$ follows from (4) literally the same way as in the proof of Theorem 4. 1.

Now we turn to the investigation of the case of edge disjoint forests

THEOREM 4. 7. *Assume G.C.H. Let $\alpha\geqq\omega$. Then*

$$\binom{\alpha^+}{\alpha^+}\nrightarrow\begin{pmatrix}1 & 1 & 2 & \alpha \\ \mathscr{B}_{\alpha^+}^3 \vee & \mathscr{B}_{\alpha^+}^7 \vee & \mathscr{B}_{\alpha^+}^2, & \mathscr{B}_{\alpha^+,\alpha,\alpha^+}\end{pmatrix}^{1,2}.$$

THEOREM 4. 8. *Assume G.C.H. and $\alpha > \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & & 1 & & 2 & & \alpha^+\\ \mathscr{B}^3_{\alpha^+} & \vee & \mathscr{B}^8_{\alpha^+} & \vee & \mathscr{B}^2_{\alpha^+} & , & \mathscr{B}_{\alpha^+,\alpha,\alpha^+}\end{pmatrix}^{1,2}.$$

Both theorems are generalizations of Theorem 4. 3. Theorem 4. 2 shows that on the right hand side in $B_{\alpha^+,\alpha,\alpha^+}$ α can not be replaced by anything smaller. Theorem 4. 8 is sharper than Theorem 4. 7 but for $\alpha=\omega$ it is false by Theorem 4. 5.

We describe here the proof of Theorem 4. 3. The proofs of Theorems 4. 7 and 4. 8 can be obtained from this proof with a sligth modification using the tricks of Theorem 4. 1 and 4. 4 respectively. We will omit this.

PROOF OF THEOREM 4. 3. As in the preceeding proofs we will define a sequence I_ξ, $\xi<\alpha^+$; $I_\xi\subset[\alpha^+]^2$ with the intention that the partitions

(1) $I_0^\xi \cup I_1^\xi = [\alpha^+]^2, \quad I_0^\xi = I_\xi, \quad I_1^\xi = [\alpha^+]^2 - I_0^\xi \quad \xi<\alpha^+$

should establish the required counterexample.

For each $\alpha \leqq \varrho < \alpha^+$ we will define a function $\beta_\varrho \in {}^\varrho\varrho$ and we put

(2) $I_\xi = \{\{\beta_\varrho(\xi), \varrho\}: \xi<\varrho,\ \alpha\leqq\varrho<\alpha^+\}$ for every $\xi<\alpha^+$.

Let $\{R_\nu\}_{\nu<\alpha^+} = [\alpha^+]^\alpha$ be a well-ordering of type α^+ of $[\alpha^+]^\alpha$. For every fixed ϱ, $\alpha\leqq\varrho<\alpha^+$ we will define β_ϱ so that it should satisfy the following requirements (3) and (4):

(3) Assume $\nu, \mu<\varrho$, $R_\nu\subset\varrho$, $R_\mu\subset\varrho$. Then there is a $\xi<\varrho$ such that $\beta_\varrho(\xi)\in R_\nu$, $\xi\in R_\mu$

(4) β_ϱ is one-to-one.

To do this we need the following lemma. If H is a set of power $\alpha\geqq\omega$, $\mathscr{H}_0, \mathscr{H}_1\subset[H]^\alpha$; $|\mathscr{H}_0|, |\mathscr{H}_1|\leqq\alpha$ then there is an $f\in{}^H H$ such that f is one-to-one and for every $H_0\in\mathscr{H}_0$, $H_1\in\mathscr{H}_1$ there are $h_0\in H_0$, $h_1\in H_1$ such that $f(h_0)=h_1$. The proof can be carried out by an easy transfinite induction; we omit it.

Applying this for $H=\varrho$, $\mathscr{H}_0-\mathscr{H}_1=\{R_\nu: R_\nu\subset\varrho\wedge\nu<\varrho\}$ we obtain an f and put $f=\beta_\varrho$. We prove that I_0^ξ, I_1^ξ satisfy the requirements of our theorem. If $\sigma<\varrho$, by (4), there is at most one $\xi<\alpha^+$ for which $\sigma=\beta\ (\xi)$, hence by (2) the I_ξ are disjoint, i.e. there is no $X\in\mathscr{B}^2_{\alpha^+}$, $X\subset I_{\xi_0}\cap I_{\xi_1}$ for $\xi_0\neq\xi_1<\alpha^+$. By (2) for each $\xi, \varrho<\alpha^+$. There is at most one $\sigma<\varrho$ for which $\beta_\varrho(\xi)=\sigma$. It follows that the I_ξ are forests, i.e. $I_\xi\notin\mathscr{B}^3_{\alpha^+}$.

Let now $X=[B, D]\in\mathscr{B}_{\alpha^+,\alpha,\alpha^+}$, $C\subset\alpha^+$, $|C|=\alpha$. Then there are $\nu,\mu<\alpha^+$ such that $B=R_\nu$, $C=R_\mu$. There is a $\varrho>\max[\nu,\mu,\alpha]$ such that $\varrho\in D$, R_μ, $R_\nu\subset\varrho$. Then by (3) there is a $\xi\in R_\mu$ such that $\beta_\varrho(\xi)\in R_\nu$. Hence there is a $\xi\in C$ such that $\{\beta_\varrho(\xi), \varrho\}\subset[B, D)$ i.e. by (2) $X\cap I_\xi\neq 0$.

By (1) this means

$$X\subset\bigcap_{\xi\in C} I_1^\xi.$$

§ 5. A remark on matrices of sets of Ulam type

Whenever we have a negative relation $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \nrightarrow \begin{pmatrix}\beta_{0,0} & \beta_{01}\\ \beta_{1,0}, & \beta_{11}\end{pmatrix}^{1,2}$ there are at least three natural ways to obtain matrices of sets having certain properties.

We give one example. As a corollary of Theorem 4. 1 we have

(1) Assume G.C.H. $\alpha \geqq \omega$. There exists a sequence I_ξ, $\xi < \alpha^+$; $I_\xi \subset [\alpha^+]^2$ satisfying the following conditions.

(2) For each $\varrho, \xi < \alpha^+$ there is at most one $\sigma < \varrho$ for which $\{\sigma, \varrho\} \in I_\xi$.

(3) Whenever $X \subset [\alpha^+]^2$, $C \subset \alpha^+$, $|X| = \alpha^+$, $|C| = \alpha$ then there is a $\xi \in C$ such that

$$X \cap I_\xi \neq 0.$$

We define a matrix $\{A_{\xi,\eta}\}_{\xi<\alpha^+,\eta<\alpha^+}$ of subsets of α^+ by the following stipulation:

(4) $A_{\xi,\eta} = \{\zeta : \eta < \zeta < \alpha^+ \wedge \{\eta, \zeta\} \in I_\xi\}$.

We obtain

COROLLARY 5. 1. *Assume G.C.H., $\alpha \geqq \omega$. There exists a matrix*

$$\{A_{\xi,\eta}\}_{\xi<\alpha^+,\eta<\alpha^+} \quad \text{of subsets of} \quad \alpha^+$$

satisfying the following conditions.

(5) *For every $\xi < \alpha^+$ the sets $A_{\xi,\eta}$, $\eta < \alpha^+$ are disjoint.*

(6) *For every $C \subset \alpha^+$, $|C| = \alpha$, $f \in {}^C\alpha^+$*

$$|\alpha^+ - \bigcup_{\xi \in C} A_{\xi, f(\xi)}| < \alpha^+.$$

In fact if I_ξ, $\xi < \alpha^+$ satisfies (2) *and* (3) *then $\{A_{\xi,\eta}\}_{\xi<\alpha^+,\eta<\alpha^+}$ defined by* (4) *satisfies* (5) *and* (6), *respectively.*

Corollary 5. 1 has been stated and proved in a paper of P. ERDŐS and S. ULAM [4] independently.

§ 7. Positive results and some further counterexamples

In [3], Problem 59, we stated in an other notation the following problems:

(1) $$\begin{pmatrix}\omega_1\\ \omega\end{pmatrix} \rightarrow \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2} ?$$

(2) $$\begin{pmatrix}\omega\\ \omega_1\end{pmatrix} \rightarrow \begin{pmatrix}1 & \omega\\ 4, & \omega\end{pmatrix}^{1,2} ?$$

We mentioned that if 4 is replaced by 3 we can prove a positive result.

Both problems are solved now.

As to (2) one can prove the following:

THEOREM 6. 1. *Assume $\alpha \geqq \omega$ and α is 0, 1-measurable, $\beta < \alpha$. Then*

$$\begin{pmatrix} \alpha \\ \alpha^+ \end{pmatrix} \to \begin{pmatrix} \alpha & 1 & \alpha \\ \beta & \vee & \alpha, \alpha \end{pmatrix}^{1,2}.$$

In fact one can prove the following slightly stronger result:

THEOREM 6. 2. *Assume $\alpha \geqq \omega$, α is 0, 1-measurable. Let I_ζ, $\zeta < \alpha$ be an arbitrary sequence $I_\zeta \subset [\alpha^+]^2$, $\xi < \alpha$. Then one of the following conditions* (3), (4), (5) *holds.*

(3) *There are $X \subset \alpha^+$, $C \subset \alpha$, $|X| = \beta$, $|C| = \alpha$ such that*

$$[X]^2 \subset \bigcap_{\xi \in C} I_\xi.$$

i.e. there are α I_ξ whose intersection contains a complete β.

(4) *There are $X \subset \alpha^+$, $C \subset \alpha$, $|X| = \alpha^+$, $|C| < \alpha$ such that*

$$[X]^2 \subset \bigcup_{\xi \in C} I_\xi,$$

i.e. there are fewer than α I_ξ whose union contains a complete α^+ (and as a corollary one of them contains a complete α) (note that $\alpha \to (\alpha)^2_\gamma$ for $\gamma < \alpha$ if α is measurable).

(5) *There are $X \subset \alpha^+$, $C \subset \alpha$, $|X| = |C| = \alpha$ such that*

$$[X]^2 \cap \bigcup_{\xi \in C} I_\xi = 0,$$

i.e. there are α I_ξ such that the intersection of the complement of them contains a complete α.

As to the problem (1) we proved

(6) Assume $\alpha \geqq \omega$, α is 0, 1-measurable, $\beta < \alpha$, then

$$\begin{pmatrix} \alpha^+ \\ \alpha \end{pmatrix} \to \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^{1,2}_\beta.$$

F. Galvin proved the following generalization of (6) for $\alpha = \omega$:

THEOREM OF GALVIN.

$$\begin{pmatrix} \omega_1 \\ \omega \end{pmatrix} \to \begin{pmatrix} \omega \\ \omega \end{pmatrix}^{1,r}_k \quad \textit{for every} \quad k, r < \omega;$$

but this method of proof breaks down for $\alpha > \omega$ α 0, 1-measurable.

One can prove the following generalization of Galvin's theorem.

THEOREM 6. 3. *Assume $\alpha \geqq \omega$, α is 0, 1-measurable. Then*

$$\begin{pmatrix} \alpha^+ \\ \alpha \end{pmatrix} \to \begin{pmatrix} \alpha \\ \alpha \end{pmatrix}^{1,r}_\beta \quad \textit{for} \quad r < \omega, \ \beta < \alpha.$$

DEFINITION 6. 1. Let $\alpha_0, \alpha_1, \beta_0, \beta_1, \gamma$ cardinals. $\begin{pmatrix}\alpha_0\\ \alpha_1\end{pmatrix} \to \begin{pmatrix}\beta_0\\ \beta_1\end{pmatrix}^{1,<\omega}_{\gamma}$ denotes that the following statement is true. Assume that for every $r<\omega$ $\alpha_0 X[\alpha_1]^r = \bigcup_{\nu<\gamma} I^r_\nu$. Then there are $A_0 \subset \alpha$, $A_1 \subset \alpha$, $f \in {}^\omega\gamma$ such that $|A_0| = \beta_0$, $|A_1| = \beta_1$ and

$$A_0 \times [A_1]^r \subset I^r_{f(r)}, \quad \text{for} \quad r<\omega.$$

The symbol defined above is a common generalization of the symbols defined in 1. 1. 2 and 1. 1. 3.

Galvin conjectured that as a generalization of the author's results [5] that for $\alpha>\omega$, α 0, 1-measurable $\alpha \to (\alpha)^{<\omega}_\beta$ $(\beta<\alpha)$ the following result will hold.

THEOREM 6. 4. *Assume $\alpha>\omega$, α is 0, 1-measurable. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha\end{pmatrix} \to \begin{pmatrix}\alpha\\ \alpha\end{pmatrix}^{1,<\omega}.$$

This was proved by the second author. The proofs of the theorems 6. 1, 6. 2, 6. 3, 6. 4 will appear in a forthcoming paper [6] in the *Fundamenta Mathematicae* containing the results of the lecture given by the second author on the symposium held in Warsaw August 27—September 2, 1968.

We now give some counterexamples to show that Theorems 6. 1, 6. 3 cannot be improved in certain directions.

THEOREM 6. 5. *Assume $\alpha \geqq \omega$, α is a strong limit cardinal. Then*

$$\begin{pmatrix}\alpha\\ 2^\alpha\end{pmatrix} \not\to \begin{pmatrix}1 & \alpha\\ \alpha & 2\end{pmatrix}^{1,2}.$$

PROOF. We define a sequence I_ζ, $\zeta<\alpha$; $I_\zeta \subset [{}^\alpha 2]^2$ as follows.

(7) Assume $f \neq g \in {}^\alpha 2$. Put

$$\{f, g\} \in I_\zeta \quad \text{iff} \quad \min\{\xi : f(\xi) \neq g(\xi)\} < \zeta.$$

Assume $[X]^2 \subset I_\zeta$ for some $X \subset {}^\alpha 2$, $\zeta<\alpha$. Then, by (7), for $f \in X$ $f: \to f \upharpoonright \zeta$ is a one-to-one mapping of X into ${}^\zeta 2$. Hence, α being strong limit, $|X| \leqq 2^{|\zeta|} < \alpha$. That means none of the I_ζ contains a complete α.

Assume now that $\{f, g\} \in [{}^\alpha 2]^2$. Then $f(\xi) \neq g(\xi)$ for some $\xi<\alpha$ and by (7) $\{f, g\} \in I_\zeta$ for every $\xi<\zeta<\alpha$. This proves the theorem.

Assuming G.C.H., Theorem 6. 5 says that $\begin{pmatrix}\alpha\\ \alpha^+\end{pmatrix} \not\to \begin{pmatrix}1 & \alpha\\ \alpha & 2\end{pmatrix}^{1,2}$ if α is a limit cardinal.

Strangely enough this very weak counter-example cannot be proved if α is a successor cardinal.

THEOREM 6. 7. *Assume G.C.H., $\alpha \geqq \omega$. Then*

$$\begin{pmatrix}\alpha^+\\ \alpha^{++}\end{pmatrix} \to \begin{pmatrix}1 & \alpha^+\\ \alpha^+ & 2\end{pmatrix}^{1,2}$$

and $\alpha^{++} \to [\alpha^+]^2_{\alpha^+,\alpha}$ are equivalent.

Note that $\alpha^{++} \to [\alpha^+]^2_{\alpha^+,\alpha}$ is known to be independent of the axioms of set theory and the G.C.H. (see e.g.[7]).

PROOF OF THEOREM 6. 7. Assume $\alpha^{++} \nrightarrow [\alpha^+]^2_{\alpha^+,\alpha}$ and let $I_\xi < [\alpha^{++}]^2$, $\xi < \alpha^+$ be a sequence establishing this negative relation. Put $I_0^\xi = \bigcup_{\zeta<\xi} I_\zeta$, $I_1^\xi = [\alpha^{++}]^2 - I_0^\xi$ for $\xi < \alpha^+$. Then I_0^ξ obviously does not contain a complete α^+ and $\bigcap_{\xi \in C} I_1^\xi = 0$ for $C \subset \alpha^+$, $|C| = \alpha^+$ since $\bigcup_{\xi<\alpha^+} I_\xi = [\alpha^{++}]^2$. Hence the partitions $[\alpha^{++}]^2 = I_0^\xi \cup I_1^\xi$ $\xi < \alpha^+$ establish

$$\begin{pmatrix}\alpha^+ \\ \alpha^{++}\end{pmatrix} \nrightarrow \begin{pmatrix}1 & \alpha^+ \\ \alpha^+ , & 2\end{pmatrix}^{1,2}.$$

On the other hand let $[\alpha^{++}]^2 = I_0^\xi \cup I_1^\xi$, $\xi < \alpha^+$ be a sequence of partitions establishing the negative polarized partition relation.

Then for each $X \in [\alpha^{++}]^2$ there is a $\xi(X) < \alpha^+$ such that $X \in I_0^\xi$ for $\xi(X) \leqq \xi < \alpha^+$. Put $I_\xi = \{X \in [\alpha^{++}]^2, \ \xi(X) \leqq \xi\}$ for $\xi < \alpha^+$. Then $I_\xi \subset I_0^\xi$ and the I_ξ obviously establish $\alpha^{++} \nrightarrow [\alpha^+]^2_{\alpha^+,\alpha}$.

We think it is relevant to mention here the following negative result.

THEOREM 6. 8. *Assume G.C.H., $\alpha > \omega$. Then*

$$\begin{pmatrix}\alpha^+ \\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & 2 \\ \mathscr{B}_{\alpha^+,\alpha,\alpha^+} , & \mathscr{B}^2_{\alpha^+}\end{pmatrix}^{1,2}.$$

This is a trivial reformation of Theorem 17/A of [2] saying $\alpha^+ \to [\mathscr{B}_{\alpha^+,\alpha,\alpha^+}]^2_{\alpha^+}$ where this is a self explanatory modification of the symbol defined in 1. 1. 4.

The following theorem shows that assuming G.C.H. in Theorem 6. 1 the $\frac{1}{\alpha}$ cannot be replaced by $\mathscr{B}^1_{\alpha^+,\alpha,\alpha^+}$ even if $\beta = 2$.

THEOREM 6. 9. *Assume G.C.H., $\alpha \geqq \omega$, $\alpha \to (\alpha)^2_2$. Then*

$$\begin{pmatrix}\alpha \\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}\alpha & 1 & 2 \\ 2 & \vee \ \mathscr{B}_{\alpha^+,\alpha,\alpha^+} , & \alpha\end{pmatrix}^{1,2}.$$

PROOF. By the assumption α is a (strong) limit cardinal. Hence by Theorem 6. 5 there exists a sequence $I_0^{0,\xi}, I_1^{0,\xi}$ $\xi < \alpha$ establishing $\begin{pmatrix}\alpha \\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}\alpha & 1 \\ 2 , & \alpha\end{pmatrix}^{1,2}$. By Theorem 6. 8 there exists a sequence $I_0^{1,\xi}, I_1^{1,\xi}$, $\xi < \alpha$ establishing $\begin{pmatrix}\alpha \\ \alpha^+\end{pmatrix} \nrightarrow \begin{pmatrix}1 & 2 \\ \mathscr{B}_{\alpha^+,\alpha,\alpha^+} , & \mathscr{B}^2_{\alpha^+}\end{pmatrix}^{1,2}$. Put $I_0^\xi = I_0^{0,\xi} \cap I_0^{1,\xi}$, $I_1^\xi = [\alpha^+]^2 - I_0^\xi$. It is obvious from the construction that each $\{\zeta, \eta\} \in [\alpha^+]^2$ is contained only in less than α I_0^ξ, and that none of the I_0^ξ contain an $X \in \mathscr{B}_{\alpha^+,\alpha,\alpha^+}$.

Assume $X \subset \alpha_1^+$ $|X| = \alpha$ and let $\xi \neq \zeta < \alpha$. Considering that $I_1^{1,\xi} \cap I_1^{1,\zeta} = 0$ it follows from the assumption $\alpha \to (\alpha)^2_2$, that there exists a $Y \subset X$, $|Y| = \alpha$ such that either $[Y]^2 \subset I_0^{1,\xi}$ or $[Y]^2 \subset I_0^{1,\zeta}$. Hence $[X]^2 \subset I_1^\xi \cap I_1^\zeta$ would imply either $[Y]^2 \subset I_1^{0,\xi}$ or $[Y]^2 \subset I_1^{0,\zeta}$, a contradiction. Thus $X \subset \alpha$, $|X| = \alpha$ implies $[X]^2 \not\subset I_1^\xi \cap I_1^\zeta$ for every pair $\xi \neq \zeta < \alpha$ and the theorem is proved.

§ 7. A result for singular strong limit cardinals

DEFINITION 7. 1. Let $\mathscr{A}=(A_\xi)_{\xi<\beta}$ be a sequence of disjoint sets. Put $A=\bigcup_{\xi<\beta} A_\xi$. Let $X, Y\subset A$. Put $X|-|_{\mathscr{A}}Y$ if $|X\cap A_\xi|=|Y\cap A_\xi|$ for each $\xi<\beta$. Let $H\subset\mathscr{P}(A)$. H is said to be *canonical with respect to* $\mathscr{A}$ if for every $X, Y\subset A$ we have

$$X\in H \quad \text{iff} \quad Y\in H.$$

Let $I=(I_\nu)_{\nu<\gamma}$ be an r-partition of A. I is said to be *canonical with respect to* $\mathscr{A}$ if I_ν is canonical with respect to $\mathscr{A}$ for every $\nu<\gamma$.

CANONIZATION LEMMA. *Let α be a singular strong limit cardinal. Put $\beta=\mathrm{cf}(\alpha)$. Let $r<\omega$, $\gamma<\alpha$ and let $[\alpha]^r=\bigcup_{\nu<\gamma} I_\nu$ be an r-partition of type γ of α. Let further $(B_\xi)_{\xi<\beta}$ be a sequence of disjoint subsets of α such that the cardinality of B_ξ increase rapidly enough e.g. satisfies the following conditions.*

$$(1) \qquad \sum_{\xi<\beta}|B_\xi| = \alpha \quad \textit{and} \quad |B_\xi| \geqq \exp_{r-1}\ \Big(\big|\bigcup_{\zeta<\xi} B_\zeta\big|\Big)^+$$

where $\exp_0(\alpha)=\alpha$, $\exp_{s+1}(\alpha)=2^{\exp_s(\alpha)}$ $s<\omega$. Then there exists a sequence $\mathscr{A}=(A_\xi)_{\xi<\beta}$ of subsets of α such that

$$A_\xi\subset B_\xi,\quad |A_\xi|\leqq|A_\zeta| \quad \textit{for} \quad \xi<\zeta<\beta,\quad |A|=\alpha \quad \textit{for} \quad A=\bigcup_{\xi<\beta} A_\xi$$

and the r-partition is canonical with respect to $\mathscr{A}$.

The canonization lemma is proved in [2] assuming G.C.H. for every singular α, but the proof yields the result as stated above. A detailed proof will appear in a forthcoming book of the three of us. As a corollary we will prove the following

LEMMA. *Let α be a singular strong limit cardinal. Put $\beta=\mathrm{cf}(\alpha)$. Let $r<\omega$ and I_ν, $\nu<\alpha$ be a sequence such that $I_\nu\subset[\alpha]^r$ for $\nu<\alpha$. Then there are sequences $\mathscr{A}=(A_\xi)_{\xi<\beta}$, $\mathscr{C}=(C_\xi)_{\xi<\beta}$ of disjoint subsets of α satisfying the following conditions: For $A=\bigcup_{\xi<\beta}A_\xi$, $C=\bigcup_{\xi<\beta}C_\xi$, $|A_\xi|\leqq|A_\zeta|$, $|C_\xi|\leqq|C_\zeta|$ for $\xi<\zeta<\alpha$, $|A|=|C|=\alpha$. I_ν is canonical with respect to $\mathscr{A}$ for every $\nu\in C$ and $I_\nu\cap[A]^r=I_\mu\cap[A]^r$ for $\mu, \nu\in C_\xi$ for every $\xi<\beta$.*

PROOF. Considering that α is strong limit there is a sequence $(B_\xi)_{\xi<\beta}$ of type β of disjoint subsets of α satisfying the cardinality condition (1) of the canonization lemma. Put $B=\bigcup_{\xi<\beta}B_\xi$. Then $|B|=\alpha$. We define an $r+1$ partition J of type 2 of B as follows.

Let $X\in[B]^{r+1}$. Assume first that the following condition (2) holds:

$$(2) \qquad X=Y\cup\{v\},\quad Y\in\Big[\bigcup_{\xi<\beta,\,\xi\,\text{even}} B_\xi\Big]^r,\quad v\in B_{\xi_0} \text{ and } \xi_0 \text{ is odd.}$$

Put $X\in J_0$ iff $Y\in I_v$. If (2) is false we put $X\in J_0$. $J_1=[B]^2-J_0$. By the canonization lemma there is a sequence $\mathscr{A}'=(A'_\xi)_{\xi<\beta}$ of disjoint subsets of α satisfying the following conditions: $\big|\bigcup_{\xi<\beta}A'_\xi\big|=\alpha$, $|A'_\xi|\leqq|A'_\zeta|$ and $A'_\xi\subset B_\xi$ for $\xi<\zeta<\alpha$, and the partition J is canonical with respect to $\mathscr{A}'$. Put $A_\xi=A'_{\xi\cdot 2}$, $C_\xi=A'_{\xi 2+1}$, $A=\bigcup_{\xi<\beta}A_\xi$, $C=\bigcup_{\xi<\beta}C_\xi$.

Considering that the cardinality of the A'_ξ increase we have $|A|=|C|=\alpha$. Assume $X, Y\in[A]^r$, $X|\!-\!|_{\mathscr{A}}Y$ and let $v, \mu\in C_\xi$ for a $\xi<\beta$. Then $X\cup\{v\}$, $Y\cup\{\mu\}\in\in[A']^{r+1}$, $X\cup\{v\}|\!-\!|_{\mathscr{A}}$, $Y\cup\{\mu\}$ and $X\cup\{v\}$, $Y\cup\{\mu\}$ both satisfy (2). By the canonicity of J we have $X\cup\{v\}\in J_0$ iff $Y\cup\{\mu\}\in J_0$. Hence by the definition of J $X\in I_v$ iff $Y\in I_\mu$. Applying this for $v=\mu$ we get that I_v is canonical with respect to $\mathscr{A}$ for $v\in C$. Applying the above result for $X=Y$ we get that $I_v\cap[A]^r=I_\mu\cap[A]^r$ for every $v, \mu\in C_\xi$ for every $\xi<\beta$.

Using the Lemma we can reduce a number of problems concerning singular strong limit cardinals to problems for regular cardinals already discussed. Before doing this as a converse of the Lemma we describe a construction for definings partition canonically.

DEFINITION 8. 2. Let α be a singular limit cardinal $\beta=\mathrm{cf}(\alpha)<\alpha$. Let $(\alpha_\xi)_{\xi<\beta}$ be a sequence of cardinals less than α tending to α, $\alpha=\bigcup_{\xi<\beta}A_\xi$ is a disjoint partition of α where $|A_\xi|=\alpha_\xi$ for $\xi<\beta$.

Let further $[\beta]^2=I_0^\xi\cup I_1^\xi$ be a sequence of type β of 2-partitions of β. We define two sequences of canonical partitions of α as follows:

(1) The sequence $[\alpha]^2=I_0^{*,\xi}\cup I_1^{*,\xi}$, $\xi<\beta$ of type β is defined by the following stipulation. For every $\sigma, \varrho<\alpha$, $\xi<\beta$ $\{\sigma,\varrho\}\in I_0^{*,\xi}$ iff $\sigma\in A_\zeta$, $\varrho\in A_\eta$ and $\{\eta,\zeta\}\in I_0^\xi$ $\{\sigma,\varrho\}\in I_1^{*,\xi}$ iff $\sigma\in A_\zeta$ $\varrho\in A_\eta$ and $\{\eta,\zeta\}\in I_1^\xi$ or $\zeta=\eta$.

(2) The sequence $[\alpha]^2=I_0^{**,v}\cup I_1^{**,v}$ $v<\alpha$ of type α is defined by the stipulation

$$I_i^{**,v}=I_i^{*,\xi}\quad\text{for}\quad i<2,\quad v\in A_\xi.$$

THEOREM 7. 1. *Let α be a singular limit cardinal,* $\mathrm{cf}(\alpha)=\beta$. *Assume that* $\begin{pmatrix}\beta\\ \beta\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \beta\\ \gamma & \vee\ \mathscr{B}_\alpha^1, & \beta\end{pmatrix}^{1,2}$ *holds for some γ. Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\ \mathscr{B}_\alpha^1, & \alpha\end{pmatrix}^{1,2}$$

holds as well.

PROOF. Let $[\beta]^2=I_0^\xi\cup I_1^\xi$ $\xi<\beta$ be a sequence of 2-partitions of β establishing the assumed negative relation. Then the second canonical sequence $I_0^{**,v}, I_1^{**,v}$ defined in 8. 2 satisfies the requirement of the theorem.

Considering that $\begin{pmatrix}\beta\\ \beta\end{pmatrix}\nrightarrow\begin{pmatrix}1 & 1 & \beta\\ \beta & \vee\ \mathscr{B}_\beta^1, & \beta\end{pmatrix}$ holds trivially e.g. by 2. 4 for every $\beta\geqq\omega$. Theorem 7. 1 yields a proof of 2. 6.

The following theorem is the main result of this §.

THEOREM 7. 2. *Let α be a singular strong limit cardinal. Put* $\mathrm{cf}(\alpha)=\beta$. *Let* $\gamma<\mathrm{cf}(\alpha)$, $\delta<\alpha$. *Assume*

$$\begin{pmatrix}\beta\\ \beta\end{pmatrix}\to\begin{pmatrix}1 & 1 & \beta\\ \gamma & \vee\ \mathscr{B}_{\beta,1,\beta}, & \beta\end{pmatrix}^{1,2}.$$

Then

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix}\to\begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\ \mathscr{B}_{\alpha,\delta,\alpha}, & \alpha\end{pmatrix}^{1\ 2}$$

holds as well.

PROOF. Let $[\alpha]^2 = I_0^\nu \cup I_1^\nu$, $\nu < \alpha$ be a sequence of type α of 2-partitions of α. We may assume $I_0^\nu \cap I_1^\nu = 0$ for $\nu < \alpha$. By the Lemma there exist sequences $\mathscr{A} = (A_\xi)_{\xi<\beta}$, $C = (C_\xi)_{\xi<\beta}$ satisfying the requirements of the Lemma. Put $A = \bigcup_{\xi<\beta} A_\xi$, $C = \bigcup_{\xi<\beta} C_\xi$. Considering that the cardinality of A_ξ is increasing and tends to α we may assume that $|A_\xi| > \max[\gamma, \delta]$ for every $\xi < \beta$. Assume that

(1) $X \subseteqq \alpha$, $|X| = \gamma$ implies $[X]^2 \subset I_0^\nu$ and $Y \subset I_0^\nu$ implies that $Y \notin B_{\alpha,\delta,\alpha}$ for $\nu < \alpha$.

By the canonicity for every $\nu \in C$, $\xi < \beta$ we have either $[A_\xi]^2 \subset I_0^\nu$ or $[A_\xi]^2 \subset I_1^\nu$, hence by (1) we have

(2) $[A_\xi]^2 \subset I_1^\nu$ for every $\nu \in C$.

Using again the canonicity we define a sequence $[\beta]^2 = \tilde{I}_0^\xi \cup \tilde{I}_1^\xi$, $\xi < \beta$ of type β of disjoint 2-partitions of β by the following stipulation. For every $\zeta, \eta, \xi < \beta$ $i < 2$

(3) $\{\zeta, \eta\} \in \tilde{I}_i^\xi$ iff $[A_\zeta, A_\eta] \subset I_i^\mu$ for every $\mu \in C_\xi$.

Considering that $|A_\xi| > \delta$ it follows from (1) that $X \in B_{\beta,1,\beta}$ implies $X \not\subset \tilde{I}_0^\xi$ for $\xi < \beta$. On the other hand (1) and (2) obviously imply that $\tilde{I}_0^\xi$ does not contain a complete γ for $\xi < \beta$.

Thus it follows from the assumption $\begin{pmatrix}\beta\\ \beta\end{pmatrix} \to \begin{pmatrix}1 & 1 & \beta\\ \beta & \vee\, B_{\beta,1,\beta}, & \beta\end{pmatrix}^{1,2}$ that there are $U, V \subset \beta$, $|U| = |V| = \beta$ such that

(4) $[U]^2 \subset \tilde{I}_1^\xi$ for every $\xi \in V$.

Put

$$X = \bigcup_{\xi \in U} A_\xi, \quad Y = \bigcup_{\xi \in V} C_\xi.$$

Then $|X| = |Y| = \alpha$, (2) and (4) imply that $[X]^2 \subset I_1^\nu$ for every $\nu \in Y$. This proves the theorem.

We obtain from Theorem 1. 1 and 7. 2 the following

COROLLARY 7. 3. *Assume* cf(α) *is* 0, 1*-measurable and* α *is a singular strong limit cardinal,* $\gamma < \mathrm{cf}(\alpha)$, $\delta < \alpha$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \to \begin{pmatrix}1 & 1 & \alpha\\ \gamma & \vee\, \mathscr{B}_{\alpha,\delta,\alpha}, & \alpha\end{pmatrix}^{1,2}.$$

We mention one more (very easy) positive result

THEOREM 7. 4. *Let* α *be a singular strong limit cardinal,* $\gamma < \alpha$. *Then*

$$\begin{pmatrix}\alpha\\ \alpha\end{pmatrix} \to \begin{pmatrix}1 & \alpha\\ \mathscr{B}_{\alpha,\gamma,\gamma}, & \alpha\end{pmatrix}^{1,2}.$$

Using the same as in the proof of Theorem 7. 2, Theorem 7. 4 follows trivially from the lemma. We omit the details.

A short list of special notations

The ... symbol is defined on page

$\mathscr{B}_{\alpha,\gamma,\delta}$	p. 372 (Definition 2. 1)
$\mathscr{B}_{\alpha}^{0}$	p. 372
$\mathscr{B}_{\alpha}^{1}$	p. 372
$\mathscr{B}_{\alpha}^{2}$	p. 372
$\mathscr{B}_{\alpha}^{3}$	p. 375 (Definition 3. 1)
$\mathscr{B}_{\alpha}^{4}$	p. 375 (Definition 3. 1)
$\mathscr{B}_{\alpha}^{5}$	p. 376 (Definition 3. 2)
$\mathscr{B}_{\alpha}^{6}$	p. 376 (Definition 3. 2)
$\mathscr{B}_{\alpha}^{7}$	p. 377 (Definition 4. 1)
$\mathscr{B}_{\alpha,k}^{8}, \mathscr{B}_{\alpha}^{8}$	p. 379 (Definition 4. 2)
$\mathscr{B}_{\alpha,k}^{9}$	p. 381 (Definition 4. 3)
$\mathscr{B}_{\alpha}^{10}$	p. 382 (Definition 4. 4)

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