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ON SOME EXTREMAL PROBLEMS ON r-GRAPHS

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Abstract. Denote by $G^{(r)}(n;k)$ an r-graph of n vertices and k r-tuples. Turán's classical problem states: Determine the smallest integer $f(n;r,l)$ so that every $G^{(r)}(n;f(n;r,l))$ contains a $K^{(r)}(l)$. Turán determined $f(n;r,l)$ for $r=2$, but nothing is known for $r>2$. Put $\lim_{n=\infty} f(n;r,l)/\binom{n}{r} = c_{r,l}$. The values of $c_{r,l}$ are not known for $r>2$.

I prove that to every $\epsilon > 0$ and integer t there is an $n_0 = n_0(t,\epsilon)$ so that every $G^{(r)}(n;[(c_{r,l}+\epsilon)\binom{n}{r}])$ has lt vertices $x_i^{(j)}$, $1 \le i \le t$, $1 \le j \le l$, so that all the r-tuples $\{X_{i_1}^{(j_1)}, ..., X_{i_r}^{(j_r)}\}$, $1 \le i_s \le t$, $1 \le j_1 < ... < j_r \le l$, occur in our $G^{(r)}$. Several unsolved problems are posed.

By an r-graph $G^{(r)}$ $(r \ge 2)$ we shall mean a graph whose basic elements are its vertices and r-tuples; for $r=2$ we obtain the ordinary graphs.

$G^{(r)}(n)$ denotes an r-graph of n vertices.

$G^{(r)}(n;m)$ denotes an r-graph of n vertices and m r-tuples.

$K^{(r)}(n)$ will denote $G^{(r)}(n;\binom{n}{r})$, the complete r-graph of n vertices.

$K_l^{(r)}(n_1, ..., n_l)$ will denote the r-graph of $\Sigma_{i=1}^{l} n_i$ vertices $x_i^{(j)}$, $1 \le j \le l$, $1 \le i \le n_j$, and the r-tuples of our graph are all the r-tuples $(x_{i_1}^{(j_1)}, ..., x_{i_r}^{(j_r)})$, $1 \le j_1 < ... < j_r \le l$, $1 \le i_1 \le n_{j_1}, ..., 1 \le i \le n_{j_r}$.

$K_l^{(r)}(t)$ will denote $K_l^{(r)}(t, ..., t)$.

$e(G^{(r)})$ denotes the number of r-tuples in $G^{(r)}$. Thus $e(K_l^{(r)}(n_1, ..., n_l))$ equals the rth elementary symmetric function formed from $n_1, ..., n_l$.

$f(n;G^{(r)}(u;v))$ is the smallest integer for which every $G_1^{(r)}(n;f(n;G^{(r)}(u;v))$ contains $G^{(r)}(u;v)$ as a subgraph. Put

$$f(n;K_l^{(r)}(t)) = f_l^{(r)}(n;t), \qquad f(n;K^{(r)}(l)) = f_l^{(r)}(n;1) = f_l^{(r)}(n).$$

In other words $f_l^{(r)}(n)$ is the smallest integer for which every $G^{(r)}(n;f_l^{(r)}(n))$ contains a $K^{(r)}(l)$.

The function $f(n; G^{(2)}(u; v))$ was extensively studied in several recent papers [2, 4, 11]. Turán ([13], see also [12]), who started these investigations, determined $f_l^{(2)}(n)$ for every l and n (e.g. $f_3^{(2)}(n) = [\frac{1}{4}n^2] + 1$). He proved

$$(1) \qquad \lim_{n\to\infty} f_l^{(2)}(n)/\binom{n}{2} = 1 - \frac{1}{l-1}\,.$$

The values of $f_l^{(r)}(n)$ are unknown for every $r \geq 3$ and $l > r$, though Turán made many years ago several plausible conjectures. He conjectured, among others, that

$$f_4^{(3)}(3n) = 3n\binom{n}{2} + n^3 + 1\,, \qquad f_5^{(3)}(2n) = n^2(n-1) + 1\,.$$

It is known and easy to see that

$$(2) \qquad \lim_{n\to\infty} f_l^{(r)}(n)/\binom{n}{r} = c_{r,l}$$

exists, in fact it is shown in [9] that $f_l^{(r)}(n)/\binom{n}{r}$ is a nonincreasing sequence. The values of $c_{r,l}$ are not known for $r > 2$, $l > r$.

Stone and I [7] proved that for every $t \geq 1$ and $l > 2$,

$$(3) \qquad \lim_{n\to\infty} f_l^{(2)}(n; t)/\binom{n}{2} = 1 - \frac{1}{l-1}\,.$$

Let $G^{(2)}$ be an ordinary graph of chromatic number l. Simonovits and I [6] proved

$$(4) \qquad f(n; G^{(2)})/\binom{n}{2} = 1 - \frac{1}{l-1}\,.$$

(4) is an easy consequence of (3), since every l-chromatic graph $G^{(2)}$ is a subgraph of some $K_l^{(2)}(t)$.

Very little is known about $f(n; G^{(r)})$ for $r > 2$. I proved [3] that for every $r \geq 2$ and $t \geq 1$,

$$(5) \qquad f_r^{(r)}(n; t) < c_1 n^{r-\epsilon_{r,t}}\,.$$

For $r = 2$ this is a result of Kövari and the Turáns [10], who proved that $\epsilon_{2,t} \leq 1/t$. It seems likely that $\epsilon_{2,t} = 1/t$, but this is known only for

$t = 2$ and $t = 3$ ([1], see also [8]). The best possible values for $\epsilon_{r,t}$ are not known for $r > 2$.

Let

$$\frac{1}{2}\left(1-\frac{1}{l-2}\right) < \alpha \leq \frac{1}{2}\left(1-\frac{1}{l-1}\right).$$

(3) immediately implies that every $G^{(2)}(n; [\alpha n^2])$ contains a subgraph of $m = m(n)$ ($m \to \infty$ as $n \to \infty$) vertices which has at least $\frac{1}{2}m^2(1-1/(l-1))$ edges (it suffices to take the subgraph $K_l^{(2)}(t)$). It is easy to see that $\frac{1}{2}(1-1/(l-1))$ cannot be replaced by a larger number.

Let $G^{(r)}(n)$ be any graph having the vertices $x_1, ..., x_n$. $G^{(r)}(x_{i_1}, ..., x_{i_m})$ is the subgraph spanned by the vertices $x_{i_1}, ..., x_{i_m}$. By probabilistic methods [5], the following result can be proved:

Let $0 < \alpha < \frac{1}{2}$ and let $n \to \infty$. Then there is a $G^{(2)}(n; [\alpha n^2])$ so that for every $(m/\log n) \to \infty$ every subgraph $G^{(2)}(x_{i_1}, ..., x_{i_m})$ spanned by m vertices has $(\alpha + o(1))m^2$ edges. In other words, the edges are in a certain sense uniformly distributed over all large subgraphs. It can be shown that this result is also best possible in the following sense: Let $0 < \alpha < \frac{1}{2}$ and $G^{(2)}(n; [\alpha n^2])$ any graph. Then to every c there is an ϵ so that our graph has a spanned subgraph $G^{(2)}(x_{i_1}, ..., x_{i_m})$, $m > c \log n$ for which

$$(\alpha-\epsilon)m^2 < e(G^{(2)}(x_{i_1}, ..., x_{i_m})) < (\alpha+\epsilon)m^2$$

is not satisfied. We do not discuss the proof of these results in this paper.

(5) clearly implies that every $G^{(r)}(n; [\epsilon n^r])$ contains a subgraph of m vertices ($m = m(n)$, $m \to \infty$ as $n \to \infty$) which has at least (m^r/r^r) r-tuples. (To see this, it suffices to consider the subgraph $K_r^{(r)}(t)$ the existence of which is guaranteed by (5).) Unfortunately, this is the only result of this type which I can prove for $r > 2$. I am certain that the following result is true:

There is an absolute constant $c > 1/r^r$ so that every $G^{(r)}(n; [(n^r/r^r)(1+\epsilon)])$ contains a subgraph $G^{(r)}(m; [cm^r])$ where $m = m(n)$, $m \to \infty$ as $n \to \infty$.

I cannot even prove this conjecture for $r = 3$. On the other hand I can generalise (3) for r-graphs. In fact, I can prove the following:

Theorem. *For every $r \geq 2$, $l \geq r$ and $t \geq 1$,*

$$\lim_{n\to\infty} f_l^{(r)}(n; t)/\binom{n}{r} = c_{r,l}.$$

The only blemish is that we do not know the value of $c_{r,l}$ for $r > 2$.

To prove the Theorem we have to show that for every $r \geq 2$, $t \geq 1$ and $\epsilon > 0$, $G^{(r)}(n; [(c_{r,l}+\epsilon)\binom{n}{r}]$ always contains a $K_l^{(r)}(t)$ if $n > n_0(r, t, l, \epsilon)$. First we prove the following

Lemma. *For every $G^{(r)}(n; [(c_{r,l}+\epsilon)\binom{n}{r}])$ and every $m \geq r$ there is a sufficiently small $\eta = \eta(\epsilon) > 0$ so that for at least $\eta\binom{n}{m}$ m-tuples $x_{i_1}, ..., x_{i_m}$,*

$$e(G^{(r)}(x_{i_1}, ..., x_{i_m})) > (c_{r,l} + \tfrac{1}{2}\epsilon)\binom{m}{r}. \tag{6}$$

Proof of the Lemma. We evidently have (the summation is extended over all the $\binom{n}{m}$ m-tuples of n)

$$\sum e(G^{(r)}(x_{i_1}, ..., x_{i_m})) = \binom{n-r}{m-r}(c_{r,l}+\epsilon)\binom{n}{r}, \tag{7}$$

since each r-tuple of our $G^{(r)}(n; (c_{r,l}+\epsilon)\binom{n}{r})$ occurs in exactly $\binom{n-r}{m-r}$ m-tuples.

On the other hand, if our Lemma would not be true then for all but $\eta\binom{n}{m}$ of the r-tuples, the r-graph $G^{(r)}(x_{i_1}, ..., x_{i_m})$ has at most $(c_{r,l} + \tfrac{1}{2}\epsilon)\binom{m}{r}$ r-tuples and the remaining $\eta\binom{n}{m}$ graphs $G^{(r)}(x_{i_1}, ..., x_{i_m})$ can of course each have at most $\binom{m}{r}$ r-tuples. Thus we would have

$$\sum e(G^{(r)}(x_{i_1}, ..., x_{i_m}) < \binom{n}{m}\binom{m}{r}(c_{r,l} + \tfrac{1}{2}\epsilon) + \eta\binom{n}{m}\binom{m}{r} \tag{8}$$

$$< \binom{n}{m}\binom{m}{r}(c_{r,l} + \tfrac{3}{4}\epsilon)$$

for sufficiently small $\eta = \eta(\epsilon)$. (8) clearly contradicts (7) since $\binom{n}{m}\binom{m}{r} = \binom{n-r}{m-r}\binom{n}{r}$. This contradiction proves the Lemma.

Now we are ready to prove the Theorem. An l-tuple $(l > r)$ of our $G^{(r)}$ is called *good* if all its r-tuples occur in $G^{(r)}$.

Proof of the Theorem. Let $G^{(r)}(x_{i_1}, ..., x_{i_m})$ be any of the $\eta\binom{n}{m}$ subgraphs of $G^{(r)}(n; [(c_{r,l}+\epsilon)\binom{n}{r}])$ which satisfy (6). By the definition of $c_{r,l}$ this graph contains a $K^{(r)}(l)$ if $m > m_0(\epsilon)$, i.e. an l-tuple all whose r-tuples occur in the graph, in other words a good l-tuple. Thus there are at least $\eta\binom{n}{m}$ l-tuples all whose r-tuples occur in $G^{(r)}$. These good l-tuples are not, of course, all distinct, but the same l-tuple can occur in at most $\binom{n-l}{m-l}$

m-tuples. Hence for $m > m_0$ our graph contains at least

$$(9) \qquad \eta\binom{n}{m}/\binom{n-l}{m-l} > \eta\binom{n}{m}^l$$

good l-tuples. The good l-tuples define a $G^{(l)}(n)$ which, by (9) and (5), contain a $K_l^{(l)}(t)$ for every t if n is sufficiently large. By the definition of good l-tuples, the $K_r^{(l)}(t)$ having the same vertices as $K_l^{(l)}(t)$ occurs in $G^{(r)}(n; [(c_{r,l}+\epsilon)\binom{n}{r}])$ (i.e. all its r-tuples occur in $G^{(r)}$) and this completes the proof of the Theorem.

By the same method we can prove the following slightly more general result:

Let $G^{(r)}$ be any r-graph whose vertices are $x_1, ..., x_n$. $G^{(r)}(t)$ is defined as follows: Its vertices are $x_i^{(j)}$, $1 \le i \le n$, $1 \le j \le t$; an r-tuple $(x_{i_1}^{(j_1)}, ..., x_{i_r}^{(j_r)})$, $1 \le i_1 < ... < i_r \le n$, $1 \le j_s \le t$, $s = 1, ..., r$, belongs to $G^{(r)}(t)$ if and only if $(x_{i_1}, ..., x_{i_r})$ belongs to $G^{(r)}$. We then have for every $t \le 1$,

$$\lim_{n\to\infty} f(n; G^{(r)})/\binom{n}{r} = \lim_{n\to\infty} f(n; G^{(r)}(t)/\binom{n}{r} = G^{(r)}(c)\,.$$

Unfortunately, $G^{(r)}(c)$ is known only if $G^{(r)}$ is a subgraph of $K_r^{(r)}(t)$ for some t, in which case $G^{(r)}(c) = 0$. However, we can give a lower bound for $G^{(r)}(c)$ as follows:

$G^{(r)}$ defines an ordinary graph $G^{(2)}(G^{(r)})$ by: $G^{(2)}(G^{(r)})$ has the same vertices as $G^{(r)}$; two vertices of $G^{(r)}$ are joined in $G^{(2)}(G^{(r)})$ if and only if they belong to the same r-tuple of $G^{(r)}$. Let l be the chromatic number of $G^{(2)}(G^{(r)})$. If $l = r$, then $G^{(r)}$ is a subgraph of some $K_r^{(r)}(t)$ and $G^{(r)}(c) = 0$. In general, it is easy to see that

$$(10) \qquad G^{(r)}(c) \ge \prod_{s=0}^{r-1} \left(1 - \frac{s}{l-1}\right)\,.$$

In general, (10) is certainly not best possible.

Perhaps the following result holds: Every $G^{(3)}(3n; n^3+1)$ contains either a $G^{(3)}(4, 3)$ (the structure of this graph is unique) or a graph of 5 vertices $x_1, ..., x_5$ and four triples (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_2, x_5), (x_3, x_4, x_5), or a graph of 5 vertices and five triples (x_1, x_2, x_3), (x_1, x_2, x_4), (x_1, x_3, x_5), (x_2, x_4, x_5), (x_3, x_4, x_5).

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