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One some general problems in the theory of partitions, I

by

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To the memory of H. Davenport

1. In our fourth paper on statistical group theory (see [2]) we needed and proved that "almost all" sums of *different* prime powers not exceeding x consist essentially of

$$(1.1)\qquad (1+o(1))\frac{2\sqrt{6}}{\pi}\log 2\cdot\sqrt{\frac{x}{\log x}}$$

summands. Further needs of this theory make it necessary to find general theorems in this direction, i.e. when the summands are taken from a given sequence

$$(1.2)\qquad \Lambda\colon\ 0<\lambda_1<\lambda_2<\ldots$$

of integers. The only result we know in this direction refers to the case when Λ is the sequence of all positive integers. In this case Erdös and Lehner (see [1]) proved even the stronger result that almost all "unequal" partitions of n (i.e. with exception of at most $o(q(n))$ partitions of n into unequal parts) consist of

$$(1.3)\qquad (1+o(1))\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}$$

summands; here $q(n)$ stands for the number of unequal partitions of n for which according to Hardy and Ramanujan (see [3]) the relation

$$(1.4)\qquad q(n)=\frac{1+o(1)}{4\sqrt[4]{3}}\,n^{-\frac{3}{4}}e^{\frac{\pi}{\sqrt{3}}\sqrt{n}}$$

holds. Now we have found that having *only* asymptotical requirement on the counting function

$$(1.5)\qquad \Phi_\Lambda(x)=\sum_{\lambda_\nu\le x}1$$

we can prove general theorems. More exactly we assert

THEOREM I. *If with an $0 < a \leqslant 1$ and real β the relation*

$$\lim_{x\to\infty} \Phi_A(x) x^{-a} \log^{\beta} x = A$$

holds then for almost all systems

$$\lambda_{i_1} + \lambda_{i_2} + \ldots \leqslant N, \quad 1 \leqslant i_1 < i_2 < i_3 < \ldots, \tag{1.6}$$

the number of summands is

$$(1+o(1)) C_1 N^{a/(a+1)} \log^{-\beta/(a+1)} N, \quad C_1 = C_1(a, \beta, A) \tag{1.7}$$

for $N \to \infty$.

The explicit value of C_1 is

$$A^{1/(a+1)} \frac{\Gamma(a+1)\left(1-\dfrac{2}{2^a}\right)\zeta(a)(a+1)^{\beta/(a+1)}}{\left\{a\left(1-\dfrac{1}{2^a}\right)\zeta(a+1)\right\}^{a/(a+1)}}; \tag{1.8}$$

for $a = 1$ $\left(1-\dfrac{2}{2^a}\right)\zeta(a)$ means $\log 2$. "Almost all" means in this case that (1.7) holds with exception of $o(g(N))$ solutions of (1.6) at most where $g(n)$ stands for the total number of solutions of (1.6).

The proof will follow *mutatis mutandis* from that of

THEOREM II. *If for $x \to +\infty$*

$$\Phi_A(x) = A\frac{x^a}{\log^{\beta} x}\left\{1+O\left(\frac{1}{\log x}\right)\right\} \tag{1.9}$$

then for almost all solutions of (1.6) *the number of summands is*

$$C_1 N^{a/(a+1)} \log^{-\beta/(a+1)} N \{1+O(\log^{-1/4(a+1)} N)\}. \tag{1.10}$$

Moreover we remark that the number of solutions of (1.6) not satisfying (1.10) cannot exceed

$$\exp\{C_2 N^{a/(a+1)} \log^{-\beta/(a+1)} N (1 - C_3 \log^{-1/(2a+2)} N)\}$$

where $C_3 = C_3(a, \beta, A) > 0$ and

$$C_2 = a^{-a/(a+1)}(1+a)^{1+\beta/(a+1)}\{A(1-2^{-a})\zeta(a+1)\Gamma(a+1)\}^{1/(1+a)}. \tag{1.11}$$

For the sake of orientation we remark that in our case (1.9) the total number of solutions of (1.6) is

$$\exp\{C_2 N^{a/(a+1)} \log^{-\beta/(a+1)} N (1 + O(\log^{-1/(a+1)} N \log\log N))\}. \tag{1.12}$$

2. In the proof of Theorem II the fact that the λ_ν's are integers will not be used; it holds for real λ_ν's. Applying it with $\lambda_\nu = \log(\nu+1)$, $\nu = 1, 2, \ldots$ and $N = \log Y$ we get the

COROLLARY I. *Almost all factorizations*

$$x_1x_2x_3\ldots \leqslant Y, \quad 2 \leqslant x_1 < x_2 < \ldots$$

in different factors consist of

$$\frac{2\sqrt{3}\log 2}{\pi}\sqrt{\log Y}\{1+O(\log\log Y)^{-1/8}\}$$

factors.

3. Though it is not concerned with statistical group theory, Erdös–Lehner's theorem raises the natural question whether or not a general theorem analogous to Theorem II exists for the unequal Λ-partitions of n (of course the λ_ν's are positive integers again). Denoting by $p_\Lambda(n)$ the number of these partitions the easy combination of Theorem II and (1.12) we get

THEOREM III. *If beside the limes relation* (1.9) *the inequality* (C_2 in (1.11))

$$\log p_\Lambda(n) > C_2 n^{a/(a+1)}\log^{-\beta/(a+1)} n\,(1-\log^{-1/(2a+2)} n(\log\log n)^{-1}) \tag{3.1}$$

holds then the number of summands is

$$C_1 n^{a/(a+1)}\log^{-\beta/(a+1)} n\{1+O(\log^{-1/(4a+4)} n)\} \tag{3.2}$$

in every "unequal" Λ-partition of n with $o(p_\Lambda(n))$ exceptions at most.

As (1.4) shows (3.1) is in the case when Λ consists of all natural integers, amply satisfied; hence for almost all unequal partitions of n the number of summands is

$$\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}\{1+O(\log^{-1/8} n)\}. \tag{3.3}$$

Erdös–Lehner's proof gives the stronger estimation

$$\frac{2\sqrt{3}\log 2}{\pi}\sqrt{n}\{1+n^{-1/4}\omega(n)\}$$

if only $\omega(n) \nearrow \infty$ arbitrarily slowly; we got however (3.3) from a general theorem and used (1.4) very weakly. As shown by Ingham (see [5], p. 1086) the inequality (3.1) is amply satisfied for the Λ-sequence

$$1^k, 2^k, \ldots, \quad k \geqslant 1, \text{ integer}.$$

In this case we have

$$A = 1, \quad a = 1/k, \quad \beta = 0,$$

$$C_1 = \frac{\Gamma(1+1/k)(1-2^{1-1/k})\zeta(1/k)}{\{(1/k)(1-2^{-1/k})\zeta(1+1/k)\}^{1/(k+1)}} \overset{\text{def}}{=} C_1^*; \tag{3.4}$$

hence we got the

Corollary II. *Almost all partitions of n with different k-th powers of positive integers consists of*

$$C_1^* n^{1/(k+1)}\{1+O(\log^{-k/(4k+4)} n)\} \tag{3.5}$$

summands $(k \geq 1)$.

As to the requirement (3.1) in Theorem III this can be probably weakened. However some additional restriction on the sequence beyond (1.9) is necessary; (1.9) alone cannot assure even the existence of a single unequal Λ-partition of n.

4. It is again natural to ask the corresponding questions for partitions permitting repetition of the same summand, too. In the special case when Λ consists of all natural numbers, Erdös–Lehner l.c. found that almost all such partitions consist of

$$\frac{\sqrt{6}}{2\pi}\sqrt{n}\log n\left\{1+O\left(\frac{\omega(n)}{\log n}\right)\right\}$$

summands if only $\omega(x) \nearrow \infty$ arbitrarily slowly. For general Λ-sequences however — in contrast to Theorem II — asymptotical formulae like (1.9) are no more sufficient to assure a similar statistical law for the number of summands. We shall return to these seemingly more delicate problems as well as to finer laws of the distribution of summands in later papers of this series.

5. As told it is enough to prove Theorem II (with λ_j's not necessarily integers). Let $D(y)$ monotonically increasing so that

$$f(x) = \int_0^\infty e^{-xy}\,dD(y) \tag{5.1}$$

exists for $x > 0$. Then we state the

Lemma I. *Suppose that with an* $0 < a_1 \leq 1$, $A_1 > 0$ *and real* β_1, *the relation*

$$\log f(x) = \frac{A_1}{x^{a_1}\log^{\beta_1}(1/x)}\left\{1+O\left(\frac{\log\log(1/x)}{\log(1/x)}\right)\right\}$$

holds for $x \to +0$. *Then we have for* $y \to +\infty$

$$\log D(y) = C_4 y^{a_1/(a_1+1)}\log^{-\beta_1/(1+a_1)} y\,\{1+O(\log^{-1/(a_1+1)} y \log\log y)\}$$

with

$$C_4 = A_1^{1/(1+a_1)}(1+a_1)^{1+\beta_1/(1+a_1)}a_1^{-a_1/(a_1+1)}.$$

Without remainder term this is due to Hardy and Ramanujan (see [4]). A detailed proof for the case $a_1 = \beta_1 = 1$ can be found in our paper [2]; the present more general case follows *mutatis mutandis*.

6. Next let $Q(N)$ stand for the number of solutions of (1.6) and

$$F_Q(x) = \int_0^\infty e^{-xy}\,dQ(y). \tag{6.1}$$

Then we have evidently

$$F_Q(x) = \prod_{\nu=1}^\infty (1+e^{-\lambda_\nu x}). \tag{6.2}$$

Let further with a positive integer m

$$Q_m(y) = \sum_{\substack{\lambda_{l_1}+\lambda_{l_2}+\ldots<y \\ l_1<l_2<\ldots<l_m}} 1 \tag{6.3}$$

and

$$F_{Q_m}(x) = \int_0^\infty e^{-xy}\,dQ_m(y). \tag{6.4}$$

Putting for $r>0$

$$G_Q(x, r) = 1+\sum_{m=1}^\infty e^{-mr} F_{Q_m}(x) \tag{6.5}$$

we have evidently

$$G_Q(x, r) = \prod_{\nu=1}^\infty (1+e^{-r-\lambda_\nu x}). \tag{6.6}$$

7. We shall need the

LEMMA II. (1.9) *implies for* $x\to+0$

$$\log F_Q(x) = C_5 x^{-a}\log^{-\beta}\frac{1}{x}\left\{1+O\left(\log^{-1}\frac{1}{x}\operatorname{loglog}\frac{1}{x}\right)\right\} \tag{7.1}$$

with

$$C_5 = A\left(1-\frac{1}{2^a}\right)\Gamma(a+1)\zeta(a+1). \tag{7.2}$$

For the proof we remark that representation (6.2) gives at once

$$\log F_Q(x) = \int_0^\infty \log(1+e^{-xy})\,d\Phi_A(y) = x\int_0^\infty \frac{\Phi_A(y)}{1+e^{xy}}\,dy.$$

(1.9) gives from this

$$\log F_Q(x) = Ax\int_0^\infty \frac{y^a}{\log^\beta(y+2)}\cdot\frac{dy}{1+e^{xy}} + O(x)\int_0^\infty \frac{y^a}{\log^{\beta+1}(y+2)}\cdot\frac{dy}{1+e^{xy}}. \tag{7.3}$$

The contribution of the range $y < x^{-1}\log^{-1/\alpha}(1/x)$ to both integrals is (roughly)

$$O\left(\frac{1}{x^{\alpha}\log^{\beta+1}(1/x)}\right). \tag{7.4}$$

The same holds as is easy to see, for $y > 10x^{-1}\log(1/x)$. The remaining part of the second term in (7.3) is evidently

$$O\left(\frac{x}{\log^{\beta+1}(1/x)}\right)\int_0^{\infty}\frac{y^{\alpha}}{1+e^{xy}}\,dy = O\left(\frac{1}{x^{\alpha}\log^{\beta+1}(1/x)}\right). \tag{7.5}$$

Replacing in the remaining part of the first term in (7.3) $\log^{\beta}(y+2)$ by $\log^{\beta}(1/x)$ the error is

$$O\left(\frac{1}{x^{\alpha}}\cdot\frac{\log\log(1/x)}{\log^{\beta+1}(1/x)}\right).$$

A further easy reasoning gives — with the same error term — for the main term

$$\frac{Ax}{\log^{\beta}(1/x)}\int_0^{\infty}\frac{y^{\alpha}}{1+e^{xy}}\,dy = \frac{Ax^{-\alpha}}{\log^{\beta}(1/x)}\int_0^{\infty}\frac{y^{\alpha}}{1+e^{y}}\,dy = C_5x^{-\alpha}\log^{-\beta}\frac{1}{x}$$

indeed (C_5 in (7.2)).

Combining Lemmas I and II we obtain

$$\log Q(N) = C_2N^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}N\{1+O(\log^{-1/(\alpha+1)}N\log\log N)\} \tag{7.6}$$

indeed (C_2 in (1.11)).

8. Let further

$$R(x) = \sum_{\nu=1}^{\infty}\frac{1}{e^{\lambda_\nu x}+1}. \tag{8.1}$$

We shall need the

Lemma III. *For $x \to +0$ the relation*

$$R(x) = C_6x^{-\alpha}\log^{-\beta}\frac{1}{x}\left\{1+O\left(\frac{\log\log(1/x)}{\log(1/x)}\right)\right\} \tag{8.2}$$

holds with

$$C_6 = A\Gamma(\alpha+1)\left(1-\frac{2}{2^{\alpha}}\right)\zeta(\alpha). \tag{8.3}$$

The proof of this lemma follows that of Lemma II *mutatis mutandis*; instead of the integral formula

$$A\int_0^{\infty}\frac{y^{\alpha}}{1+e^{y}}\,dy = C_5$$

we need

$$A\int_0^\infty \frac{y^a e^y}{(1+e^y)^2}\,dy = C_6.$$

9. Now we may turn to the proof of Theorem II. Let

(9.1) $$M = M(N) \nearrow \infty, \quad r_0 = r_0(N) \searrow 0, \quad x_0 = x_0(N) \searrow 0$$

to be determined later and we start from (6.5). This gives

$$1+\sum_{1\leqslant m\leqslant M} F_{Q_m}(x_0)e^{-mr_0} \leqslant G_Q(x_0, r_0)$$

and *a fortiori*

(9.2) $$\sum_{m\leqslant M} F_{Q_m}(x_0) \leqslant G_Q(x_0, r_0)e^{Mr_0}.$$

Since for each fixed m (6.4) gives

$$F_{Q_m}(x_0) \geqslant \int_0^N e^{-x_0 y}\,dQ_m(y) \geqslant e^{-Nx_0}\int_0^N dQ_m(y) = e^{-Nx_0}Q_m(N),$$

we get from (9.2)

(9.3) $$\sum_{m\leqslant M} Q_m(N) \leqslant G_Q(x_0, r_0)e^{Mr_0+Nx_0} = F_Q(x_0)\left\{\frac{G_Q(x_0, r_0)}{F_Q(x_0)}\right\}e^{Mr_0+Nx_0}.$$

The expression in curly bracket is

$$\prod_{\nu=1}^\infty \frac{1+e^{-r_0-\lambda_\nu x_0}}{1+e^{-\lambda_\nu x_0}} = \prod_{\nu=1}^\infty \left\{1-\frac{(1-e^{-r_0})e^{-\lambda_\nu x_0}}{1+e^{-\lambda_\nu x_0}}\right\}$$

$$< \exp\left\{(e^{-r_0}-1)\sum_{\nu=1}^\infty \frac{1}{e^{\lambda_\nu x_0}+1}\right\} < \exp\left\{-r_0\left(1-\frac{r_0}{2}\right)R(x_0)\right\}.$$

From this and Lemma III we obtain from (9.3)

$$\sum_{m\leqslant M} Q_m(N) \leqslant F_Q(x_0)e^{Nx_0}\times$$

$$\times\exp\left(r_0\left\{M-\left(1-\frac{r_0}{2}\right)C_6 x_0^{-a}\log^{-\beta}\frac{1}{x_0}\left(1+O\left(\frac{\log\log(1/x_0)}{\log(1/x_0)}\right)\right)\right\}\right).$$

Applying Lemma II this gives

(9.4) $$\sum_{m\leqslant M} Q_m(N) \leqslant \exp\left(Nx_0+\frac{C_5}{x_0^a\log^\beta(1/x_0)}\left\{1+O\left(\frac{\log\log(1/x_0)}{\log(1/x_0)}\right)\right\}+\right.$$

$$\left.+r_0\left\{M-\left(1-\frac{r_0}{2}\right)\frac{C_6}{x_0^a\log^\beta(1/x_0)}\left(1+O\left(\frac{\log\log(1/x_0)}{\log(1/x_0)}\right)\right)\right\}\right).$$

10. Now we choose with a constant λ to be determined later

(10.1) $$\frac{1}{x_0} = \lambda N^{1/(1+a)} \log^{\beta/(1+a)} N.$$

Then

$$Nx_0 + \frac{C_5}{x_0^a \log^\beta (1/x_0)} = N^{a/(1+a)} \log^{-\beta/(a+1)} N \left\{ \frac{1}{\lambda} + C_5 \lambda^a (1+a)^\beta \left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)\right\}.$$

We want to determine λ so that

(10.2) $$\frac{1}{\lambda} + C_5 \lambda^a (1+a)^\beta = C_2 = a^{-a/(a+1)} (1+a)^{1+\beta/(1+a)} C_5^{1/(1+a)}$$

(using (7.2) and (1.11)). This can however be written in the form

$$\frac{a}{(\lambda C_5^{1/(1+a)} (1+a)^{\beta/(1+a)} a^{1/(1+a)})} + \{\lambda C_5^{1/(1+a)} (1+a)^{\beta/(1+a)} a^{1/(1+a)}\}^a = a+1,$$

which means that

$$x = \lambda C_5^{1/(1+a)} (1+a)^{\beta/(1+a)} a^{1/(1+a)}$$

satisfies the equation

$$\frac{a}{x} + x^a = a+1$$

which is satisfied with $x = 1$. Thus choosing

(10.3) $$\lambda = C_5^{-1/(1+a)} (1+a)^{-\beta/(1+a)} a^{-1/(1+a)}$$

and using (10.1), (9.4) can be written as

$$\sum_{m \leqslant M} Q_m(N) \leqslant \exp\Big(C_2 N^{a/(a+1)} \log^{-\beta/(a+1)} N \left\{1 + O\left(\frac{\log\log N}{\log N}\right)\right\} + $$
$$ + r_0 \left\{M - \left(1 - \frac{r_0}{2}\right) C_6 \lambda^a (1+a)^\beta N^{a/(a+1)} \log^{-\beta/(a+1)} N \left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)\right\}\Big).$$

Taking (7.6) into account this takes the form

(10.4) $$\sum_{m \leqslant M} Q_m(N) \leqslant Q(N) \exp\Big(O(N^{a/(a+1)} \log^{-(\beta+1)/(a+1)} N \log\log N) + $$
$$ + r_0 \left\{M - \left(1 - \frac{r_0}{2}\right) C_1 N^{a/(a+1)} \log^{-\beta/(a+1)} N \left(1 + O\left(\frac{\log\log N}{\log N}\right)\right)\right\}\Big)$$

owing to (8.3), (10.3), (7.2) and (1.8). Choosing

(10.5) $$r_0 = \log^{-1/(4a+4)} N$$

and

(10.6) $$M = M_0 \overset{\text{def}}{=} C_1 N^{\alpha/(\alpha+1)} \log^{-\beta/(\alpha+1)} N (1 - 2\log^{-1/(4\alpha+4)} N)$$

(10.4) takes the form

(10.7) $$\sum_{m \leqslant M_0} Q_m(N) \leqslant Q(N) \exp\{-cN^{\alpha/(\alpha+1)} \log^{-(\beta+\frac{1}{2})/(\alpha+1)} N\}$$

with an unspecified p o s i t i v e constant c. This proves the first half of the Theorem II, concerning the solutions of (1.6) with "few" summands.

11. Now we have to dispose with the solution of (1.6) with "too many" summands. The form of $G_Q(x, r)$ in (6.6) shows that $G_Q(x_0, r)$ (with the x_0 in (10.1)) is an entire function of r and hence if

(11.1) $$\tfrac{1}{2} \geqslant r_1 = r_1(N) \searrow 0$$

to be determined later, then Cauchy's coefficient estimation can be applied to the segment

(11.2) $$\operatorname{Re} r = -r_1, \quad 0 \leqslant \operatorname{Im} r < 2\pi.$$

This gives for each integer m

$$e^{m|r_1|} F_{Q_m}(x_0) \leqslant G_Q(x_0, -r_1)$$

and hence as in Section 9

(11.3) $$Q_m(N) \leqslant e^{Nx_0 - m|r_1|} G_Q(x_0, -r_1).$$

If

(11.4) $$M_1 = M_1(N) \nearrow \infty$$

to be determined later then summation with respect to $m \geqslant M_1$ gives

(11.5) $$\sum_{m \geqslant M_1} Q_m(N) \leqslant e^{Nx_0 - M_1 r_1} G_Q(x_0, -r_1) \frac{1}{1 - e^{-r_1}}$$
$$\leqslant \frac{2}{r_1} e^{Nx_0 - M_1 r_1} G_Q(x_0, -r_1).$$

The representations (6.2) and (6.6) give

(11.6) $$\sum_{m \geqslant M_1} Q_m(N) \leqslant \frac{2}{r_1} \{F_Q(x_0) e^{Nx_0}\} \left\{ e^{-M_1 r_1} \prod_{\nu=1}^{\infty} \frac{1 + e^{r_1 - \lambda_\nu x_0}}{1 + e^{-\lambda_\nu x_0}} \right\}$$
$$= \frac{2}{r_1} \{F_Q(x_0) e^{Nx_0}\} \left\{ e^{-M_1 r_1} \prod_{\nu=1}^{\infty} \left(1 + \frac{e^{r_1} - 1}{e^{\lambda_\nu x_0} + 1}\right) \right\}$$
$$< \frac{2}{r_1} \{F_Q(x_0) e^{Nx_0}\} \exp\{-M_1 r_1 + (e^{r_1} - 1) R(x_0)\}$$
$$< \frac{2}{r_1} \{F_Q(x_0) e^{Nx_0}\} \exp\left(r_1\{-M_1 + (1 + r_1) R(x_0)\}\right).$$

Repeating the reasoning in Section 10 we can derive from (11.6)

$$(11.7)\qquad \sum_{m\geqslant M_1} Q_m(N)\leqslant \frac{2}{r_1}Q(N)\exp\Big(O(N^{\alpha/(\alpha+1)}\log^{-(\beta+1)/(\alpha+1)}N\log\log N)+ +r_1\Big\{-M_1+(1+r_1)C_1N^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}N\Big(1+O\Big(\frac{\log\log N}{\log N}\Big)\Big)\Big\}\Big).$$

Now choosing

$$(11.8)\qquad \begin{aligned} M_1 &= C_1N^{\alpha/(\alpha+1)}\log^{-\beta/(\alpha+1)}N(1+2\log^{-1/(4\alpha+4)}N),\\ r_1 &= \log^{-1/(4\alpha+4)}N \end{aligned}$$

(11.7) gives

$$\sum_{m\geqslant M_1} Q_m(N)\leqslant Q(N)\exp(-cN^{\alpha/(\alpha+1)}\log^{-(\beta+1)/(\alpha+1)}N)$$

with an unspecified positive c. This completes the proof.

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