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# ORDINARY PARTITION RELATIONS FOR ORDINAL NUMBERS

by

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*To* RICHARD RADO *for his 65th birthday*

## § 1. Introduction

The ordinary partition symbol invented by R. RADO and first introduced in [1] enables us to study systematically the possible generalizations of RAMSEY's theorem.

Let $\alpha$, $\beta_0, \ldots, \hat{\beta}_\gamma$ be either order types or cardinals $\gamma$ an ordinal and $r$ a cardinal and assume that $\beta_0, \ldots, \hat{\beta}_\gamma$ are cardinals if $\alpha$ is a cardinal. Then

(1.1) $$\alpha \to (\beta_0, \ldots, \hat{\beta}_\gamma)^r \text{ or equivalently } \alpha \to (\beta_\nu)^r_{\nu<\gamma}$$

denotes that the following statement is true.

Let $(S, \prec)$ be an ordered set; tp $S(\prec) = \alpha$ or let $S$ be a set $|S| = \alpha$ if $\alpha$ is an order type or $\alpha$ is a cardinal respectively. Let $[S]^r = \{X : X \subset S \wedge \wedge |X| = r\} = \bigcup (\nu < \gamma) J_\nu$ be an arbitrary $r$-partition of length $\gamma$ of $S$. Then there exist a subset $S' \subset S$ and an ordinal $\nu < \gamma$ such that

$$[S']^r \subset J_\nu$$

and tp $S'(\prec) = \beta_\nu$ if $\beta_\nu$ is an order type or $|S'| = \beta_\nu$ if $\beta_\nu$ is a cardinal respectively. $\alpha \nrightarrow (\beta_0, \ldots, \hat{\beta}_\gamma)^r$ denotes that the negation of the above statement is true.

In [1], [2] and [3] several generalisations of (1.1) had been defined and a general partition calculus had been developed. In [3] an almost complete discussion of (1.1) is given in case the entries $\alpha$, $\beta_0, \ldots, \hat{\beta}_\gamma$ are cardinals and G. C. H. is assumed. In a forthcoming book of R. Rado and the authors this discussion will be given without using G. C. H.

In this paper we will consider some special problems for the ordinary partition relation in case the entries are ordinals. We will only consider the case $r = 2$, and in most of the cases we assume $\gamma = 2$ too.

Even the problems concerning these special cases are rather ramified. In our paper [4] we gave a collection of typical unsolved problems. Here we will consider only one type of these problems.

We will investigate the problem

$$\alpha \to (\beta, \gamma)^2 \tag{1.2}$$

where $\alpha$, $\beta$, $\gamma$ are ordinals. We will assume $|\alpha| = \aleph_\xi$, $\beta \geq \gamma$. It is easy to see that if in this case $\beta = \omega_\xi$ the problem (1.2) can be reduced to a problem involving only cardinals. In case $\beta < \omega_\xi$ difficult problems arise which we do not consider here. See Problems 10, 10/A, 10/B of [4]. We will consider the problems where $\beta > \omega_\xi$.

On the other hand with an easy SIERPIŃSKI-type argument one can establish the negative relation

$$\alpha \nrightarrow (\omega_\xi + 1, \aleph_0)^2 \qquad \text{for every } \xi \geq 0. \tag{1.3}$$

Thus we will be interested in problems of the following type

$$\begin{gathered} \alpha \to (\beta, k)^2, \quad 3 \leq k < \omega, \quad \xi > 0 \\ \omega_\xi + 1 \leq \beta \leq \alpha < \omega_{\xi+1}. \end{gathered} \tag{1.4}$$

There are many results and problems concerning (1.4) even in the simplest case $\xi = 0$. We collected these results and problems in [4] 3.2 and Problems 6 and 7. We only mention that one of the most difficult problems (Problem 7) of [4] has been recently solved by CHANG[1]. He proved $\omega^\omega \to (\omega^\omega, 3)^2$ but his proof does not yield

$$\omega^\omega \to (\omega^\omega, 4)^2.$$

All the positive results for (1.4) in case $\xi = 0$ make use of Ramsey's theorem

$$\aleph_0 \to (\aleph_0)^r_k \qquad r, k < \omega.$$

This is the case with SPECKER's result $\omega^2 \to (\omega^2, k)^2$; $k < \omega$. For references see [4]. Thus in case $\xi > 0$ one of the first questions was if

$$\omega_1^2 \to (\omega_1^2, 3)^2 \qquad \text{holds or not?}$$

See problem 13 of [4].

The second author proved recently the following result (see [5]):

$$\omega_\xi^2 \nrightarrow (\omega_\xi^2, 3)^2 \tag{1.5}$$

provided $\xi = \eta + 1$ and $\aleph_\eta$ is regular.

On the other hand, P. ERDŐS and R. RADO proved in [6] the following result

Let $\xi$ be arbitrary, $k, l < \omega$. Then there is $f(k, l) < \omega$ such that

$$\omega_\xi \,.\, f(k, l) \to (\omega_\xi k, l)^2. \tag{1.6}$$

[1] See C. C. CHANG, A theorem in combinatorial set theory (to appear). Using CHANG's method E. MILNER proved $\omega^\omega \to (\omega^\omega, k)^2$ for all $k < \omega$.

Our real aim in this paper is to push further the results (1.5) and (1.6) for the cases $\xi > 0$, to establish several consequences and to fill up some of the gaps. We are going to state a number of related problems and results which can not be formulated in terms of the ordinary partition symbol.

In certain cases we only outline or entirely omit the proofs.

We use the usual notation of set theory. We mention that each ordinal is considered to be the set of smaller ordinals. The notation $\omega_\xi$ and $\aleph_\xi$ mean the same ($\omega_0 = \omega$). We agree that $\omega_\xi^{\omega_\eta}$ denotes the ordinal power while $\aleph_\xi^{\aleph_\eta}$ denotes the cardinal power i.e. $\omega^\omega = \omega_0^{\omega_0}$ is a denumerable ordinal while $\aleph_0^{\aleph_0}$ is the cardinality and the initial number of the continuum.

## § 2. Some negative results using G.C.H.

First we introduce some special notation. Let $\xi, \eta$ be ordinals and $f$ a cardinal valued function with $D(f) \subset \omega_\eta$, $f(\nu) \leqq \omega_\xi$ for $\nu \in D(f)$. Put $A = \omega_\xi \times \omega_\eta$, $A_\nu = \omega_\xi \times \{\nu\}$ for $\nu \in \omega_\eta$, $S(\xi, \eta, f) = \{X \subset A : |X \cap A_\nu| = f(\nu)$ for $\nu \in D(f)\}$.
We further put

$$S(\xi, \eta, \varkappa, \beta) = \bigcup \{f : |D(f)| = \omega_\beta \wedge f(\nu) = \omega_\varkappa\}\, S(\xi, \eta, f).$$

We prove

THEOREM 1.[2] *Assume* G.C.H. *Let $\xi = \eta + 1$, $\eta$ arbitrary. Put $S_0 = S(\eta + 1, \eta, \eta + 1, \eta)$. Then there exists $I \subset [A]^2$ satisfying the following conditions:*

a) $X \subset A$, $[X]^2 \subset I$ *imply* $|X| < 3$

b) $X \subset A$, $X \in S_0$ *imply* $[X]^2 \cap I \neq 0$.

Theorem 1 is obviously equivalent to

COROLLARY 1. Assume G.C.H. and $\varrho < \omega_{\eta+1}$. Then

$$\omega_{\eta+1}\, \varrho \nrightarrow (\omega_{\eta+1}\, \omega_\eta, 3)^2.$$

This should be compared with (1.5) and (1.6).

We will also prove the rather special result

COROLLARY 2. Assume G.C.H, $\sigma < \omega_2$. Then

$$\sigma \nrightarrow (\omega_1^\omega, 3)^2.$$

The proof of Corollary 2 will be given on p. 175.

[2] This result was already stated in [4] without proof. See the remarks concerning Problem 13.

Strangely enough this result does not generalize and e.g. it is not known if

$$\sigma \nrightarrow (\omega_2^{\omega_1}\ ;\ 3)^2$$

holds for $\omega_2^{\omega_1} < \sigma < \omega_3$. We shall return to this problem on p. 176.

Instead of Theorem 1 we prove the stronger

THEOREM 1/A. *Assume* G.C.H. *Let* $\xi = \eta + 1$, $\eta$ *arbitrary. Put* $S_1 = S(\eta + 1, \eta, \eta, \eta)$. *Then there exists* $I \subset [A]^2$ *satisfying the following conditions*

a) $X \subset A$, $|X| = 3$ *imply* $[X]^2 \not\subset I$,

b) $X \subset A$, $|X \cap A_\nu| = \aleph_{\eta+1}$ *for some* $\nu < \omega_\eta$ *and* $Y \in S_1$ *for some* $Y \subset X$

*imply* $[X]^2 \cap I \neq 0$,

c) *For every* $\nu < \nu' < \omega_\eta$ *and for every* $x \in A_\nu$ *there is at most one* $y \in A_{\nu'}$ *with* $\{x, y\} \in I$.

Theorem 1/A is obviously stronger than Theorem 1 since $X \subset A$, $X \in S_0$ implies both $|X \cap A_\nu| = \aleph_{\eta+1}$ for some $\nu < \omega_\eta$ and $Y \in S_1$ for some $Y \subset X$.

PROOF of Theorem 1/A.

(1) Let $\{x_\alpha\}_{\alpha<\omega_{\eta+1}} = A$ be the well-ordering of $A$ satisfying $x_\alpha = (\mu_\alpha, \nu_\alpha)$ for $\alpha < \omega_{\eta+1}$ and $\alpha < \beta$ iff either $\mu_\alpha < \mu_\beta$ or $\mu_\alpha = \mu_\beta$ and $\nu_\alpha < \nu_\beta$. The type of this well ordering is indeed $\omega_{\eta+1}$.

For every $\alpha < \omega_{\eta+1}$ we put

$$Z_\alpha = \{(\mu, \nu) : \mu < \mu_\alpha \wedge \nu > \nu_\alpha\} = \mu_\alpha \times \big(\omega_\eta - (\nu_\alpha + 1)\big).$$

We are going to define a function $f$ such that $f(x_\alpha) \subset Z_\alpha$ for every $\alpha < \omega_{\eta+1}$ by transfinite induction on $\alpha$. Our intention is to put

$$I = \big\{\{x_\alpha, x_\beta\} : \beta < \alpha \wedge x_\beta \in f(x_\alpha)\big\}. \tag{2}$$

By G.C.H. there is a well-ordering $\{Y_\tau\}_{\tau<\omega_{\eta+1}} = S_1$ of type $\omega_{\eta+1}$ of $S_1$. Put $\mathcal{H}_\alpha = \{Y_\tau \in S_1 : \tau < \alpha \wedge Y_\tau \subset Z_\alpha\}$.

Assume that $\alpha < \omega_{\eta+1}$ and $f(x_\beta)$ is defined for every $\beta < \alpha$ in such a way that $f(x_\beta) \subset Z_\beta$ and $|f(x_\beta) \cap A_\nu| \leq 1$ holds for every $\nu < \omega_\eta$.

Now we claim that there exists a set $B$ satisfying the following conditions

$$\begin{gathered} B \subset Z_\alpha,\ \ |B \cap A_\nu| \leq 1 \quad \text{for every } \nu < \omega_\eta, \\ B \cap Y_\tau \neq 0 \text{ for every } Y_\tau \in \mathcal{H}_\alpha \end{gathered} \tag{3}$$

and $x_\gamma \notin f(x_\beta)$ for every pair $\beta, \gamma < \alpha$, $x_\beta, x_\gamma \in B$.

We only outline the proof of (3). By a well-known theorem of BERNSTEIN and by $|\mathcal{H}_\alpha| \leq \aleph_\eta$ the elements of $\mathcal{H}_\alpha$ can be represented in different

$A_\nu$-s, i.e. For every $\nu_\alpha < \nu < \omega_\eta$ one can define a subset $F_\nu \subset A_\nu$, $|F_\nu| = \aleph_\eta$ such that for each $Y_\tau \in \mathcal{H}_\alpha$, $F_\nu \subset Y_\tau$ for some $\nu_\alpha < \nu < \omega_\eta$. Then by transfinite induction on $\nu$, one can pick an element $b_\nu$ of $F_\nu$ in such a way that $b_\nu \in F_\nu - \bigcup (\nu' < \nu) f(b_{\nu'})$, $b_\nu \in Z_\alpha$. The set $B = \{b_\nu\}_{\nu_\alpha < \nu < \omega_\eta}$ obviously satisfies the requirements of (3).

We now put $f(x_\alpha) = B$ for a set $B$ satisfying (3). Then $f(x_\alpha) \subset Z_\alpha$ and $|f(x_\alpha) \cap A_\nu| \leq 1$ for $\nu < \omega_\eta$ hence $f(x_\alpha)$ is defined for every $\alpha < \omega_{\eta+1}$.

We prove that the $I$ defined by (2) satisfies the requirements a), b) and c). Assume $X \subset A$, $|X| = 3$, $[X]^2 \subset I$. We can put $X = \{x_\alpha, x_\beta, x_\gamma\}$. We can assume that $\nu_\alpha = \min\{\nu_\alpha, \nu_\beta, \nu_\gamma\}$. Then, by (2), (3) and by $[X]^2 \subset I$, we must have $x_\beta, x_\gamma \in f(x_\alpha) \subset Z_\alpha$. Thus $\mu_\beta, \mu_\gamma < \mu_\alpha$, and as a corollary $\beta, \gamma < \alpha$. Then, by (3), $x_\beta \notin f(x_\gamma)$, $x_\gamma \notin f(x_\beta)$ hence by (2), $\{x_\beta, x_\gamma\} \notin I$. This contradicts $[X]^2 \subset I$ and a) follows.

Let now $X \subset A$, $\nu < \omega_\eta$, $Y \subset X$ be such that $|X \cap A_\nu| = \aleph_{\eta+1}$ and $Y \in S_1$. $Y \in S_1$ implies that there are $Y_\tau \subset Y$ and $\mu < \omega_{\eta+1}$ such that $Y_\tau \subset Z_{\alpha_0}$, $x_{\alpha_0} = (\mu, \nu)$. By $|X \cap A_\nu| = \aleph_{\eta+1}$, there is $\alpha > \tau$ with $x_\alpha \in$ $\in X \cap A_\nu$ and $\mu_\alpha > \mu$. Then $\nu_\alpha = \nu$, hence $Y_\tau \subset Z_\alpha$ and thus $Y_\tau \in \mathcal{H}_\alpha$. By (3), $f(x_\alpha) \cap Y_\tau \neq 0$ and, by (2) this means $[X]^2 \cap I \neq 0$. This proves b). c) follows from (2) considering that, by (3), $|f(x_\alpha) \cap A_\nu| \leq 1$ for every $\nu < \omega_\eta$.

To prove Corollary 2 we need the following result often cited as the MILNER—RADO paradox [7]. In terms of (1.1) this can be expressed as follows

MILNER—RADO THEOREM. *Let* $\alpha > 0$, $\sigma < \omega_{\alpha+1}$. *Then*

$$\sigma \nrightarrow (\omega_\alpha^n)^1_{n<\omega}. \tag{2.1}$$

To prove Corollary 2 we apply (2.1) in its first nontrivial instance $\alpha = 1$.

PROOF of Corollary 2.

Let now $\sigma < \omega_2$. Let $B_0 \cup \ldots \cup \hat{B}_\omega$ be a disjoint 1-partition of $\sigma$ establishing the negative relation (2.1). We can assume $|B_n| = \aleph_1$ for every $n < \omega$, and that the $B_n$ are disjoint.

By Theorem 1, there is an $I \subset [\sigma]^2$ such that

(i) $X \subset \sigma$, $|X| = 3$ imply $[X]^2 \not\subset I$,

(ii) $X \subset \sigma$, $[X]^2 \cap I = 0$ imply $|X \cap B_n| < \aleph_1$ for all but finitely many $n < \omega$.

Put $\mathfrak{I}_0 = [\sigma]^2 - I$, $\mathfrak{I}_1 = I$. Then the partition $[\sigma]^2 = \mathfrak{I}_0 \cup \mathfrak{I}_1$ establishes

$$\sigma \nrightarrow (\omega_1^\omega, 3)^2.$$

By (i) it is sufficient to see that

$$X \subset \sigma,\ [X]^2 \subset \mathfrak{I}_0 \text{ imply } \operatorname{tp} X\,(<) < \omega_1^\omega.$$

By (ii) $[X]^2 \subset \mathfrak{I}_0$ implies that $X = X_0 \cup \ldots \cup X_{n-1} \cup Y$ where $|Y| < \aleph_1$; $\operatorname{tp} X_i\,(<) < \omega_1^i$ for $i < n$ $\omega_1^\omega$ being indecomposable this yields $\operatorname{tp} X\,(<) < \omega_1^\omega$.

Remarks and problems.

PROBLEM 1. Assume G.C.H. Is it true that

$$\omega_2^\omega \nrightarrow (\omega_2^\omega, 3)^2 \ ?$$

Is it true that

$$\sigma \nrightarrow (\omega_2^{\omega_1}, 3)^2 \ \text{ for } \ \omega_2^{\omega_1} \leq \sigma < \omega_3 \ ?$$

This would be a straightforward generalization of the special result stated in Corollary 2.

PROBLEM 2. Assume G.C.H. Is it true that

$$\omega_2\omega \nrightarrow (\omega_2\,\omega, 3)^2 \ ?$$

Is it true that $\alpha\,\omega \nrightarrow (\alpha\,\omega, 3)^2$ for every cardinal $\omega < \alpha < \alpha_0$, $cf(\alpha) \neq \omega$ where $\alpha_0$ is the first cardinal $> \omega$ for which $\alpha_0 \nrightarrow (\alpha_0, \alpha_0)^2$ ?

The first part of Problem 2 was already stated in [4] Problem 13. It is obvious that a positive solution of the first part of Problem 2 would imply a positive solution of the first stronger part of Problem 1. It is easy to see that $\alpha_0\omega \rightarrow (\alpha_0\omega, 3)^2$ holds. It is intrigueing that we have no information for any smaller cardinal with $cf(\alpha) \neq \omega$. We mention that as a generalisation of Specker's result $\omega^2 \rightarrow (\omega^2, k)^2$ for $k \subset \omega$ we can prove

THEOREM 2. *Let $\alpha$ be a strong limit cardinal (i.e. $2^{|\beta|} < \alpha$ for $\beta < \alpha$) and assume $cf(\alpha) = \omega$.*

*Then $\alpha\,\omega \rightarrow (\alpha\,\omega, k)^2$ holds for $k < \omega$.*

The proof can be carried out using the canonization method described in [3], Lemma 3 and applying $\omega^2 \rightarrow (\omega^2, k)^2$. We omit the details but we mention that the argument breaks down if we want to apply it for the proof of

$$\text{G.C.H.} \Rightarrow \omega_\omega^2 \rightarrow (\omega_\omega, 3)^2$$

and we can not decide if this relation holds or not.

As to the second part of Problem 1 we mention that assuming G.C.H.

$$\sigma \nrightarrow (\omega_2^{\omega_1}, 3)^2$$

holds for $\sigma < \omega_2^{\omega_2}$. This is connected with a possible generalization of (2.1). To be able to formulate this we define another symbol.

(2.2) DEFINITION. Let $\alpha, \beta, \gamma, \delta$ be ordinals $r$ a cardinal, $\alpha \rightarrow [\beta]^r_{\gamma,\delta}$ denotes the following statement. If $[\alpha]^r = \bigcup (\nu < \gamma)\, \mathfrak{I}_\nu$ is an arbitrary $r$-partition of length $\gamma$ of $\alpha$ then there are

$$B \subset \alpha, \ \ D \subset \gamma$$

such that tp $D(<) = \delta$, tp $B(<) = \beta$ and

$$[B]^2 \subset \bigcup (\nu \in D)\, \mathfrak{I}_\nu .$$

For the definition of this symbol see [3] and [4].

$\omega_\alpha^\omega$ being indecomposable (2.1) can be written in the following form.

*Let* $\alpha > 0$, $\sigma < \omega_{\alpha+1}$ *then*

$$\sigma \nrightarrow [\omega_\alpha^\omega]^1_{\omega, n} \quad \textit{for every } n < \omega. \tag{2.3}$$

In the proof of Corollary 2 we applied that $\sigma \nrightarrow [\omega_1^\omega]^1_{\omega, n}$ holds for every $n > 0$ and for $\sigma < \omega_2$

A straightforward generalisation would be the following assertion.

*Assume G.C.H. Then*

$$\sigma \nrightarrow [\omega_2^{\omega_1}]^1_{\omega_1, \nu} \tag{2.4}$$

*holds for* $\nu < \omega_1$, $\sigma < \omega^3$ ?

We have discussed this problem in [4], see Problems 19 – 21/A. We know that (2.2) is true for $\sigma < \omega_2$. As we have mentioned in [4] for $\sigma = \omega_2^{\omega_1}$ (2.4) seems to be independent of the axioms of set theory. Obviously, if for a given $\sigma$ there is a positive answer to (2.4) then this together with Theorem 1 gives a positive answer to the second part of Problem 1.

We did not investigate if Problem 1 is equivalent to the corresponding case of (2.4) or to any of the known unsolvable problems. Finally we discuss a further possible refinement or Theorem 1 by stating some results without proofs.

THEOREM 3.

A) *Assume* G.C.H. *Let* $\xi = 1$, $\eta = 0$. *Put*

$$S_2 = \bigcup \{f : D(f) \subset \omega \wedge |D(f)| = \omega \wedge \forall k(k < \omega \Rightarrow \exists n(\omega > f(n) > k))\}\, S(1, 0, f).$$

*Then there is* $I \subset [A]^2$ *satisfying the requirements*

a) $X \subset A$, $|X| = 3$ *imply* $[X]^2 \not\subset I$,

b) $X \subset A$, $|X \cap A_n| = \omega_1$ *for some* $n < \omega$

*and* $Y \subset X$ *for some* $Y \in S_2$ *imply* $[X]^2 \cap I \neq 0$.

B) *Let* $\xi = 1$, $\eta = 0$, $l, k < \omega$ *and let* $I \subset [A]^2$ *satisfy the condition a) of part* A). *Then there is* $X \subset A$ *satisfying the following conditions. There are* $Y, Z \subset X$ *with* $Y \in S(1, 0, f)$, $Z \in S(1, 0, g)$ *where*

$$|D(f)| = l, \quad f(n) = \omega_9 \quad \textit{for } n \in D(f)$$

$$|D(g)| = \omega, \quad g(n) = k \quad \textit{for } n \in D(g) \textit{ and}$$

$$[X]^2 \cap I = 0.$$

We formulated Theorem 3 because parts A) and B) together give a surprisingly sharp result.

Part A) can be proved quite similarly to Theorem 1/A. In § 4 we are going to establish a weaker result than part B). We will not prove the existence of $Z \subset X$ but we will do it generally for $\xi = \eta + 1$, $\eta$ arbitrary.

The proof of B) is quite tricky but we omit it because it is very special. We mention that we can generalize neither Part A) nor Part B) of Theorem 2 for the case $\xi = 2$, $\eta = 1$. The proofs already break down in the most special cases.

## § 3. Consequences of the negative result (1.5)

First we state (1.5) in a slightly stronger form and using the notation of the previous §. For this stronger form see also [5].

THEOREM 4 (HAJNAL). *Assume* G.C.H. *Let* $\xi = \eta = \zeta + 1$ *and assume* $\aleph_\zeta$ *is regular. Put* $S_3 = S(\zeta + 1, \zeta + 1, \zeta + 1, \zeta + 1)$. *Then there is* $I \subset [A]^2$ *satisfying the following conditions:*

a) $$X \subset A,\ |X| = 3\ \textit{imply}\ [X]^2 \not\subset I,$$

b) $$X \subset A,\ X \in S_3\ \textit{imply}\ [X]^2 \cap I \neq 0,$$

c) *for every* $\nu < \nu' < \omega_{\zeta+1}$ *and for every* $x \in A_\nu$ *there is at most one* $y \in A_{\nu'}$ *with* $\{x, y\} \in I$,

d) $$[A_\nu]^2 \cap I = 0\ \textit{for}\ \nu < \omega_{\zeta+1}\,.$$

COROLLARY 3. Assume G.C.H. and let $\aleph_\zeta$ be regular, $k, t < \omega$. Then

$$\omega_{\zeta+1}^{(t+1)(k+1)} \nrightarrow (\omega_{\zeta+1}^{k+2}, t+2)^2.$$

COROLLARY 4. Under the conditions of Corollary 3

$$\omega_{\zeta+1}^{(t+1)(k+1)+1} \nrightarrow (\omega_{\zeta+1}^{k+2} + 1, t+2)^2.$$

PROOF of Corollary 3. We prove the statement by induction on $t$. For $t = 0$ the statement

$$\omega_{\zeta+1}^{k+1} \nrightarrow (\omega_{\zeta+1}^{k+2}, 2)^2$$

is trivially true. Assume that $t > 0$ and the statement is true for $t - 1$:

$$\omega_{\zeta+1}^{(t+1)(k+1)} = \omega_{\zeta+1}^{t(k+1)} \,.\, \omega_{\zeta+1}^{k+1}.$$

Put briefly

$$\sigma = \omega_{\zeta+1}^{t(k+1)},\ \varrho = \omega_{\zeta+1}^{k+1},\ \tau = \sigma . \varrho\,.\ \text{By}\ t > 0$$

we have $|\sigma| = |\varrho| = \aleph_{\zeta+1}$. Moreover there are sets

$$B_0 < \ldots < \hat{B}_\varrho \text{ such that}^3$$

(1) $$\tau = B_0 \cup \ldots \cup \hat{B}_\varrho \text{ and } \mathrm{tp} B_\alpha (<) = \sigma \text{ for } \alpha < \varrho.$$

Let now $C_0, \ldots, \hat{C}_{\omega_{\zeta+1}}$ be a reordering of type $\omega_{\zeta+1}$ of the sequence $B_0, \ldots, \hat{B}_\varrho$. By Theorem 4, there is $I \subset [\tau]^2$ satisfying conditions a), b), c), d). $A$ replaced by $\tau$ and $A_\tau$ replaced by $C_\tau$ respectively.

By (1) and by the induction hypothesis, for every $\alpha < \varrho$ there is a 2-partition of length 2 of $B_\alpha$,

(2) $$[B_\alpha]^2 = \mathfrak{J}_0^\alpha \cup \mathfrak{J}_1^\alpha \text{ establishing } \sigma \nrightarrow (\omega_{\zeta+1}^{k+2}, t+1)^2.$$

Define the 2-partition of length 2 of $\tau$, $[\tau]^2 = \mathfrak{J}_0 \cup \mathfrak{J}_1$ as follows

(3) $$\mathfrak{J}_0 = \bigcup (\alpha < \varrho)\, \mathfrak{J}_0^\alpha \cup (([\tau]^2 - \bigcup (\alpha < \varrho)[B_\alpha]^2) - I),$$

$$\mathfrak{J}_1 = \bigcup (\alpha < \varrho)\, \mathfrak{J}_1^\alpha \cup I.$$

By (2), this is really a 2-partition of $\tau$.

Let now $X \subset \tau$, $[X]^2 \subset \mathfrak{J}_1$. Put $N = \{\nu < \omega_{\zeta+1}: C_\nu \cap X \neq 0\}$. By (2), (3) and d) we have $|C_\nu \cap X| < t+1$ for $\nu \in N$. By (3) and a) we have $|N| \leq 2$. Thus we may assume $N = \{\nu, \nu'\}$, $\nu < \nu'$. Then, by c), we have $|C_{\nu'} \cap X| \leq 1$ and thus $|X| < t+2$.

Next, let $X \subset \tau$, $[X]^2 \subset \mathfrak{J}_0$. Put

$$M = \{\nu < \omega_{\zeta+1}: |X \cap C_\nu| > \aleph_\zeta\}.$$

By (3) and b) we have $|M| \leq \aleph_\zeta$.

On the other hand, by (3), $[X]^2 \subset \mathfrak{J}_0$ implies $[X \cap C_\nu]^2 \subset \mathfrak{J}_0^\alpha$ for the $\alpha$ satisfying $B_\alpha = C_\nu$. Hence, by (2), $\mathrm{tp} X \cap C_\nu (<) < \omega_{\zeta+1}^{k+2}$ for every $\nu < \omega_{\zeta+1}$.

Put

$$X_0 = \bigcup (\nu \in M)\, C_\nu \cap X,\ X_1 = \bigcup (\nu \in \omega_{\zeta+1} - M)\, C_\nu \cap X.$$

We have $X = X_0 \cup X_1$. By the definition of $M$, $\mathrm{tp} X_0 (<) < \omega_{\zeta+1}^{k+2}$ because of $\omega_{\zeta+1}^{k+2} \to (\omega_{\zeta+1}^{k+2})^1_\aleph$ and

$$\mathrm{tp} X_1 (<) \leq \omega_{\zeta+1}^{k+1} < \omega_{\zeta+1}^{k+2} \text{ because of } \varrho = \omega_{\zeta+1}^{k+1}.$$

$\omega_{\zeta+1}^{k+2}$ being indecomposable we have $\mathrm{tp} X (<) < \omega_{\zeta+1}^{k+2}$. Thus the partition defined by (3) establishes $\tau \nrightarrow (\omega_{\zeta+1}^{k+2}, t+2)^2$.

Proof of Corollary 4. Put $\tau = \omega_{\zeta+1}^{(t+1)(k+1)+1}$, $\sigma = \omega_{\zeta+1}^{(t+1)(k+1)}$. Then there are $B_0 < \ldots < \hat{B}_{\omega_{\zeta+1}}$ such that $\tau = B_0 \cup \ldots \cup \hat{B}_{\omega_{\zeta+1}}$

$$\mathrm{tp} B_\nu (<) = \sigma \quad \text{for} \quad \nu < \omega_{\zeta+1}.$$

[3] $X < Y$ means that $\varkappa < \lambda$ for $\varkappa \in X$, $\lambda \in Y$.

Thus, by Corollary 3, there are partitions $[B_\nu]^2 = \mathfrak{I}_0^\nu \cup \mathfrak{I}_1^\nu$ establishing $\sigma \nrightarrow (\omega_{\xi+1}^{k+2}, t+2)^2$ for $\nu < \omega_{\xi+1}$.

Put $$\mathfrak{I}_1 = \bigcup (\nu < \omega_{\xi+1})\, \mathfrak{I}_1^\nu,\ \mathfrak{I}_0 = [\tau]^2 - \mathfrak{I}_1$$

Then $[\tau]^2 = \mathfrak{I}_0 \cup \mathfrak{I}_1$, obviously establishes

$$\tau \nrightarrow (\omega_{\xi+1}^{k+2} + 1,\ t+2)^2.$$

REMARK. Though Corollary 4 is a trivial consequence of Corollary 3 it gives a best possible result in the first nontrivial case $t = 1$ for every $k < \omega$. We will discuss this after having established some positive results.

## § 4. Some positive relations

Let now $S$ be a set and $I \subset [S]^2$. As usual we may consider the pair $\mathcal{G} = \langle S, I \rangle$ as a graph where $S$ and $I$ are the set of vertices and edges, respectively. We will put

$$I(x) = \{y:\ \{x, y\} \in I\}\ \text{ for } x \in S.$$

$|I(x)|$ is the valency of the vertex $x$ in $\mathcal{G}$.

We will now state a lemma which in spite of its triviality has some sharp consequences.

LEMMA. *Let $(S, \prec)$ be an ordered set, $\alpha > 0$, $|\varrho| < \omega_\alpha$. Assume*

$$S = S_0 \cup \ldots \cup \hat{S}_\varrho, \quad S_0 \prec \ldots \prec \hat{S}_\varrho$$

*and assume that $I \subset [S]^2$ is such that*

$$|S_\sigma - \bigcup (x \in S')\, I(x)| \geq \omega_\alpha$$

*or every $\sigma < \varrho$ and for every $S' \subset S$, $|S'| < \omega_\alpha$.*

*Then there is $X \subset S$ satisfying the following conditions:*

$$(4.1) \qquad [X]^2 \cap I = 0;\ |X \cap S_\sigma| \geq \omega_\alpha \text{ for every } \sigma \leq \varrho$$

*and as a corollary* $\text{tp}X\,(\prec) \geq \omega_\alpha \cdot \varrho$.

(4.1) can be proved by an obvious zig-zaguing so we omit the details. As a corollary we have

COROLLARY. For every $\xi$ and for every $t, k < \omega$

$$(4.2) \qquad \omega_{\xi+1}^{(t+1)(k+1)+1} \to (\omega_{\xi+1}^{k+2},\ t+2)^2.$$

(This should be compared with Corollary 4 of Theorem 4.)

PROOF. By induction on $t$. For $t = 0$ the statement

$$\omega_{\zeta+1}^{k+2} \rightarrow (\omega_{\zeta+1}^{k+2}, 2)^2$$

is trivial.

Assume $t > 0$ and the statement is true for $t-1$. Put $\gamma = \omega_{\zeta+1}^{(t+1)(k+1)+1}$ and let $I \subset [\gamma]^2$ be such that

(1) $$X \subset \gamma, \ [X]^2 \subset I \text{ imply } |X| < t+2.$$

Then, by (1), for every $x \in \gamma$ we have

(2) $$Y \subset I(x), \ [Y]^2 \subset I \text{ imply } |Y| < t+1.$$

We have to prove, that there is $Z \subset \gamma$ such that

(3) $$[Z]^2 \cap I = 0 \quad \text{and} \quad \mathrm{tp} Z\,(<) = \omega_{\zeta+1}^{k+2}.$$

By (2) and by the induction hypothesis we may assume that $\mathrm{tp} I(x)\,(<) < \omega_{\zeta+1}^{t(k+1)+1}$ for every $x \in \gamma$ otherwise (3) holds.

Considering that

$$\gamma = \omega_{\zeta+1}^{t(k+1)+1} \cdot \omega_{\zeta+1}^{k+1}$$

and that

$$\omega_{\xi+1}^{t(k+1)+1} \rightarrow (\omega_{\zeta+1}^{t(k+1)+1})^1_{\aleph_\xi}$$

(3) follows trivially from (4.1).

Corollary 4 and (4.2) together determine the smallest ordinal $\pi$ for which

$$\omega_{\zeta+1}^{m} \nrightarrow (\pi, t+2)^2$$

holds in case $m \not\equiv 1 \pmod{t+1}$.

In case $m \not\equiv 0 \pmod{t+1}$ corollary 3 and the following result give a complete discussion.

THEOREM 5. *Let $\zeta$ be arbitrary, $k < \omega$, $1 \leq t < \omega$. Then*

$$\omega_{\zeta+1}^{(t+1)(k+1)} \rightarrow (\mu, t+2)^2 \textit{ for every } \mu < \omega_{\zeta+1}^{k+2}.$$

For the proof of Theorem 5 we need some preliminaries.

LEMMA. *Let $\zeta$ be arbitrary $m < \omega$, $r + s = m$;*

$$\alpha = \omega_{\zeta+1}^{m}, \ \beta = \omega_{\zeta+1}^{r}, \ \gamma = \omega_{\zeta+1}^{s}, \ \nu < \omega_{\zeta+1};$$

$$\alpha = B_0 \cup \ldots \cup \hat{B}_\gamma, \ B_0 < \ldots < \hat{B}_\gamma, \ \mathrm{tp} B_\varrho\,(<) = \beta \textit{ for } \varrho < \gamma.$$

*Let $F$ be a function such that $D(F) = \alpha$ and, for every $x \in \alpha$, $F(x) \subset \gamma$, $\mathrm{tp} F(x)\,(<) < \nu$ moreover if $x \in B_\varrho$ then $\varrho \notin F(x)$ for every $\varrho < \gamma$, $x \in \alpha$. Then there exists $X \subset \alpha$ with the following properties*

$$\mathrm{tp} X\,(<) = \alpha.$$

For every $x, y \in X$, $y \in B_\varrho$ implies $\varrho \notin F(x)$.

In [8] we proved the following result.

THEOREM. (ERDŐS, HAJNAL, MILNER). *Let $\zeta$ be arbitrary, $m < \omega$, $\alpha = \omega_{\zeta+1}^{m}$, $\nu < \omega_{\zeta+1}$. Let $f$ be a set mapping defined on $\alpha$ with $\operatorname{tp} f(x) (<) < \nu$ for $x \in \alpha$. Then there exists a free subset $X \subset \alpha$ of type $\alpha$. ($X$ is free if for every pair $x, y \in X$, $x \notin f(y)$).*

This Theorem is obviously equivalent to the special case $r = 0$, $s = m$ of the Lemma. Its proof is quite lengthy. Considering that the Lemma can be obtained with a routine modification of the proof of our Theorem, we omit it.

Using the Lemma we prove the following

THEOREM. *Let $\zeta$, $m$, $r$, $s$, $\alpha$, $\beta$, $\gamma$, $B_0, \ldots, \hat{B}_\gamma$ have the same meaning as in the Lemma. Assume moreover that $r > 0$ and let $I \subset [S]^2$ be given in such a way that*

(i) $\operatorname{tp} I(x)(<) < \mu$ *for every* $x \in S$ *for a fixed* $\mu < \omega_{\zeta+2}^{r+2}$

(ii) $\operatorname{tp}(I(x) \cap B_\varrho) (<) < \beta$ *for every* $x \in B_\varrho$, $x \in S$, $\varrho < \gamma$.

*Then there is* $Y \subset \alpha$, $\operatorname{tp} Y(<) \geq \omega_{\zeta+1}^{s+1}$ *such that*

(4.3) $$[Y]^2 \cap I = 0.$$

PROOF. Put $F(x) = \{\varrho < \gamma \colon \operatorname{tp}(I(x) \cap B_\varrho)(<) = \beta\}$ for $x \in S$. By (ii), $x \in B_\varrho$ implies $\varrho \notin F(x)$ for $x \in \alpha$, $\varrho < \gamma$. There is $\nu < \omega_{\zeta+1}$ such that $\mu < \beta \cdot \nu$. Then, by (i), $\operatorname{tp} F(x) (<) < \nu$ for every $x \in \alpha$. By the Lemma there is $Z \subset \alpha$, $\operatorname{tp} Z(<) = \alpha$ such that $\varrho \notin F(x)$ for $y \in B_\varrho$, $x, y \in Z$.

Hence, for every $x \in Z$ and for every $\varrho < \gamma$ $Z \cap B_\varrho \neq 0$ implies $\operatorname{tp}(I(x) \cap B_\varrho) (<) < \beta$.

Using $\beta \to (\beta)^1_{\aleph_\xi}$, (4.1) implies the existence of $Y \subset Z$ satisfying the requirements of (4.3).

PROOF of Theorem 5. By induction on $t$.

CASE 1. $t = 1$. Put $\alpha = \omega_{\zeta+1}^{2(k+1)}$ we have to prove $\alpha \to (\mu, 3)^2$ where $\mu < \omega_{\zeta+1}^{k+2}$. Assume this is false.

Let $I \subset [\alpha]^2$ be then such that

(1) $X \subset \alpha$, $[X]^2 \subset I$ imply $|X| < 3$ and

(2) $X \subset \alpha$, $\operatorname{tp} X (<) = \mu$ imply $[X]^2 \cap I \neq 0$.

(1) and (2) imply

(3) $\operatorname{tp} I(x) (<) < \mu$ for $x \in \alpha$.

Let $\beta = \omega_{\zeta+1}^{k+1}$. We can set

$$\alpha = B_0 \cup \ldots \cup \hat{B}_\beta,\ B_0 < \ldots < \hat{B}_\beta,\ \mathrm{tp} B_\varrho (<) = \beta \text{ for } \varrho < \beta.$$

By (1), there is $B'_\varrho \subset B_\varrho$, $\mathrm{tp} B'_\varrho = \beta$ such that $\mathrm{tp}\,(I(x) \cap B'_\varrho)\,(<) < \beta$ for $x \in B'_\varrho$. Hence we may as well assume that

(4) $\mathrm{tp}\,(I(x) \cap B_\varrho)\,(<) < \beta$ holds for $\varrho < \beta$, $x \in B_\varrho$.

(3) and (4) imply by (4.3) the existence of a $Y \subset \alpha$, $\mathrm{tp}\, Y\,(<) = \omega_{\zeta+1}^{k+2} > \mu$ with $[Y]^2 \cap I = 0$. This contradicts (2).

CASE 2. $t > 1$. Put $\alpha = \omega_{\zeta+1}^{(t+1)(k+1)}$ and assume the result is true for $t - 1$.

Let $I \subset [\alpha]^2$ be such that $X \subset \alpha$, $[X]^2 \subset I$ imply $|X| < t + 2$. By the induction hypothesis we may assume $\mathrm{tp}\, I(x)\,(<) < \omega_{\zeta+1}^{t(k+1)}$ for every $x \in \alpha$.

Considering $\alpha = \omega_{\zeta+1}^{t(k+1)} \cdot \omega_{\zeta+1}^{k+1}$ and $\omega_{\zeta+1}^{t(k+1)} \to (\omega_{\zeta+1}^{t(k+1)})^1_{\aleph_\zeta}$, by (4.1), we then even have a

$$Y \subset \alpha,\ [Y]^2 \cap I = 0 \text{ with } \mathrm{tp} Y(<) \geq \omega_{\zeta+1}^{k+2} > \mu.$$

Note that in cases $m \not\equiv 0,1 \pmod{t+1}$ we do not know a best possible result for $\omega_{\zeta+1}^m \to (\pi, t+2)^2$.

The following seem to be the most interesting unsolved problems.

PROBLEM 3.

$$\omega_1^2 \to (\omega_1\, \tau, 4)^2 \text{ for } \omega \leq \tau < \omega_1\ ?$$

$$\omega_1^5 \to (\omega_1^2 \tau, 4)^2 \text{ for } \omega \leq \tau < \omega_1\ ?$$

Note that, by (4.2), $\omega_1^4 \to (\omega_1, 4)^2$ and this easily implies $\omega_1^5 \to (\omega_1^2 . \tau, 4)^2$ for $\tau < \omega$. On the other hand $\omega_1^5 \nrightarrow (\omega_1^3, 4)^2$ follows from Corollary 3 which even implies

$$\omega_1^6 \nrightarrow (\omega_1^3, 4)^2.$$

We establish one more positive result relevant to Corollary 1 and Problem 2.

THEOREM 6. *Let* $\eta \leq \zeta + 1$. *Then*

$$\omega_{\zeta+1}\omega_\eta \to (\omega_{\zeta+1} . \alpha, 3)^2 \textit{ for } \alpha < \omega_\eta.$$

This should be compared with Theorem 2/B.

PROOF. It is obviously sufficient to prove the theorem for regular $\omega_\eta$. The special case $t = 0$, $k = 0$ of Theorem 5 yields this for $\eta = \zeta + 1$. We assume $\eta < \zeta$. We use the notation introduced in § 2 with $\xi = \zeta + 1$. Let

$I \subset [A]^2$ be given, let $\prec$ be the usual antilexicographical ordering of $A$ and assume

(1) $$X \subset A,\ [X]^2 \subset I \text{ imply } |X| < 3.$$

We prove the following statement: there is

(2) $$Y \subset A,\ \text{tp} Y (\prec) = \omega_{\zeta+1} \cdot \alpha,\ [Y]^2 \cap I = 0.$$

We assume (2) is false. Then by (1), we have

(3) $$\text{tp}\, I(x)\,(\prec) < \omega_{\zeta+1} \cdot \alpha \text{ for } x \in A.$$

By $\omega_{\zeta+1} \to (\omega_{\zeta+1}, 3)^2$ and by (1) we may assume

(4) $$[A_\nu]^2 \cap I = 0 \text{ for } \nu < \omega_\eta.$$

Put

$$F(x) = \{\nu < \omega_\eta :\ |I(x) \cap A_\nu| = \omega_{\zeta+1}\}.$$

By (3), we have

(5) $$\text{tp} F(x)\,(<) < \alpha \text{ for } x \in A.$$

By $|A_\nu| = \omega_{\zeta+1}$, by the regularity of $\omega_\eta$ and by (5) for every $\nu < \omega_\eta$ there are $A'_\nu \subset A_\nu$ and $\tau(\nu) < \omega_\eta$ such that

6) $$|A'_\nu| = \omega_{\zeta+1} \text{ and } F(x) \subset \tau(\nu) \text{ for } x \in A'_\nu.$$

Then, by transfinite induction, one can choose a subset $N \subset \omega_\eta$ such that for every $\nu < \nu'$; $\nu, \nu' \in N$, $x \in A'_\nu$ we have

(7) $$\nu' \notin F(x), \text{ and } |N| = \omega_\eta.$$

We prove:

There is $N' \subset N$, $|N'| < \omega_\eta$ such that for every $M \subset N - N'$, $|M| < \omega_\eta$ there are $M < \{\nu\}$, $\nu < \omega_\eta$ and $B \subset A'_\nu$ such that

(8) $$|B| = \omega_{\zeta+1} \text{ and } M \cap F(x) = 0 \text{ for } x \in B.$$

If (8) is false define by transfinite induction a sequence $M_0, \ldots, \hat{M}_\alpha$ of subsets of $N$ so that for every $\varrho < \alpha$, $N_\varrho = \bigcup\, (\sigma < \varrho)\, M_\sigma$; $M_\varrho = M$ is a counterexample for (8) when $N'$ is replaced by $N_\varrho$.

Then $M_0 < \ldots < \hat{M}_\alpha$ and there is $\nu$ with $\bigcup\, (\varrho < \alpha)\, M_\varrho < \{\nu\}$ and there is $x \in A'_\nu$ with $F(x) \cap M_\varrho \neq 0$ for every $\varrho < \alpha$. This contradicts (5).

Using (8) one can define by transfinite induction a subset $M \subset N$, $|M| = \omega_\eta$ and $A''_\nu$ for $\nu \in M$ such that

(9) $$|A''_\nu| = \omega_{\zeta+1},\ A''_\nu \subset A'_\nu \text{ for } \nu \in M$$

and for every $\nu, \nu' \in M$, $\nu < \nu'$, $x \in A'_{\nu'}$, we have $\nu \notin F(x)$.

By (4), (7) and (9), it follows from (4.1) that there is $Y \subset A$ $\text{tp}\, Y\,(<) = \omega_{\zeta+1} \cdot \omega_\eta$, $[Y]^2 \cap I = 0$. This contradicts (2) and proves the theorem.

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