Some Unsolved Problems in Graph Theory and Combinatorial Analysis

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In the present note I discuss some unsolved problems in graph theory and combinatorial analysis which I have thought about in the recent past. I hope that at least a good proportion of them are new.

First I introduce some notation. $G(n;k)$ will denote a graph of n vertices and k edges. C_r denotes a circuit having r vertices, K_l denotes the complete graph of l vertices. A graph is said to have girth k if it contains a C_k but no C_l for $l < k$. A clique is a maximal complete subgraph (i.e. it is a complete subgraph which is not properly contained in any larger complete subgraph). $K_r(p_1, \ldots, p_r)$ denotes the complete r-chromatic graph where there are p_i vertices of the ith colour and any two vertices of different colour are joined by an edge. By an r-graph $G^{(r)}$ we shall mean a graph whose basic elements are its vertices and r-tuples; for $r = 2$ we obtain the ordinary graphs. $G^{(r)}(n;k)$ will denote an r-graph of n vertices and k r-tuples. A set of r-tuples is called independent if no two of them have a vertex in common. $c, c_1, c_2, \ldots,$ will denote position absolute constants.

1. In the colloquium on graph theory at Tihany, Bollobás and I stated the following problem: is it true that every graph of n edges contains a subgraph of at least $cn^{3/4}$ edges which has no rectangle? Folkman (in a letter) gave the following counter-example. Let the vertices of our G be $x_1, \ldots, x_m; y_1, \ldots, y_{m^2}$, every x is joined to every y. This graph has m^3 edges and it is easy to see that every subgraph having $m^2 + \binom{m}{2} + 1$ edges contains a rectangle (this statement is false for $m^2 + \binom{m}{2}$ edges)

Perhaps our conjecture is true with $cn^{2/3}$ instead of $cn^{3/4}$, but I cannot even prove it with $cn^{\frac{1}{2}+\varepsilon}$ ($cn^{\frac{1}{2}}$ is trivial).†

2. Is it true that every graph of $5n$ vertices which contains no triangle can

† *Note added in proof:* Szemerédi proved $cn^{2/3}$.

be made two-chromatic by the omission of at most n^2 edges? It is easy to see that if this is true it is best possible.

3. I proved that if $k < cn$ then every $G(n; [\frac{1}{4}n^2] + k)$ contains k edge-disjoint triangles. The proof uses the following theorem of Gallai and myself: every $G(n; [\frac{1}{4}(n-1)^2] + 2)$ which has chromatic number 3 contains a triangle.

I first thought that the theorem might hold for very much larger values of k, but Sauer showed by a simple example that this is not so. Let the vertices of G be $x_1, \ldots, x_n; y_1, \ldots, y_n; z_1, \ldots, z_4$. Every x is joined to every y and z, every y is joined to every z, and any two z's are also joined: This is a $G(2n+4; (n+1)^2 + 4n + 2)$ or $k = 4n + 2$, and it is not difficult to prove that G contains only $4n + 1$ edge disjoint triangles.

It would be interesting to determine the largest value of k for which our result holds; our proof only gives small values of $k < cn (c < \frac{1}{2})$. Perhaps the following result holds: to every c_1 there is an $f(c_1)$ so that every $G(n; [\frac{1}{4}n^2] + k), k < c_1 n$ contains at least $k - f(c_1)$ edge disjoint triangles.

In view of my theorem with Gallai the following question could be asked: what is the smallest integer u_r so that every $G(n; u_r)$ which has chromatic number $\geqslant r$, contains a triangle? $u_2 = [\frac{1}{4}n^2] + 1$ (this is the well known theorem of Turán) and $u_3 = [\frac{1}{4}(n-1)^2] + 2$. u_4 is unknown [5].†

4. We proved that every $G\left(n; \binom{n-1}{2} + 2\right)$ is Hamiltonian and that this is best possible. I showed that for $n > n_0(k)$ every

$$G\left(n; \binom{n-k}{2} + \binom{k+1}{2} + 1\right)$$

contains a C_{n-k}. My proof is not quite trivial. The result is easily seen to be the best possible. It would be interesting to determine or estimate $n_0(k)$ [6].

5. Rényi and I considered the following problem. Determine or estimate the smallest $f(n)$ for which all but $o\left(\binom{\binom{n}{2}}{f(n)}\right)$ of the graphs $G(n; f(n))$ on n labelled vertices are Hamiltonian. This question seems to be difficult. Recently Moon and Moser and I. Palásti proved $f(n) < cn^{3/2}$, but it seems certain that $f(n) < n^{1+\varepsilon}$ [24].

6. Denote by $g(3, n)$ the smallest integer for which there is a graph of $g(3, n)$ vertices which contains no triangle and has chromatic number n. It is known that

$$c_1 n^2 \log n / \log\log n < g(3, n) < c_2 n^2 (\log n)^2. \quad (1)$$

It would be desirable to improve (1) and to give an asymptotic formula for

† *Note added in proof:* Simonovits determined u_r.

$g(3, n)$, also it might be of some interest to prove

$$\lim_{n=\infty} g(3, n+1)/g(3, n) = 1.$$

Denote by $f(l, n)$ the smallest integer so that every graph of $f(l, n)$ vertices either contains a K_l or a set of n independent points. It is known that

$$c_3 n^2 \log n/\log\log n < f(3, n) < c_4 n^2(\log n)^2. \qquad (2)$$

It would be desirable to improve (2) and to obtain an asymptotic formula for $f(l, n)$. I cannot even prove

$$\lim_{n=\infty} f(l, n+1)/f(l, n) = 1.$$

Recently Yackel proved

$$f(n, n) < c_2\binom{2n}{n}\Big/\sqrt{n} \qquad (3)$$

I proved by probabilistic methods

$$f(n, n) > 2^{n/2}. \qquad (4)$$

It would be very interesting to prove that

$$\lim_{n=\infty} f(n, n)^{1/n}$$

exists and to obtain a non trivial lower bound for $f(n, n)$ by non-probabilistic methods [4], [13], [27], [30].

Yackel's proof of (3) will appear in the *Journal of Combinatorial Theory.*

7. Clearly every graph can be directed so that it should contain no directed circuit. The following problem is due to Ore. Let G be a graph. We want to direct its edges so that it should contain no directed circuit, and further, that it should contain no circuit which becomes directed if one reverses the direction of one of its edges. What is the necessary and sufficient condition that G can be directed in such a way?

Clearly G can contain no triangle. At first one may guess that every G which contains no triangle can be directed in such a way, but Ore showed that this is not true.

Gallai showed that the graph of Grötsch (Fig. 1) gives a counterexample.

I could find no graph whose girth is greater than four and which cannot be directed in such a way.

The graph of Grötsch is 4-chromatic and has no triangle. Perhaps it is the only graph of not more than 11 vertices with this property.†

† *Note added in proof:* I learned that this is known to several mathematicians.

8. Hajnal and I conjectured that every graph of infinite chromatic number contains a subgraph of infinite chromatic number which contains no triangle (or more generally has girth $> k$). If our graph is the infinite complete graph this is a well known theorem of Tutte.) We also formulated the finite version of these conjectures: is it true that there is an $f(k, r)$ so that every G having chromatic number $\geqslant f(k, r)$ contains a subgraph of girth $\geqslant k$ and chromatic number r?

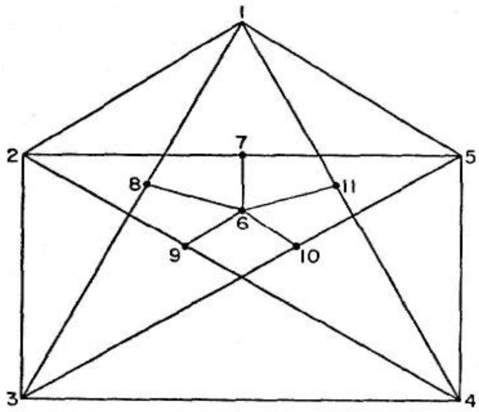

FIGURE 1

Finally we asked: let m be an infinite cardinal and G be an m-chromatic graph. It is true that G has a subgraph of chromatic number m which contains no triangle? If G is the complete graph of power m this is a result of Rado and myself.

Perhaps these problems could be formulated for connectivity instead of chromatic number, e.g. is it true that if G remains connected after the omission of any finite set of its vertices and all the edges incident to them, then G has a subgraph with the same property and which has girth $\geqslant k$? [3], [22].

9. Moon [32] proved that if $n = 3k + 2 \geqslant 8$ and we colour the edges of a K_n with two colours, then there are always k vertex disjoint triangles whose edges have the same colour (different triangles can have different colours).

Let $f(l)$ be the smallest integer with the property that, if we colour the edges of a K_n with two colours, then there are always $[n/l] - f(l)$ vertex disjoint K_l's all of whose edges have the same colour. Ramsey's theorem implies $f(l) < 4^b$ but it seems certain that $f(l)^{1/l} \to 1$ and perhaps $f(l) < c\, l$, or $f(l)$ is bounded.

How many vertex disjoint K_l's are there, all edges of which have the same colour? (Here different K_l's must have the same colour.) It is easy to see that for $l = 2$ the answers is $\frac{1}{3}n + O(1)$ [31].

10. Denote by $\log_r n$ the r-fold iterated logarithm and let $L(n)$ be the smallest integer k for which $1 < \log_k n \leqslant e$. I state two simple problems in graph theory, which seem to lead to this function $L(n)$. Moon and Moser proved that if $f(n)$ is the largest integer for which there is a graph of n vertices having $f(n)$ cliques of different sizes, then

$$n - \frac{\log n}{\log 2} - \log\log n < f(n) < n - \frac{\log n}{\log 2}. \tag{1}$$

I improved the lower bound to $n - \dfrac{\log n}{\log 2} - L(n)$, but could not improve the upper bound.

Bondy considered the following problem. Denote by $h(n)$ the smallest integer for which there exists a $G(n; n + h(n))$ which contains a C_k for every $3 \leqslant k \leqslant n$. Bondy proved (not yet published)

$$\frac{\log n}{\log 2} < h(n) < \frac{\log n}{\log 2} + L(n). \tag{2}$$

It seemed to us that in (1) the lower bound, and in (2) the upper bound, is close to the truth but we could not even prove [12]

$$h(n) - \frac{\log n}{\log 2} \to \infty, \qquad n - \frac{\log n}{\log 2} - f(n) \to \infty.$$

Bondy's paper is not yet published.†

11. Goodman, Pósa and I [18] proved that every $G(n;k)$ is the union of at most $[\frac{1}{4}n^2]$ edge-disjoint complete graphs, where in fact the complete graphs can be chosen as edges and triangles. It is easy to see that $[\frac{1}{4}n^2]$ is best possible. We thought that for $k > \frac{1}{4}n^2$ our theorem could be sharpened. Lovász proved in this direction the following result: put $e = \binom{n}{2} - k$ and let t be the largest integer for which $t^2 - t \leqslant e$. Then $G(n;k)$ is the union of $e + t$ complete subgraphs. For $e = t^2$ and $e = t^2 - t$ the result is sharp.

Lovász does not assume that the complete graphs are edge disjoint and observes that the result no longer holds if edge disjointness is insisted upon. In this case no satisfactory non-trivial sharpening of our theorem is known.

Gallai and I considered the following further problems. Is it true that every connected graph of n vertices is the union of $[\frac{1}{2}(n + 1)]$ edge disjoint paths? Lovász proved this if all the vertices of G have odd valency.

Denote by $h(n)$ the smallest integer so that every graph of n vertices is the

† *Note added in proof:* Spencer proved $f(n) > n - \dfrac{\log n}{\log 2} - c$.

union of $h(n)$ edge-disjoint edges and circuits. We showed $h(n) < cn \log n$, but probably $h(n) < c_1 n$. $K_2(3, n-3)$ shows that $h(n) > (1 + c_2)n$. Perhaps every graph of n vertices is the union of $n - 1$ edges and circuits if we do not require them to be edge disjoint [29].

12. Is it true that every $G(n; [n^{1+\varepsilon}])$ contains a subgraph which is non-planar and has at most c_ε vertices? It is not difficult to see that $c_\varepsilon \to \infty$ as $\varepsilon \to 0$.

13. It is well known that $G(n; k)$ can be planar only if $k \leqslant 3n - 6$. A planar graph $G(n; 3n - 6)$ is called saturated. A theorem of Turán implies that every $G(n; [\frac{1}{4}n^2] + 1)$ contains a triangle, i.e. a saturated planar graph of three vertices. It is easy to construct a $G(n; [\frac{1}{4}n^2] + [\frac{1}{2}(n-1)])$ which contains no saturated planar graph of more than three vertices; perhaps this example is the best possible, and every $G(n; [\frac{1}{4}n^2] + [\frac{1}{2}(n+1)])$ contains a saturated planar graph of more than three vertices. Simonovits has just proved this conjecture [15], [34].

14. T. Gallai and I proved that if

$$l > (k-1)n - \binom{k+1}{2} \tag{1}$$

then $G(n; l)$ contains k independent edges.

I proved that if $n > 400k^2$ and

$$l > \left(\frac{(n-k+1)^2}{4}\right) + (k-1)n - (k-1)^2 + \binom{k-1}{2} \tag{2}$$

then $G(n; l)$ contains k vertex independent triangles.

Pósa and I proved that if $n > 24k$ and

$$l > (2k-1)n - 2k^2 + k \tag{3}$$

then $G(n; l)$ contains k independent circuits.

It is easy to see that (1), (2) and (3) become false if the inequality is replaced by equality.

It would be desirable to improve (2) and (3) so that like (1) they would be valid for all n, but I have not succeeded in doing this, and did not even formulate a reasonable conjecture.

Denote by $f(n; r, k)$ the smallest integer so that every $G^{(r)}(n; f(n; r, k))$ contains k independent r-tuples. The value of $f(n; 2, k)$ is given by (1).

Denote by $g(n; r, k-1)$ the number of those r-tuples formed from the elements $x_1, x_2, \ldots, x_n$ each of which contains at least one of the elements $x_1, \ldots, x_{k-1}$. Clearly $f(n; r, k) > g(n; r, k-1)$.

I proved that for $n > c_r k$

$$f(n; r, k) = 1 + g(n; r, k-1).$$

Perhaps

$$f(n; r, k) = 1 + \max\left(\binom{rk-1}{r}, g(n; r, k-1)\right). \tag{4}$$

If $r = 2$, (4) is true and becomes (1), but I could not even settle $r = 3$ [11], [17], [21].

J. Moon, found a simpler proof of (1) and (2).

15. Recently several papers were published on extremal problems in graph theory; here I want to mention only a few of them. Let G' be a graph. $f(n; G')$ is the smallest integer so that every $G(n; f(n; G'))$ contains G' as a subgraph. Kövári, the Turáns and independently I, proved that

$$f(n; K_2(r, r)) < c_r n^{2-1/r}. \tag{1}$$

It would be very desirable to prove that

$$f(n; K_2(r, r)) > c_r' n^{2-1/r}. \tag{2}$$

(2) is known for $r = 2$ and $r = 3$ but no good lower bound is known for $r \geqslant 4$. For $r = 2$, Brown, Mrs Turán, Rényi and I in fact proved

$$f(n; K_2(2, 2)) = (1 + o(1))\, n^{3/2}/2. \tag{3}$$

Perhaps if G is a graph of n vertices which contains no triangle and rectangle, then it has at most $(1 + o(1))\, n^{3/2}/2\sqrt{2}$ edges.

Very likely

$$c_r^{(1)} n^{1+1/r} < f(n; C_{2r}) < c_r^{(2)} n^{1+1/r}. \tag{4}$$

The upper bound is not hard to prove but the lower bound is not known for $r > 2$.

Let G_k be the following graph: its vertices are $x; y_1, \ldots, y_k; z_1, \ldots, z_{\binom{k}{2}}$. x is joined to every y, and every z is joined to two y's; distinct z's are joined tö different pairs. Perhaps

$$f(n; G_k) < c_k n^{3/2}. \tag{5}$$

I proved (5) for $k = 3$, but had no success with $k > 3$. $f(n; G_k) > cn^{3/2}$ is trivial, since for $k \geqslant 3$, G_k contains a rectangle (i.e. a $K_2(2, 2)$).

Simonovits and I tried to investigate $f(n; G_k - x)$ (i.e. we omit from G_k the vertex x and all edges incident to it). It seems likely that $f(n; G_k - x) = o(n^{3/2})$ and in fact we conjecture

$$f(n; G_k - x) < n^{(3/2)-\varepsilon_k} \tag{6}$$

but we could prove none of these results. We showed that to every $\varepsilon > 0$ there is a K_ε so that

$$f(n; G_{K_\varepsilon} - x) > n^{(3/2)-\varepsilon} \tag{7}$$

The proof of (7) is not published.

Perhaps the most interesting unsolved problems in this field are the original problems of Turán which are perhaps not sufficiently well known. Therefore I restate them here: denote by $g(n; k, l)$ the smallest integer so that if $|S| = n$ and $A_1, \ldots, A_s, s = g(n; k, l)$ are subsets of $S, |A_i| = k, 1 \leqslant i \leqslant s$, then there is a $B \subset S, |B| = l$ all of whose subsets of k elements occur amongst the A's. Turán determined $g(n; 2, l)$ for every l, but for $k > 2$ the problem is unsolved. It is easy to see that

$$\lim_{n=\infty} g(n; k, l) \Big/ \binom{n}{k}$$

exists for every k and l, but for $k > 2$ the value of the limit is not known. In particular Turán conjectured

$$g(3n; 3, 4) = 3n\binom{n}{2} + 1, \qquad g(2n; 3, 5) = 2n\binom{n}{2} + 1 \tag{8}$$

but the proof of (8) seems elusive [2], [10], [14], [25], [33], [34].

16. Rado and I considered the following question: let $f(n;k)$ be the smallest integer so that if $A_i, 1 \leqslant i \leqslant f(n;k)\, |A_i| = n$ are sets, then one can always find k. A's which have pairwise the same intersection. We proved

$$(k-1)^{n+1} < f(n;k)$$
$$< n!(k -)^{n+1}\left(1 - \frac{1}{2!(k-1)} - \frac{2}{3!(k-1)^2} - \cdots - \frac{n-1}{n!(k-1)^{n-1}}\right) \tag{1}$$

We conjectured that

$$f(n;k) < c_k^n \tag{2}$$

Abbott improved (1), but (2) is far from being settled. Recently Sauer determined $f(2;k)$ (not yet published).

(2), if true would have several applications in number theory.

Rado and I also investigated these questions if n and k are infinite cardinal numbers, but all the problems can then be solved completely.

Finally, the following question could be considered: let $g(n;k)$ be the smallest integer so that if $|S| = n$, $A_i \subset S$, $1 \leqslant i \leqslant g(n;k)$ then one can always find k A's which have pairwise the same intersection. Determine or estimate $g(n;k)$; compute $\lim_{n=\infty} g(n;k)^{1/k}$ [1], [23].

17. Let $\{A_i\}$, $\bigcup_i A_i = S$, $|A_i| \geqslant 2$ be a set system. This set system is said to be k-chromatic if one can divide S into k disjoint sets S_i, $\bigcup_{i=1}^{k} S_i = S$ so that no A_i is contained in any S_i and such a division into less than k sets is impossible. (A set system is thus two-chromatic if it has property B according to Miller.) I conjectured and Lovász proved that if $|A_i| \geqslant 2$ and for every $S_1 < S$ there are fewer than S_1 A's contained in S_1 then the chromatic number of the system is 2. The Steiner triplets of $n = 7$ show that this result is best possible (even if $|A_i| > 2$).

Probably the following result holds: there is a constant c_t, $c_t \to \infty$ as $t \to \infty$ so that if $\{A_i\}$ is a set-system, $|A_i| > t$, and for every $S_1 \subset S$ there are fewer than $c_t|S_1|$ A's contained in S_1, then the system has chromatic number 2.

Lovász proved that if $|A_i| = r > 2$ and the set system is k-chromatic and does not contain all the r-tuples of a set of $(k-1)(r-1)+1$ elements, then there are k A's any two of which intersect in the same element. This is a generalization of a well-known theorem of Brooks.

Is it true that if $|A_i| \geqslant t$ and the system is three-chromatic then there is an element which is contained in at least $(1+c)^t$ A_i's? [20], [29].

18. Let $|S| = n$, $A_i \subset S$, $1 \leqslant i \leqslant l_n$. Assume that no A_i is the union of other A's. Kleitman and I observed that (unpublished)

$$c_1 2^n / n^{3/2} < l_n < c_2 2^n / n^{3/2}$$

Probably there is a c so that

$$\max l_n = \big(1 + o(1)\big)\, c 2^n / n^{3/2}.$$

19. Let $|S| = n$, $A_i \subset S$, $1 \leqslant i \leqslant u_n$, $|A_i| = 3$. Assume that any subset $S_i \subset S$, $|S_i| = 6$ contains at most two A's. How large can u_n be? No doubt $u_n = o(n^2)$ and I expect that $u_n < n^{2-c}$ for some $c > 0$.

Let $|S| = n$, $A_i \subset S$, $|A_i| = 3$, $|A_i \cap A_j| \leqslant 1$. It is easy to see that there always exists an $S_1 \subset S$, $|S_1| > c_1\sqrt{n}$ so that no A_i is contained in S_1, but the above result becomes false if $c_1\sqrt{n}$ is replaced by $c_2 n^{2/3}$. A set S_1 is called independent if no A is contained in it. Denote by $f(n)$ the minimum of the largest independent set where the minimum is extended over all possible choices of the sets A_i, $|A_i| = 3$, $|A_i \cap A_j| \leqslant 1$. As stated $c_1\sqrt{n} < f(n) < c_2 n^{2/3}$. Improve the estimation for $f(n)$.

Hajnal and I considered the following question: let S have the elements $x_1, \ldots, x_n$. To each couple (x_i, x_j) make correspond an element x_i, $x_r = p(x_i, x_j)$, $r \neq i$, $r \neq j$. A set $S_1 \subset S$ is said to be independent if for any $x_i \in S_1$, $x_j \in S_1$, $p(x_i, x_j) \notin S_1$. Denote by $g(n)$ the minimum of the largest independent set where the minimum is taken over all functions $p(x_i, x_j)$. We proved

$$c_1 n^{1/3} < g(n) < c_2\sqrt{n \log n}.$$

Improve the estimations of $g(n)$ [20].

20. Kleitman proved the following conjecture of mine. Let $|S| = n$, $A_i \subset S$, $1 \leqslant i \leqslant k$. Assume that the union of two A's never equals a third. Then

$$\max k = (1 + o(1)) \binom{n}{[\frac{1}{2}n]}. \tag{1}$$

Moser now asked: let $A_1, \ldots, A_l$ be k arbitrary sets. Denote by $f(k)$ the largest integer so that for every choice of the k sets there always are $f(k)$ of them $A_{i_1}, \ldots, A_{i_{f(k)}}$ so that the union of two of them never equals a third. Riddel observed $f(k) > c_1\sqrt{k}$ and recently Komlós and I showed $f(k) < c_2\sqrt{k}$.

Moser's beautiful question can clearly be modified in various ways. It can also be given a number-theoretic interpretation, e.g. let $a_1 < \ldots < a_k$ be k integers, let $g(k)$ be the largest integer so that one can always find $g(k)$ integers $a_{i_1} < \ldots$ so that the sums (or products) of any two are distinct (or never equals a third etc.) [9].

21. Let $|S| = n$, $k < n/2$, $A_i \subset S$, $1 \leqslant i \leqslant r$, $|A_i| = k \leqslant n/2$, $|A_i \cap A_j| \geqslant 1$, $1 \leqslant i \leqslant j \leqslant r$. Ko, Rado and I proved that then $\max r = \binom{n-1}{k-1}$. In fact the same result holds if instead of $|A_i| = k$ we only assume $|A_i| \leqslant k$, $A_i \not\subset A_j$.

Let us now assume $|A_i \cap A_j| \geqslant s > 1$. What can be said about max r? If n is sufficiently large we proved that $\max r = \binom{n-s}{k-s}$, but Min showed that this is not always true and there does not seem to be an easy way to determine max r. Denote $\max r = f(n; k, s)$. We conjectured [16]

$$f(4k; 2k, 2) = \left(\binom{4k}{2k} - \binom{2k}{k}^2\right)\Big/2.$$

It is easy to see that

$$f(4k; 2k, 2) \geqslant \left(\binom{4k}{2k} - \binom{2k}{k}^2\right)\Big/2.$$

22. Szekeres and I proved that if $x_1, \ldots, x_{m^2+1}$ is any sequence of distinct numbers one can always find $n+1$ of them which form a monotonically increasing or decreasing sequence; it is easy to see that this theorem is false for n^2 numbers.

I now asked: let $f(b)$ be the largest integer so that every sequence of distinct numbers $x_1, \ldots, x_{f(n)}$ can be decomposed into the union of n monotonic sequences. Hanani proved that

$$f(n) = \frac{n(n+3)}{2}.$$

As far as I know the following question is not yet settled.-Let $x_1, \ldots, x_n$ be a sequence of distinct numbers, determine max $(\sum x_{i_r})$ where the maximum is to be taken over all monotonic sequences [26], [28].

23. Let $|A_i| = n, 1 \leqslant i \leqslant l, l \leqslant c_1 2^n$ $\left|\bigcup_{i=1}^{l} A_i\right| = N$. Is it true that if $n > n_0|c_1|$ there are $c_2 2^N (c_2 = c_2(c_1))$ subsets B of $\bigcup_{i=1}^{l} A_i$ which have a non-empty intersection with every A_i but which contains none of the A_i's? If $l = 2^n$ I cannot even prove the existence of a single such set B [8].

24. Let the vertices of K_n be $x_1, \ldots, x_n$. Denote by (i,j) the edge joining x_i and x_j. Let $f(i,j) = \pm 1, 1 \leqslant i \leqslant j \leqslant n$. Put

$$H(n) = \min_f \left(\max_{K_r} |\textstyle\sum f(i,j)| \right) \tag{1}$$

where in (1) the maximum is taken over all complete subgraphs of $K_n (1 \leqslant r \leqslant n)$, the summation is extended over all the edges of K_r and the minimum is taken over all the $2^{\binom{n}{2}}$ functions $f(i,j)$. I proved [7]

$$\frac{n}{4} \leqslant H(n) < cn^{3/2}. \tag{2}$$

It would be desirable to improve (2).†

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† *Note added in proof:* Spencer and I proved $H(n) > cn^{3/2}$.

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