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ON A RAMSEY TYPE THEOREM

by

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To the memory of A. RÉNYI

$n \to (u)_k^2$ is the well known arrow symbol introduced by ERDŐS and RADO [1]. It means that if we color the edges of a complete graph of n vertices, k_n, by k colors there is always a k_u whose edges all have the same color.

The symbol $n \to [v]_k^2$ introduced by ERDŐS, HAJNAL and RADO [2] means that if we color the edges of a k_n by k colors there is always a k_v whose edges contain only $k - 1$ colors. These symbols were studied extensively for infinite cardinals in [1] and [2]. In this paper we will only consider finite n. It is well known ([3], [4]) that

$$n \to \left[\frac{\log n}{2 \log 2}\right]_2^2 \tag{1}$$

$$n \not\to \left(\frac{2 \log n}{\log 2}\right)_2^2. \tag{2}$$

It would be very desirable to determine the constant c so that

$$n \to ((c - \varepsilon) \log n)_2^2, \quad n \not\to ((c + \varepsilon) \log n)_2^2 . \tag{3}$$

It is not even known if such a c exists and we can make no contribution towards (3).

By the methods of [3] and [4] it is easy to see that there are absolute constants c_1 and c_2 so that for every k and n

$$n \to \left(\frac{c_1 \log n}{k \log k}\right)_k^2 \tag{4}$$

and

$$n \not\to \left(\frac{c_2 \log n}{\log k}\right)_k^2. \tag{5}$$

It would be very desirable to bring (4) and (5) closer together, but it seems difficult to guess what happens for large k.

The methods of [4] immediately give that for a sufficiently large absolute constant c_3

$$n \nrightarrow [c_3 k \log n]^2_k . \tag{6}$$

ERDŐS, HAJNAL and RADO conjectured that for every c there is a $k_0 = k_0(c)$ so that for $k > k_0$

$$n \rightarrow [c \log n]^2_k . \tag{7}$$

In this note we prove (7). In fact, we prove

THEOREM 1. *There is an absolute constant c_4 so that*

$$n \rightarrow \left[c_4 \frac{k}{\log k} \log n \right]^2_k . \tag{8}$$

(6) and (8) are not so far apart, but at present we cannot even guess where the real truth lies.

Let $G(n)$ be a graph of n vertices. $\overline{G(n)}$, the complementary graph, has the same vertices and two vertices are joined in $\overline{G(n)}$ if and only if they are not joined in $G(n)$.

Color the edges of K_n by k colors. At least one of the colors has fewer than $\frac{1}{k}\binom{n}{2}$ edges. Thus, Theorem 1 is an immediate consequence of the following

THEOREM 2. *Let $G(n; r)$ be a graph of n vertices and $r < \frac{n^2}{k}$ edges. Then either $\overline{G(n; r)}$ or $G(n; r)$ contains a K_s for $s > c_5 \frac{k}{\log k} \log n$.*

The result is best possible as far as the order of magnitude is concerned since it fails for $s > c_6 \frac{k}{\log k} \log n$ if c_6 is a sufficiently large absolute constant.

Before we prove Theorem 2, we show that it is best possible. First we show that

$$n \nrightarrow \left(\frac{c_7 \log n}{\log k}, c_8 k^{3/4} \log n \right)^2_2 . \tag{9}$$

We do not give all the details since the proof closely follows [4].[1] Decompose K_n in all possible ways as the edge disjoint union of two graphs G_1 and G_2 (i.e. $K_n = G_1 \cup G_2$) where G_1 has $\frac{1}{\sqrt{k}}\binom{n}{2}$ edges — in other words roughly

[1] (9) in fact is contained in a paper of Rényi, Probabilistic methods in combinatorial mathematics, Combinatorial Math. and its Appl., Proc. Confer. held at Chapel Hill, Univ. of North Carolina, Univ. of North Carolina Prepr., 1967, pp. 1—13, see p. 5.

speaking an edge belongs to G_1 with probability $\frac{1}{\sqrt{k}}$ and to G_2 with probability $1-\frac{1}{\sqrt{k}}$. We use the probability language to shorten the computation — it will be clear to the reader that we could work as in [4] and [5] simply using binomial coefficients.

Now we show that the probability that G_1 contains a K_{s_1}, $s_1 = c_7 \log n/\log k$ is less than $\frac{1}{10}$ if c_7 is sufficiently large. A simple argument shows that this probability is less than

$$\binom{n}{s_1}\left(\frac{1}{k^{1/2}}\right)^{\binom{s_1}{2}} < n^{s_1}\exp\left(-\frac{s_1^2\log k}{10}\right) < \frac{1}{10} \tag{10}$$

if $c_7 > 20$ (we are not interested in trying to obtain small values for our constants, since the best possible values are beyond our reach even in (3)).

Now we prove that for sufficiently large c_8 the probability that G_2 contains a

$$K_{s_2}, \quad s_2 = c_8 k^{3/4}\log n$$

is also less than $\frac{1}{10}$. The probability that G_2 contains a K_{s_2} is less than

$$\binom{n}{s_2}\left(1-\frac{1}{k^{1/2}}\right)^{\binom{s_2}{2}} < n^{s_2}\exp\left(-\frac{s_2^2}{4\,k^{1/2}}\right) < \frac{1}{10}. \tag{11}$$

(10) and (11) implies that with probability greater than $\frac{4}{5}$, G_1 contains no K_{s_1} and G_2 no K_{s_2} and (9) is proved.

(9) easily implies that Theorem 2 is best possible. We now construct a decomposition $K_n = G_1 + G_2$, so that the number of edges in G_1 $e(G_1) < \frac{n^2}{k}$ and neither G_1 nor G_2 contains a

$$K_s \text{ if } s > \frac{c_6 k}{\log k}\log n.$$

Put $n = u_1 + \ldots + u_k$ where the u_i are as nearly equal as possible $\left(u_i = \left[\frac{n}{k}\right] \text{ or } \left[\frac{n}{k}\right]+1\right)$. Decompose the set of n vertices of K_n into sets S_i, $|S_i| = u_i$. Define G_2^* by $(x, y) \in G_2^*$ if $x \in S_i$, $y \in S_j$, $i \neq j$.

By (9) the complete graph defined by the vertices of S_i can be decomposed into two graphs $G_1^{(i)}$ and $G_2^{(i)}$ where $G_1^{(i)}$ contains no K_{s_1}, $s_1 > c_8 k^{3/4}\log n$ and $G_2^{(i)}$ contains no K_{s_2}, $s_2 = c_7 \log n/\log k$

$$G_1 = \bigcup_{i=1}^{k} G_1^{(i)}, \quad G_2 = G_2^* \bigcup_{i=1}^{k} G_2^{(i)}.$$

$G_1 \cup G_2 = K_n$, G_1 clearly contains no K_{s_1} and G_2 contains no K_{s_2}. This proves that Theorem 2 is best possible.

We now prove Theorem 2. Let $G = G(n, t)$ be a graph having n vertices and $t < \frac{n^2}{k}$ edges. At least half of the vertices, say $x_1, x_2, \ldots, x_m$ $\left(m > \frac{n}{2}\right)$ have valency $v(x_i) < \frac{10n}{k}$. We consider the subgraph G' spanned by these m vertices.

Suppose the largest complete graph in $\bar{G}'$ (i.e. independent set in G') has l vertices. We can assume that these are $x_1, \ldots, x_l$. By assumption $v(x_i) < \frac{10n}{k} < \frac{20m}{k}$, and so

$$\sum_{i=1}^{l} v(x_i) < \frac{20ml}{k}. \tag{12}$$

It follows from (12) that at most $\frac{m}{2}$ of the vertices $x_{l-1}, \ldots, x_m$ are joined to $s = \frac{40l}{k}$ or more of the vertices $x_1, \ldots, x_l$. We can assume that

$$s < \frac{c_{10} \log n}{\log k} \tag{13}$$

where c_{10} is a suitable small constant. Then the number of subsets of $\{x_1, \ldots, x_l\}$ having fewer than s elements is at most

$$s \binom{l}{s-1} < s^2 \left(\frac{l}{s}\right)^s e^s < l \left(\frac{ek}{40}\right)^{40l/k} < l k^{c_{11} \log n / \log k} < \sqrt[3]{n} \tag{14}$$

provided c_{10} is sufficiently small. It follows from (14) that there are $\{y_1, \ldots, y_p\} \subset \subset \{x_1, \ldots, x_l\}$ and $\{z_1, \ldots, z_q\} \subset \{x_{l+1}, \ldots, x_m\}$ so that $p < s, q > \dfrac{\frac{m}{2} - l}{\sqrt[3]{n}} > \sqrt{n}$ and such that each z_j $(1 \leq j \leq q)$ is joined to each y_i $(1 \leq i \leq p)$ but to no other point in $\{x_1, \ldots, x_l\} - \{y_1, \ldots, y_p\}$.

Now consider the subgraph G'' spanned by $\{z_1, \ldots, z_q\}$, $q > \sqrt{n}$. By a simple and well-known theorem of ERDŐS and SZEKERES [3]

$$\binom{u+v-2}{u-1} \to (u, v). \tag{15}$$

The graph G'' does not contain an independent set containing $s = \frac{40l}{k}$ vertices, otherwise we could exchange these for $\{y_1, \ldots, y_p\}$ (in $x_1, \ldots, x_l$) and obtain a complete subgraph in $\bar{G}'$ having more than l vertices, contrary to the definition of l. Therefore, by (14) and (15) a simple computation gives that G'' contains a complete graph having $c_{10}k\log n/\log k$ vertices. This proves Theorem 2 and hence Theorem 1.

Finally we would like to call attention to the fact that, as pointed out by V. T. Sós, Theorem 2 belongs to the class of theorems studied recently by P. Erdős and V. T. Sós [6].

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