ON THE DISTRIBUTION OF THE ROOTS OF ORTHOGONAL POLYNOMIALS

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Dedicated to Professor G. Szegő on the occasion of his 75th birthday

Let $-1 \leq x \leq 1$, $p(x) \geq 0$ an integrable function. $P_n(x)$, $n = 1, 2, \ldots$ is the sequence of orthogonal polynomials with respect to $p(x)$.

In other words $P_n(x)$ is a polynomial of degree n and

$$\int_{-1}^{+1} P_n(x) P_m(x) p(x) \, dx = 0, \qquad n \neq m.$$

It is well known that all the roots of $P_n(x)$ are simple and they all are in $(-1, +1)$. Many mathematicians obtained asymptotic formulas for $P_n(x)$ if various regularity assumptions are made about $p(x)$ and from these one can deduce results about the distribution of the roots of $P_n(x)$ [2].

Let $-1 \leq x_1 \leq \ldots < x_n \leq 1$ be the roots of $P_n(x)$ (we omitted the upper index n since there is no danger of misunderstanding). Put $\cos \vartheta_i = x_i$ and denote by $N_n(a, b)$ the number of ϑ's on the arc (a, b), $0 < a < b \leqslant \pi$. TURÁN and I proved [1] that if the roots of $p(x)$ form a set of measure 0, then

$$N_n(a, b) = n \frac{b - a}{\pi} + o(n). \tag{1}$$

In other words the roots of $P_n(x)$ are uniformly distributed in this sense. As far as I know this is the most general result on the uniform distribution of roots. More than 30 years ago I had an idea to obtain a result which in some sense is more general. Assume that $|p(x)| < C$ and denote by $\mathscr{S}$ the set in x for which $p(x) > 0$. Assume further that $\mathscr{S}$ has the following property: To every $\varepsilon > 0$ there is a $\delta > 0$ so that if we omit from $\mathscr{S}$ an arbitrary set of measure $< \delta$, the remaining set has transfinite diameter greater than $\frac{1}{2} - \varepsilon$. I conjectured that the necessary and sufficient condition that the roots of $P_n(x)$ should be uniformly distributed in the above sense (i.e. that they satisfy (1) for every $0 \leq a < b \leq \pi$) is that $\mathscr{S}$ has property P.

I have never succeeded in obtaining a satisfactory proof of this conjecture (ULLMAN and I obtained some preliminary results).

It is easy to see that the conjecture fails to hold if $|p(x)| < C$ is not assumed.

Now we investigate the interval $(-\infty, +\infty)$. Let $p(x) \geq 0$ and assume that $x^n p(x)$ is integrable in $(-\infty, +\infty)$ for every $n \geq 0$. Let $P_n(x)$ be the unique sequence of polynomials satisfying

$$\int_{-\infty}^{\infty} P_n(x)\, P_m(x)\, p(x)\, dx = 0; \qquad 1 \leq n < m < \infty$$

henceforth we normalize $P_n(x)$ to have the highest coefficient 1.

Let $\lambda_1 < \ldots < x_n$ be the roots of $P_n(x)$. As far as I know the distribution of the x's have been studied only in case of the classical polynomials (e.g. Hermite or Laguerre polynomials). Transform the interval (x_1, x_n) linearly onto $(-1, +1)$ so that x_1 goes into -1 and x_n into $+1$. The sequence $x_1 < \ldots < x_n$ goes into $y_1 < \ldots < y_n$. We say that the roots of $P_n(x)$ are uniformly distributed if the y's satisfy (1). It is of course well known that the roots of the classical polynomials (Hermite, Laguerre) are not uniformly distributed in this sense. The reason for this fact turns out to be that the weight function $p(x)$ does not tend to 0 fast enough. We prove the following

THEOREM. *Let $-\infty < x < \infty$, $0 < p(x) < C$. Assume that to every $\varepsilon < 0$ there is an x_0 so that for every $|x| > x_0$, if y is of the same sign as x and $|y| > |x(1+\varepsilon)|$ then*

$$p(y) < p(x)^2. \tag{2}$$

Then the roots of $P_n(x)$ are uniformly distributed in the above sense.

(2) implies that for every k as $|x| \to \infty$

$$p(x) = o(e^{-|x|^k}). \tag{3}$$

It seems likely that if (3) holds for every k then the roots of $P(x)$ are uniformly distributed. This would be stronger than our theorem but I have never been able to prove it. Some condition like (3) is certainly needed for the uniform distribution since if $p(x) = e^{-|x|^k}$ it is not difficult to show that the roots of $P_n(x)$ are not uniformly distributed.

The proof of our Theorem will be very similar to our old theorem with TURÁN, the rapid decrease of $p(x)$ makes our polynomials behave as if they would be orthogonal in $(-1, +1)$.

We will use the following extremal property of $P_n(x)$: Let $q_n(x) = x^n + \ldots$ be any polynomial, then

$$\int_{-\infty}^{\infty} P_n^2(x)\, p(x)\, dx \leq \int_{-\infty}^{\infty} q_n^2(x)\, p(x)\, dx, \tag{4}$$

equality in (4) holds only if $q_n(x) \equiv P_n(x)$.

Let (u_n, v_n) be the largest interval so that for every $u_n < x < v_n$, $p(x) > \dfrac{1}{2^n}$.

First we show that the roots of $P_n(x)$ are uniformly distributed in (u_n, v_n) (in the above sense).

Let us assume that the roots of $P_n(x)$ are not uniformly distributed in (u_n, v_n). Let $\varepsilon > 0$ and put

$$A_n = \max_{u_n(1-\varepsilon) < x < v_n(1-\varepsilon)} |P_n(x)|, \quad |P_n(x_0)| = A_n. \tag{5}$$

It is well known that if the roots of $P_n(x)$ are not uniformly distributed in (u_n, v_n) then

$$A_n > (1+c)^n \left(\frac{v_n - u_n}{4}\right)^n. \tag{6}$$

It follows from (6) and the well-known theorem of Markov that for

$$|P_n(x)| > \frac{1}{2} A_n$$

$$x_0 - \frac{1}{2n^2} < x < x_0 + \frac{1}{2n^2}. \tag{7}$$

From the definition of (u_n, v_n) and by a repeated application of (2) we have that for $x_0 - \dfrac{1}{2n^2} < x < x_0 + \dfrac{1}{2n^2}$

$$p(x) > (1-\eta)^n \tag{8}$$

for every $\eta > 0$ if n is large enough (choose $\varepsilon = \varepsilon(\eta)$ small enough). From (7) and (8) we obtain

$$\int_{-\infty}^{\infty} P_n^2(x)\, p(x)\, dx > \int_{x_0 - \frac{1}{2n^2}}^{x_0 + \frac{1}{2n^2}} P_n^2(x)\, p(x)\, dx > \tag{9}$$

$$> \frac{(1-\eta)^n}{n^2} (1+c)^{2n} \left(\frac{v_n - u_n}{4}\right)^{2n} > (1+c)^n \left(\frac{v_n - u_n}{4}\right)^{2n}$$

if η is sufficiently small and n is sufficiently large.

Now we show that (9) leads to a contradiction. Denote by $T_n(x)$ the normed Chebyshev polynomial belonging to (u_n, v_n). We have from (4)

$$\int_{-\infty}^{\infty} P_n^2(x)\, p(x)\, dx \leq \int_{-\infty}^{\infty} T_n^2(x)\, p(x)\, dx. \tag{10}$$

We now show that (9) contradicts (10). To see this observe that

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} T_n^2(x)\, p(x)\, dx = \int_{u_n}^{v_n} T_n^2(x)\, p(x)\, dx + \\ \displaystyle + \int_{-\infty}^{u_n} T_n^2(x)\, p(x)\, dx + \int_{v_n}^{\infty} T_n^2(x)\, p(x) dx = I_1 + I_2 + I_3. \end{cases} \tag{11}$$

Clearly from $p(x) < C$

$$I_1 < 4C(v_n - u_n)\left(\frac{v_n - u_n}{4}\right)^{2n} \tag{12}$$

since, as is well known,

$$\max_{u_n < x < v_n} |T_n(x)| = 2\left(\frac{v_n - u_n}{4}\right)^{2n}.$$

All the roots of $T_n(x)$ are in (u_n, v_n), thus

$$\begin{aligned} |T_n(x)| &< |x - u_n|^n \quad \text{for} \quad v_n < x < \infty, \\ |T_n(x)| &< |v_n - x|^n \quad \text{for} \quad -\infty < x < u_n. \end{aligned} \tag{13}$$

From (13) and (4) we obtain by a simple computation that for every $\delta > 0$ if n is sufficiently large

$$I_2 < (1+\delta)^n \left(\frac{v_n - u_n}{4}\right)^{2n}, \quad I_3 < (1+\delta)^n \left(\frac{v_n - u_n}{4}\right)^{2n} \tag{14}$$

(12), (14) and (10) implies

$$\int_{-\infty}^{\infty} P_n^2(x)\, p(x)\, dx < (1 + 2\delta)^n \left(\frac{v_n - u_n}{4}\right)^{2n}. \tag{15}$$

(15) contradicts (9), thus we proved our assertion that the roots of $P_n(x)$ are uniformly distributed in (u_n, v_n).

Now to prove our theorem we only have to show that for every $\varepsilon > 0$ if $n > n_0(\varepsilon)$

$$x_1 > u_n(1+\varepsilon), \quad x_n < v_n(1+\varepsilon). \tag{16}$$

It will suffice to show $x_n < v_n(1 + \varepsilon)$. Assume without loss of generality that

$$x_n - v_n \geq u_n - x_1 > \varepsilon v_n. \tag{17}$$

Put

$$Q_n(x) = \frac{P_n(x)}{x - x_n}(x - v_n).$$

We now show that (17) implies

$$\int_{-\infty}^{\infty} Q_n^2(x)\, p(x)\, dx < \int_{-\infty}^{\infty} P_n^2(x)\, p(x)\, dx \tag{18}$$

(18) contradicts (4). Thus (18) will imply that (17) leads to a contradiction, which completes the proof of (16) and our Theorem.

To prove (18) observe that for $x < \frac{x_n - v_n}{2}$ $|Q_n(x)| \leqslant |P_n(x)|$ and for $u_n < x < v_n$ and $n > n_0(\varepsilon)$

$$|Q_n(x)| = |P_n(x)| \left| \frac{x - v_n}{x - x_n} \right| < |P_n(x)| \left(1 + \frac{1}{n}\right)^{-1}. \tag{19}$$

The inequality in (19) follows from $x_n > v_n(1 + \varepsilon)$ and from the fact that (2) implies $v_n - u_n = o(n)$ (in fact it implies $v_n - u_n = o((\log n)^{\varepsilon})$).

Let x_0 be defined by (5). We have from the theorem of Chebyshev and (7) that for $x_0 - \frac{1}{2n^2} < x < x_0 + \frac{1}{2n^2}$

$$|P_n(x)| > \frac{A_n}{2} > (1 - \varepsilon)^n \left(\frac{v_n - u_n}{4}\right)^{2n}. \tag{20}$$

Hence from (19), (20) and (8)

$$\tag{21} \begin{cases} \displaystyle\int_{-\infty}^{\frac{x_n - v_n}{2}} P_n^2(x)\, p(n)\, dx - \int_{-\infty}^{\frac{x_n - v_n}{2}} Q_n^2(x)\, p(x)\, dx \geq \frac{1}{n} \int_{-\infty}^{v_n} P_n^2(x)\, p(x)\, dx > \\ \displaystyle > \frac{1}{n} \int_{x_0 - \frac{1}{2n^2}}^{x_0 - \frac{1}{2n^2}} P_n^2(x)\, p(x)\, dx > \frac{1}{n^3} (1 - \eta)^n (1 - \varepsilon)^{2n} \left(\frac{v_n - u_n}{4}\right)^{2n}. \end{cases}$$

On the other hand from (17) and by a repeated application of (2) we obtain by a simple computation

$$\tag{22} \begin{cases} \displaystyle\int_{\frac{x_n - v_n}{2}}^{\infty} Q_n^2(x)\, p(x)\, dx - \int_{\frac{x_n - v_n}{2}}^{\infty} P_n^2(x)\, p(x)\, dx < \int_{\frac{x_n - v_n}{2}}^{\infty} Q_n^2(x)\, p(x)\, dx < \\ \displaystyle < \int_{\frac{x_n - v_n}{2}}^{\infty} (x - x_1)^{2n}\, p(x)\, dx \leq \int_{\frac{x_n - v_n}{2}}^{\infty} (x + x_n)^{2n}\, p(x)\, dx < \left(\frac{v_n - u_n}{8}\right)^{2n}. \end{cases}$$

(To obtain the last inequality in (22) observe that by (2) $p(x)$ decreases much faster than $(x + v_n)^{2n}$ increases.)

(21) and (22) immediately imply (18) if ε and η are sufficiently small and $n > n_0(\varepsilon, \eta)$. Thus our Theorem is proved.

It is possible that the method used in this paper will permit to conclude that (3) also implies our Theorem, but as stated previously, I have not succeeded in this.

References

[1] P. Erdős and P. Turán, Interpolation III, *Annales of Math*, **41** (1940), 510–553.
[2] G. Szegő, *Orthogonal Polynomials*. Amer. Math. Soc. Colloquium Publications Vol. XXIII, 1959 (revised edition).