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PARTITION RELATIONS FOR η_α AND FOR $\aleph_\alpha$-SATURATED MODELS

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1. Introduction

We denote by R_α the set of all 0, 1 sequences of length ω_α which have a final 1, i.e. $(x_\nu)_{\nu<\omega_\alpha} \in R_\alpha$ if there is $\delta < \omega_\alpha$ such that $x_\nu \in \{0, 1\}$ $(\nu < \delta)$, $x_\delta = 1$, $x_\nu = 0$ $(\delta < \nu < \omega_\alpha)$. The order type of R_α with the natural lexicographic order is denoted by η_α. The sets R_α are the analogues of the ordered set of rationals to higher cardinals. Their most important property is that they are universal embedding sets for ordered sets of cardinal $\aleph_\alpha$ and in § 2 we mention other basic properties.

A graph is an ordered pair $G = (S, E)$ with $E \subset [S]^2 = \{X \subset S: |X| = 2\}$. The elements of S are the *points* of G and the elements of E are the *edges*. A set $X \subset S$ is *independent* if there are no edges in X, i.e. $[X]^2 \cap E = \emptyset$. Y is a *complete subgraph* if $[Y]^2 \subset E$ and $[M, N] = \{\{x, y\}: x \in M \wedge y \in N \wedge x \neq y\}$ is a *complete bipartite subgraph of* G if $[M, N] \subset E$, $M \cap N = \emptyset$. In [2] it was shown that if $\aleph_\alpha$ is regular and G is any graph on R_α, then either G contains an independent set of type η_α or (in some sense) a large complete subgraph (see (6) below). In this paper we establish analogous results which show that if G is any graph on R_α, then either there is an independent set of type η_α or there is (again in some sense) a large complete bipartite graph. In fact we shall introduce some new concepts which enable us to express our main results and the results of [3] in a more general setting.

As usual, $\mathfrak{P}(E)$ denotes the power set of E.

Definition 1. $\mathfrak{F}$ is an $\aleph_\alpha$-*quasi filter* on E if $\emptyset \notin \mathfrak{F} \subset \mathfrak{P}(E)$, $\mathfrak{F} \neq \emptyset$ and whenever $F_0 \supset F_1 \supset \cdots \supset F_\nu \supset \cdots$ $(\nu < \lambda)$ is a non-increasing sequence of length $\lambda < \omega_\alpha$ of sets in $\mathfrak{F}$, then there is $F \in \mathfrak{F}$ such that $F \subset \bigcap_{\nu<\lambda} F_\nu$.

Definition 2. Let $\mathfrak{F}$ be an $\aleph_\alpha$-quasi filter on E and let $F \in \mathfrak{F}$. A set $A \subset E$ is *dense* in F if $A \cap F' \neq \emptyset$ for every $F' \in \mathfrak{F}$ such that $F' \subset F$.

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We classify the subsets of E as being dense or non-dense and write

$$\mathrm{Dense}(\mathfrak{F}) = \{A \subset E: A \text{ is dense in some } F \in \mathfrak{F}\},$$

$$ND(\mathfrak{F}) \quad = \mathfrak{P}(E) - \mathrm{Dense}(\mathfrak{F}).$$

In an obvious sense, the members of $\mathrm{Dense}(\mathfrak{F})$ may be thought of as being 'large' subsets of E. We prove

(1) *If $\mathfrak{F}$ is an $\aleph_\alpha$-quasi filter, then $ND(\mathfrak{F})$ is an $\aleph_\alpha$-complete proper ideal.*

Proof. It is obvious that, if $B \subset A \in ND(\mathfrak{F})$, then $B \in ND(\mathfrak{F})$. Also, $E \notin ND(\mathfrak{F})$ since $\mathfrak{F} \neq \emptyset$.

Let $\varphi < \omega_\alpha$ and suppose that $A_\nu \in ND(\mathfrak{F})$ for $\nu < \varphi$. Let $A = \bigcup_{\nu<\varphi} A_\nu$, $F \in \mathfrak{F}$. We define a sequence $F_\nu \in \mathfrak{F}$ by transfinite induction as follows. Let $\nu < \varphi$ and suppose that $F_\mu \in \mathfrak{F}$ is already defined for $\mu < \nu$ so that $F \supset F_0 \supset \cdots \supset F_\mu \supset \cdots$. There is $F' \in \mathfrak{F}$ such that $F' \subset \bigcap_{\mu<\nu} F_\mu \cap F$. Since $A_\nu \in ND(\mathfrak{F})$, there is $F_\nu \in \mathfrak{F}$ such that $F_\nu \subset F'$ and $F_\nu \cap A_\nu = \emptyset$. This defines $F_\nu \in \mathfrak{F}$ for $\nu < \varphi$ so that

$$F \supset F_0 \supset \cdots \supset F_\nu \supset \cdots.$$

There is $F'' \in \mathfrak{F}$ such that $F'' \subset \bigcap_{\nu<\varphi} F_\nu \cap F$ and clearly $A \cap F'' = \emptyset$. Thus A is not dense in F. Since $F \in \mathfrak{F}$ was arbitrary, $A \in ND(\mathfrak{F})$ and (1) follows.

An important example is the following.

(2) *Let $E = R_\alpha$ and let $\mathfrak{F}$ be the set of all non-empty open intervals in R_α of the form (a, b) with $a, b \in R_\alpha \cup \{-\infty, \infty\}$. It is well known and easy to see that $\mathfrak{F}$ is an $\aleph_{cf(\alpha)}$-quasi filter. In this case $A \in \mathrm{Dense}(\mathfrak{F})$ if A is dense in some interval $(a, b) \in \mathfrak{F}$ considered as an ordered set.*

We remark here that $A \in \mathrm{Dense}(\mathfrak{F})$ does *not* in general imply that $|A| \geq \aleph_\alpha$ for an arbitrary $\aleph_\alpha$-quasi filter $\mathfrak{F}$ although this is the case in genuine applications. An obvious sufficient condition for this is the following.

(3) *Let $\mathfrak{F}$ be an $\aleph_\alpha$-quasi filter on E. Then $A \in \mathrm{Dense}(\mathfrak{F})$ implies that $|A| \geq \aleph_\alpha$ provided $\mathfrak{F}$ satisfies the condition: if $B \subset E$, $|B| < \aleph_\alpha$, $F \in \mathfrak{F}$, then there is $F' \subset F - B$ such that $F' \in \mathfrak{F}$.*

Note that this condition is satified by the example given in (2) when $\aleph_\alpha$ is regular.

In order to state our results in the language of the partition calculus we generalize the ordinary partition symbol of P. ERDŐS and R. RADO (e.g. see [3]). For any cardinal $\mathfrak{m}$, $[E]^{\mathfrak{m}} = \{X \subset E: |X| = \mathfrak{m}\}$.

Definition 3. Let E be a set, γ an ordinal number, r a positive integer and let $\mathfrak{G}_\nu \subset \mathfrak{P}([E]^r)$ $(\nu < \gamma)$. The partition symbol

(4) $$E \to (\mathfrak{G}_0, \ldots, \mathfrak{G}_\nu, \ldots)^r_{\nu<\gamma}$$

means that the following statement is true: For every r-partition $[E]^r = \bigcup_{\nu<\gamma} I_\nu$ of E of length γ, there are $X \subset [E]^r$ and $\nu < \gamma$ such that $X \in \mathfrak{G}_\nu$ and $X \subset I_\nu$. The negation of (4) is written as

$$E \nrightarrow (\mathfrak{G}_0, \ldots, \mathfrak{G}_\nu, \ldots)^r_{\nu<\gamma} .$$

If $\mathfrak{G}_\nu = \{[X]^r : X \in \mathfrak{F}_\nu\}$ for some $\mathfrak{F}_\nu \subset \mathfrak{P}(E)$, then we shall replace the $\mathfrak{G}_\nu$ in (4) by $\mathfrak{F}_\nu$, i.e. we write

(4′) $$E \rightarrow (\mathfrak{F}_0, \ldots, \mathfrak{F}_\nu, \ldots)^r_{\nu<\gamma} .$$

This does not lead to any confusion since the entries $\mathfrak{G}_\nu$ in the symbol (4) are subsets of $\mathfrak{P}([E]^r)$, whereas the $\mathfrak{F}_\nu$ in (4′) are subsets of $\mathfrak{P}(E)$ and must therefore be interpreted in the most natural way as a shorthand notation for the set $\{[X]^r : X \in \mathfrak{F}_\nu\} \subset \mathfrak{P}([E]^r)$.

In most cases statements of the form (4) or (4′) depend not so much upon the actual set E but rather upon the cardinality or order type of E. In such cases we simply write $|E| \rightarrow \cdots$ or tp $E \rightarrow \cdots$ in place of $E \rightarrow \cdots$. Similarly, if $\mathfrak{F}_\nu$ in (4′) is the set of all subsets of E of cardinality (or type) $\mathfrak{m}_\nu$ we simply replace the entry $\mathfrak{F}_\nu$ in (4) by $\mathfrak{m}_\nu$. In this way we regain the original partition symbol

$$\mathfrak{m} \rightarrow (\mathfrak{m}_0, \ldots, \mathfrak{m}_\nu, \ldots)^r_{\nu<\gamma}$$

introduced in [4]. We use one other special convention in this paper. If $r = 2$, $\nu < \gamma$, $\mathfrak{F} \subset \mathfrak{P}(E)$ and

$$\mathfrak{G}_\nu = \{[M, N] : M \subset E \wedge N \in \mathfrak{F} \wedge M \cap N = \emptyset \wedge |M| = \mathfrak{m}\} ,$$

then we replace the entry $\mathfrak{G}_\nu$ in (4) by the symbol $[\mathfrak{m}, \mathfrak{F}]$. For example, the relation (see (11))

$$\eta_\alpha \rightarrow (\eta_\alpha, [\mathfrak{m}, \eta_\alpha])^2$$

means: if $[R_\alpha]^2 = \mathfrak{T}_0 \cup \mathfrak{T}_1$ is any 2-partition of R_α of length 2, then either (i) there is a set $X \subset R_\alpha$ of type η_α such that $[X]^2 \subset \mathfrak{T}_0$ or (ii) there are sets $M, N \subset R_\alpha$ such that $|M| = \mathfrak{m}$, tp $N = \eta_\alpha$ and $[M, N] \subset \mathfrak{T}_1$.

Most of the results of this paper depend for their proof upon the generalized continuum hypothesis and when a formula or statement depends upon this hypothesis we prefix it by GCH. For example,

(5) (GCH) $$|R_\alpha| = \sum_{\nu<\omega_\alpha} 2^{|\nu|} = 2^{\aleph_\alpha} = \aleph_{\alpha+1} .$$

The main result proved in [3] is that

(6) *If $\alpha = cf(\alpha) > \beta$ and $\aleph_\gamma^{\mathfrak{m}} < \aleph_\alpha$ holds for all $\gamma < \alpha$ and $\mathfrak{m} < \aleph_\beta$ and if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, then*

$$\eta_\alpha \rightarrow (\eta_\alpha, \aleph_\beta)^2 .$$

In particular, this gives

(7) (GCH) $$\eta_{\alpha+1} \rightarrow (\eta_{\alpha+1}, \aleph_{cf(\alpha)})^2.$$

Using the concepts defined above, (5) can be generalized to the following:

(8) *Let $\mathfrak{F}$ be an $\aleph_\alpha$-quasi filter on E, $|\mathfrak{F}| \leqq \aleph_\alpha$. If $\alpha = cf(\alpha) > \beta$ and $\aleph_\gamma^{\mathfrak{m}} < \aleph_\alpha$ holds for every $\gamma < \alpha$ and $\mathfrak{m} < \aleph_\beta$, then*

$$E \rightarrow (\mathrm{Dense}(\mathfrak{F}), \aleph_\beta)^2 .$$

Also, corresponding to the simpler form (7), we have

(9) (GCH) *Let $\mathfrak{F}$ be an $\aleph_{\beta+1}$-quasi filter on E. Assume that $|\mathfrak{F}| \leqq \aleph_{\beta+1}$. Then*

$$E \rightarrow (\mathrm{Dense}(\mathfrak{F}), \aleph_{cf(\beta)})^2 .$$

We do not give the proof of (8) since this can be literally translated from [3] replacing the intervals of R_α by elements of $\mathfrak{F}$. However, it seems worthwhile mentioning these results in the more general setting in view of the possible applications we mention in § 4.

We shall prove in § 3 the following theorem.

(10) (GCH) *Let $\mathfrak{F}$ be an $\aleph_\alpha$-quasi filter on E and suppose that $|\mathfrak{F}| \leqq \aleph_\alpha$, $\mathfrak{m}^+ < \aleph_\alpha = \aleph_{cf(\alpha)}$. Then*

$$E \rightarrow (\mathrm{Dense}(\mathfrak{F}), [\mathfrak{m}, \mathrm{Dense}(\mathfrak{F})])^2 .$$

Since every dense subset of R_α contains a set of type η_α (see (21)) as a corollary of (10) we obtain:

(11) (GCH) *If $\mathfrak{m}^+ < \aleph_\alpha = \aleph_{cf(\alpha)}$, then*

$$\eta_\alpha \rightarrow (\eta_\alpha, [\mathfrak{m}, \eta_\alpha])^2 .$$

We mention that, as a corollary of Theorem 17 of [2], we know that

(12) (GCH) $\aleph_{\alpha+1} \nrightarrow (\aleph_{\alpha+1}, [\aleph_\alpha, \aleph_{\alpha+1}])^2 .$

This shows that the condition $\mathfrak{m}^+ < \aleph_\alpha$ in (10) and (11) cannot be replaced by the weaker condition $\mathfrak{m} < \aleph_\alpha$ and, in this sense, these results are best possible.

In contrast to (12) we shall prove

(13) (GCH) *Let $\mathfrak{F}$ be an $\aleph_{\beta+1}$-quasi filter on E, $|\mathfrak{F}| \leqq \aleph_{\beta+1}$. Then*

$$E \rightarrow (\mathrm{Dense}(\mathfrak{F}), [\aleph_\beta, \aleph_\beta])^2 .$$

And from this follows

(14) (GCH) $\eta_{\beta+1} \rightarrow (\eta_{\beta+1}, [\aleph_\beta, \aleph_\beta])^2 .$

Note that for regular $\aleph_\beta$, (14) is already implied by (6) and so the result is of interest only when $\aleph_\beta$ is singular. We do not know if (14) can be strengthened by replacing $[\aleph_\beta, \aleph_\beta]$ by $[\aleph_\beta, \eta_\beta]$. We could not settle even the simplest problem of this kind whether or not

(?) $\eta_1 \rightarrow (\eta_1, [\aleph_0, \eta_0])^2 .$

We remark that $\eta_1 \nrightarrow (\eta_1, \eta_0)^2$ follows from the trivial relation $\eta_1 \nrightarrow (\omega_1, \omega^*)$ ([4], Theorem 19).

In addition to the general results (10) and (13) we state some further results and problems involving η_α for singular $\aleph_\alpha$. It is easy to see that

(15) $\aleph_\omega \nrightarrow (\aleph_1, [1, \aleph_\omega])^2$

by considering the graph (S, E), where S is the union of disjoint sets S_n $(n < \omega)$, $|S_n| = \aleph_n$, and $E = \bigcup_{n<\omega} [S_n]^2$. An independent set can meet each S_n in at most one point and no point has valency $\aleph_\omega$. It follows from (15) that (11) is invalid for singular $\aleph_\alpha$. However, the following result holds for any limit cardinal $\aleph_\alpha$.

(16) (GCH) *If* $\mathfrak{m} < \aleph_{cf(\alpha)}, \lambda < \alpha, \alpha \neq \beta + 1$, *then*

$$\eta_\alpha \to (\eta_\alpha, [\mathfrak{m}, \eta_\lambda])^2 .$$

We do not know if (16) is true or false when $\mathfrak{m} = \aleph_{cf(\alpha)}$. The simplest problem of this kind is to decide whether or not the relation

(?) $\eta_\omega \to (\eta_\omega, [\aleph_0, \eta_0])^2$

holds. We do not even know if the relation

(?) $\eta_\omega \to (\eta_\omega, [\aleph_0, \aleph_0])^2$

holds. On the other hand, it is easy to prove that

$$\aleph_\omega \to (\aleph_\omega, [\aleph_k, \aleph_k'])^2 \qquad (k < \omega) .$$

It was asked in [3] if

(?) $\eta_\omega \to (\eta_\omega, 3)^2$.

We observed that if C_k denotes the class of all circuits of length k, then (16) implies that

(GCH) $\eta_\omega \to (\eta_\omega, C_{2k})^2$

holds for every $k < \omega$. However, we do not know if

(?) $\eta_\omega \to (\eta_\omega, C_{2k+1})^2$

holds for any fixed $k < \omega$. (A graph on R_ω which has no independent set of type η_ω is not 2-chromatic and does therefore contain odd circuits. The question is whether such a graph necessarily contains an odd circuit of fixed size.) Let P^∞, $P^{\infty,\infty}$ denote, respectively, the classes of 1-way and 2-way infinite paths in a graph. We mention that (7) and (16) easily imply that

(GCH) $\eta_\alpha \to (\eta_\alpha, P^{\infty,\infty})$ if $cf(\alpha) > 0$,

but we are unable to decide if

(?) $\eta_\omega \to (\eta_\omega, P^\infty)^2$.

In § 2 we discuss special properties of the sets R_α and of general η_α-sets. In § 3 we give the proofs of (10), (13) and (16) and in § 4 we state corollaries of (8), (10) and (13) for $\aleph_\alpha$-saturated models which are analogous to the respective corollaries (7), (11), (14) for η_α-sets.

2. Special Properties of R_α and of η_α-sets

If A, B are subsets of the ordered set $(S, <)$, we write $A < B$ if $a < b$ holds for all $a \in A$ and $b \in B$. We denote by $I(S)$ the set of all non-empty intervals of S having the forms

$$(a, b) = \{x \in S : a < x < b\}, \qquad S = (-\infty, \infty),$$
$$(a, \infty) = \{x \in S : a < x\}, \qquad (-\infty, a) = \{x \in S : x < a\}$$

where $a, b \in S$. When we say X is an interval of S we specifically mean that $X \in I(S)$. The order type of S is denoted by tp S and tp $S \geqq \theta$ means that there is a subset T of S having order type θ.

Hausdorff [8] called an ordered set $(S, <)$ an η_α-set if it has property P_α: *whenever $A, B \subset S$, $A < B$ and $|A|, |B| < \aleph_\alpha$, then there is $x \in S$ such that $A < \{x\} < B$.* It is well known that (e.g. [5])

(17) (i) *If $(S, <)$ is an η_α-set, then S is $\aleph_\alpha$-universal, i.e.* tp $S \geqq \theta$ *whenever* $|\theta| \leqq \aleph_\alpha$.

(ii) *If $\aleph_\alpha$ is regular then R_α is an η_α-set and if* (5) *holds then every η_α-set contains a subset similar to R_α.*

It follows from (17) that R_α is $\aleph_\alpha$-universal for regular $\aleph_\alpha$. In fact, R_α is $\aleph_\alpha$-universal even if $\aleph_\alpha$ is singular (see [5] and especially [9]).

We now give two further definitions. Our Definition 5 is motivated by the concept of an $\aleph_\alpha$-saturated model (see [10] and [11]) and the correlation between these two concepts will be explained in § 4.

Definition 4. A family of sets, $\mathfrak{F}$, has the *finite intersection property* (f.i.p.) if $\bigcap \mathfrak{F}' \neq \emptyset$ for any finite subfamily $\mathfrak{F}' \subset \mathfrak{F}$.

Definition 5. The family $\mathfrak{F}$ is *$\aleph_\alpha$-saturated* if $\bigcap \mathfrak{F}' \neq \emptyset$ whenever $\mathfrak{F}' \subset \mathfrak{F}$, $|\mathfrak{F}'| < \aleph_\alpha$ and $\mathfrak{F}'$ has the finite intersection property.

The following are two simple consequences of the definitions.

(18) *The ordered set $(S, <)$ has property P_α iff S is densely ordered and $I(S)$ is $\aleph_\alpha$-saturated.*

(19) *If $(S, <)$ has property P_α, then $I(S)$ is an $\aleph_\alpha$-quasi filter.*

We remark that the converse of (19) is not true even if we assume that S is densely ordered. For example, if tp $S = \eta_2 \omega + \eta_2 \omega_1^*$, then S is densely ordered and $I(S)$ is an $\aleph_2$-quasi filter, but $I(S)$ is not $\aleph_2$-saturated.

In order to apply (10) and (13) to η_α-sets and the sets R_α we now establish the following.

(20) *If $(S, <)$ is an η_α-set and $A \in$ Dense$(I(S))$, then A contains an η_α-set.*

(21) *If $A \in$ Dense$(I(R_\alpha))$, then* tp $A \geqq \eta_\alpha$.

Proof of (20). Suppose A is dense in the interval $X = (a, b)$ of S. Let $A', B' \subset A \cap X$, $A' < B'$, $|A'|, |B'| < \aleph_\alpha$. Since S has property P_α there are $a', b' \in S$ such that

$$A' \cup \{a\} < \{a'\} < \{b'\} < B' \cup \{b\}.$$

Since A is dense in X, it follows that there is $x \in A \cap X \cap (a', b')$ and hence $A' < \{x\} < B'$. Thus $A \cap X$ is an η_α-set.

(21) follows from (20) and (17) (ii) in the case when $\aleph_\alpha$ is regular. If $\aleph_\alpha$ is singular then P_α implies $P_{\alpha+1}$, i.e. every η_α-set is also an $\eta_{\alpha+1}$-set. Considering that R_α is not an $\eta_{\alpha+1}$-set (because tp $R_\alpha \ngeq \omega_{\alpha+1}$), the above argument fails for singular $\aleph_\alpha$. The following proof is quite general.

Proof of (21). By a result of HARZHEIM [7], $\eta_\alpha^2 = \eta_\alpha$. Suppose A is dense in the interval X of R_α. Since tp $X = \eta_\alpha$, it follows by HARZHEIM's theorem that X contains disjoint intervals I_x $(x \in R_\alpha)$ such that $I_x < I_y$ holds whenever $x < y$. (21) follows since $A \cap I_x \neq \emptyset$ $(x \in R_\alpha)$.

3. Proofs of (10), (13) and (16)

We shall begin by establishing a number of statements which depend upon some or all of the following hypotheses:

(22) (a) GCH.

(b) $\mathfrak{F}$ is an $\aleph_{cf(\alpha)}$-quasi filter on E, $|\mathfrak{F}| \leq \aleph_\alpha$.

(c) $[E]^2 = \mathfrak{T}_0 \cup \mathfrak{T}_1$ is a 2-partition of E such that $[A]^2 \not\subset \mathfrak{T}_0$ whenever $A \in \text{Dense}(\mathfrak{F})$.

If $x \in E$, and $i < 2$ we write $\mathfrak{T}_i(x) = \{y \in E: \{x, y\} \in \mathfrak{T}_i\}$. Also, for $X \subset E$, we define $\mathfrak{T}_i(X) = \bigcap_{x \in X} \mathfrak{T}_i(x)$.

(23) *Suppose that* (22) (b), (c) *hold and that* $C \subset F \in \mathfrak{F}$. *Then there are* $A \subset C$, $F' \subset F$ *such that* $A \cap F' = \emptyset$, $|A| < \aleph_\alpha$, $F' \in \mathfrak{F}$ *and* $C \cap F' \subset \bigcup_{x \in X} \mathfrak{T}_1(x)$.

Proof. By (22) (b) there is a sequence F_ν $(\nu < \omega_\alpha)$ containing all the sets $F'' \in \mathfrak{F}$ such that $F'' \subset F$. We can assume that $F_0 = F$. Suppose that (23) is false. Then we define a sequence $a_\nu \in C$ $(\nu < \omega_\alpha)$ by transfinite induction as follows. Let $\nu < \omega_\alpha$ and suppose that a_μ is already defined for $\mu < \nu$. Put $A_\nu = \{a_\mu: \mu < \nu\}$. If $A_\nu \cap F_\nu \neq \emptyset$, put $a_\nu = a_\mu$, where μ is the least index such that $a_\mu \in A_\nu \cap F_\nu$. If, on the other hand, $A_\nu \cap F_\nu = \emptyset$ then by our assumption and the fact that $|A_\nu| < \aleph_\alpha$, it follows that there is $a_\nu \in F_\nu \cap C - \bigcup_{\mu < \nu} \mathfrak{T}_1(a_\mu)$. This defines $A = \{a_\nu: \nu < \omega_\alpha\}$. By the construction $[A]^2 \subset \mathfrak{T}_0$ and A is dense in F. This contradicts (22) (c).

(24) *Suppose that* (22)(b) *and* (c) *hold*, $\mathfrak{m} < \aleph_{cf(\alpha)}$, $C \subset F \in \mathfrak{F}$. *Then there are* $A \subset C$, $F' \subset F$ *such that* $A \cap F' = \emptyset$, $|A| < \aleph_\alpha$, $F' \in \mathfrak{F}$ *and* $|A \cap \mathfrak{T}_1(x)| \geqq \mathfrak{m}$ *for all* $x \in F' \cap C$.

Proof. Let φ be the initial ordinal of cardinality $\mathfrak{m}$. We define A_ν and F_ν by transfinite induction for $\nu < \varphi$. Assume that $\nu < \varphi$ and that A_μ, F_μ have been defined for $\mu < \nu$ and suppose also that $F \supset F_0 \supset F_1 \supset \cdots \supset F_\mu \supset \cdots$. By (22)(b) there is $F'' \subset \bigcap_{\mu<\nu} F_\mu \cap F$, $F'' \in \mathfrak{F}$. Then by (23) there are $A_\nu \subset C$, $F_\nu \subset F''$ so that $A_\nu \cap F_\nu = \emptyset$, $|A_\nu| < \aleph_\alpha$, $F_\nu \in \mathfrak{F}$ and

$$A_\nu \cap \mathfrak{T}_1(x) \neq \emptyset \qquad (x \in F_\nu \cap C).$$

Then $F \supset F_\mu \supset F_\nu$ $(\mu < \nu)$ and the sets A_ν and F_ν are defined for every $\nu < \varphi$. Since $\varphi < \omega_{cf(\alpha)}$, the set $A = \bigcup_{\nu<\varphi} A_\nu$ has cardinality $|A| < \aleph_\alpha$ and $A \subset C$. Also, by (22)(b) there is $F' \in \mathfrak{F}$ such that $F' \subset \bigcap_{\nu<\varphi} F_\nu$. The sets A_ν $(\nu < \varphi)$ are disjoint since $A_\nu \cap F_\nu = \emptyset$ and $F_\nu \supset A_{\nu'}$ $(\nu < \nu' < \varphi)$. It follows that

$$|A \cap \mathfrak{T}_1(x)| \geqq |\varphi| = \mathfrak{m} \qquad (x \in F' \cap C).$$

(25) (GCH) *Let* $|X| = \aleph_\beta < \aleph_\alpha$, $\mathfrak{m}^+ < \aleph_\alpha$. *Suppose further that*

(26) *if* $\beta + 1 = \alpha$, $\mathfrak{m} = \aleph_\gamma$, *then* $cf(\gamma) \neq cf(\beta)$.

Then there is a set $U_\mathfrak{m}(X) \subset [X]^\mathfrak{m}$ *such that* $|U_\mathfrak{m}(X)| < \aleph_\alpha$ *and is such that whenever* $Y \in [X]^\mathfrak{m}$, *then* $Y \supset Z$ *for some* $Z \in U_\mathfrak{m}(X)$.

Remark. If (26) is false, i.e. if $\beta + 1 = \alpha$, $\mathfrak{m} = \aleph_\gamma$ and $cf(\gamma) = cf(\beta)$, then it is easy to show that there is no set $U_\mathfrak{m}(X)$ having the above property unless $|U_\mathfrak{m}(X)| \geqq \aleph_\alpha$.

Proof of (25). If $\beta + 1 < \alpha$, we simply put $U_\mathfrak{m}(X) = [X]^\mathfrak{m}$ since GCH implies that $|U_\mathfrak{m}(X)| \leqq \aleph_\beta^\mathfrak{m} < \aleph_\alpha$.

Now assume that $\beta + 1 = \alpha$. Then $\mathfrak{m} < \aleph_\beta$. There are sets X_ν $(\nu < \omega_{cf(\beta)} = \varrho)$ such that $|X_\nu| < \aleph_\beta$ and $X = \bigcup_{\nu<\varrho} X_\nu$. Put

$$U_\mathfrak{m}(X) = \bigcup_{\nu<\varrho} \Big[\bigcup_{\mu<\nu} X_\mu\Big]^\mathfrak{m}.$$

By GCH, $|U_\mathfrak{m}(X)| \leqq \aleph_\beta < \aleph_\alpha$. The set $U_\mathfrak{m}(X)$ has the required property since it follows from (26) that every set $Y \in [X]^\mathfrak{m}$ contains a subset of cardinal $\mathfrak{m}$ which is non-cofinal with X.

Finally we prove:

(27) *Suppose* (22)(a), (b), (c) *hold. Let* $C \subset F \in \mathfrak{F}$ *and suppose that* $\mathfrak{m} < \aleph_{cf(\alpha)}$, $\mathfrak{m}^+ < \aleph_\alpha$ *and that* (26) *holds. Then there are* $A \subset C$ *and* $F' \subset F$ *such that* $|A| < \aleph_\alpha$, $F' \in \mathfrak{F}$, $A \cap F' = \emptyset$ *and* $C \cap F' \subset \bigcup \{\mathfrak{T}_1(B) : B \in U_\mathfrak{m}(A)\}$.

Proof. By (24) there are $A \subset C$, $F' \subset F$ such that $A \cap F' = \emptyset$, $|A| < \aleph_\alpha$, $F' \in \mathfrak{F}$ and $|A \cap \mathfrak{T}_1(x)| \geqq \mathfrak{m}$ $(x \in F' \cap C)$. $U_\mathfrak{m}(A)$ exists by (25) and the result follows since for each element $x \in F'$ there is some $B \in U_\mathfrak{m}(A)$ such that $A \cap \mathfrak{T}_1(x) \supset B$, i.e. such that $x \in \mathfrak{T}_1(B)$.

We now give proofs of the main results (10), (16) and (13).

Proof of (10). From the hypothesis of (10) we have $\mathfrak{m}^+ < \aleph_{cf(\alpha)} = \aleph_\alpha$, and both (22)(a) and (b) hold. We shall assume that (22)(c) also holds and deduce that there are sets $B \in [E]^\mathfrak{m}$ and $C \in \text{Dense}(\mathfrak{F})$ such that $[B, C] \subset \mathfrak{T}_1$.

If α is a limit ordinal, then (26) holds vacuously. Suppose that $\alpha = \beta + 1$. The condition $\mathfrak{m}^+ < \aleph_\alpha$ implies that $\mathfrak{m} < \aleph_\beta$. Consequently, if $\aleph_\beta$ is regular then (26) holds. Finally, if $\aleph_\beta$ is singular we can assume (if necessary by replacing the cardinal $\mathfrak{m}$, which appears in the statement of (10), by some larger cardinal) that $\mathfrak{m}$ is regular and $\aleph_{cf(\beta)} < \mathfrak{m} < \aleph_\beta$. Therefore, we can assume that (26) holds.

Let $C = F \in \mathfrak{F}$ and let A, F' be the sets described in (27). Then $C \cap F' = F' \in \text{Dense}(\mathfrak{F})$. By (25) we have that $|U_\mathfrak{m}(A)| < \aleph_\alpha$ and therefore, by (1) and (27), $\mathfrak{T}_1(B) \in \text{Dense}(\mathfrak{F})$ for some $B \in U_\mathfrak{m}(A)$. This proves the result since $[B, \mathfrak{T}_1(B)] \subset \mathfrak{T}_1$ by the definition of $\mathfrak{T}_1(B)$.

Proof of (16). From the hypothesis that α is a limit ordinal and $\mathfrak{m} < \aleph_{cf(\alpha)}$, it follows that $\mathfrak{m}^+ < \aleph_\alpha$. Also, in this case (26) holds vacuously.

Put $E = R_\alpha$, $\mathfrak{F} = I(R_\alpha)$. (22)(a) holds by assumption and (22)(b) follows from (2). We can suppose that (22)(c) holds.

Let $C = F = R_\alpha$ and let A, $F' = (a, b)$ be the sets satisfying the requirements of (27). Since $\text{tp}\, F' = \eta_\alpha$ and $\lambda < \alpha$, it follows from (21) that there is $R \subset F'$ such that $\text{tp}\, R = \eta_\lambda$. We can assume that $|R| = \aleph_\lambda > |A|^+$ and that $\aleph_\lambda$ is regular. By (25), we have $|U_\mathfrak{m}(A)| < \aleph_\lambda$. Therefore, since $R \subset \bigcup \{\mathfrak{T}_1(B): B \in U_\mathfrak{m}(A)\}$, it follows from (1) and (21) that there is $B \in U_\mathfrak{m}(A)$ such that $\text{tp}\,(\mathfrak{T}_1(B) \cap R) \geqq \eta_\lambda$. This completes the proof since $[B, \mathfrak{T}_1(B)] \subset \mathfrak{T}_1$.

To prove (13) we shall use the so-called ramification argument described in lemma 1 of [2]. If $\nu = (\nu_0, \ldots, \nu_\tau, \ldots)_{\tau < \sigma}$ is a sequence of length σ, then (ν, ν_σ) denotes the extended sequence $(\nu_0, \ldots, \nu_\sigma)$ of length $\sigma + 1$ and $(\nu \restriction \tau)$ denotes the restricted sequence $(\nu_0, \ldots, \nu_\mu, \ldots)_{\mu < \tau}$ of length τ $(< \sigma)$.

Proof of (13). Put $\alpha = \beta + 1$. We want to show that, if $\mathfrak{F}$ is an $\aleph_\alpha$-quasi filter on E such that $|\mathfrak{F}| \leqq \aleph_\alpha$, then (assuming GCH)

$$(28) \qquad E \to \left(\text{Dense}(\mathfrak{F}), [\aleph_\beta, \aleph_\beta]\right)^2 .$$

If $\aleph_\beta$ is regular this already follows from (9) and so we can assume that $\beta > cf(\beta)$. By assumption (22)(a), (b) hold and we can suppose that (22)(c) also holds. We then have to show that there are $C, D \in [E]^{\aleph_\beta}$ such that $C \cap D = \emptyset$ and $[C, D] \subset \mathfrak{T}_1$.

We build up a ramification system of length $\varrho = \omega_{cf(\beta)}$ in the following way. First we choose regular cardinals $\mathfrak{m}_\sigma$ $(\sigma < \varrho)$ such that

$$\aleph_{cf(\beta)} < \mathfrak{m}_0 < \mathfrak{m}_1 < \cdots < \mathfrak{m}_\sigma < \cdots < \aleph_\beta = \sum_{\sigma < \varrho} \mathfrak{m}_\sigma .$$

For $\sigma \leqq \varrho$, let $N_\sigma = \{\nu : \nu = (\nu_0, \ldots, \nu_\tau, \ldots)_{\tau<\sigma}, \nu_\tau < \omega_\beta (\tau < \sigma)\}$. We shall define sets

$$F'_\sigma, S'(\nu), A(\nu) \qquad \text{for } \nu \in N_\sigma \text{ and } \sigma < \varrho$$

and also sets

$$F_{\sigma+1}, S(\nu), B(\nu) \qquad \text{for } \nu \in N_{\sigma+1} \text{ and } \sigma < \varrho .$$

Let $0 \leqq \sigma < \varrho$ and suppose that we have already defined F'_τ, $S'(\nu)$, $A(\nu)$ for $\nu \in N_\tau$ and $F_{\tau+1}$, $S(\nu)$, $B(\nu)$ for $\nu \in N_{\tau+1}$ when $\tau < \sigma$. Suppose also that our definitions are such that for $\tau < \sigma$

(29) $$F_1 \supset F_2 \supset \cdots \supset F_{\tau+1} ,$$

(30) $$S'(\nu) \cap F_{\tau+1} = \bigcup_{\nu_\tau < \omega_\beta} S(\nu, \nu_\tau) \qquad (\nu \in N_\tau) ,$$

(31) $$S(\nu) = F_{\tau+1} \cap \mathfrak{T}_1(B(\nu)) \qquad (\nu \in N_{\tau+1}) .$$

(32) $$F_{\tau+1} = \bigcup_{\nu \in N_{\tau+1}} S(\nu) ,$$

(33) $$S(\nu \upharpoonright \mu + 1) \supset S'(\nu) \supset S(\nu, \nu_\tau) \qquad (\mu < \tau, \nu \in N_\tau) .$$

(34) $$B(\nu, \nu_\tau) \subset A(\nu) \subset S'(\nu), A(\nu) \cap F_{\tau+1} = \emptyset \qquad (\nu \in N_\tau, \nu_\tau < \omega_\beta) .$$

We first define

$$S'(\nu) = E \cap \bigcap_{\tau<\sigma} S(\nu \upharpoonright \tau + 1)$$

for $\nu \in N_\sigma$ (note that if $\sigma = \tau + 1$, this implies that $S'(\nu) = S(\nu)$ by (33)). By (22) (b) and (29) there is $F'_\sigma \subset \bigcap_{\tau+1<\sigma} F_{\tau+1}$, $F'_\sigma \in \mathfrak{F}$. By GCH, $|N_\sigma| \leqq \aleph_\beta$ and so there is a 1-1 map φ from N_σ onto a section of ω_β. We define $A(\nu)$ and F^σ_ν for $\nu \in N_\sigma$ by induction on $\varphi(\nu)$. Assume that $\varphi(\nu) = \xi$ and that $A(\nu')$, $F_{\nu'}$ have been defined for $\nu' \in N_\sigma$ with $\varphi(\nu') < \xi$ so that $F'_\sigma \supset F^\sigma_{\nu'} \supset F^\sigma_{\nu''}$ holds whenever $\varphi(\nu') < \varphi(\nu'') < \xi$. By (22) (b) there is $F' \in \mathfrak{F}$ such that

$$F' \subset F'_\sigma \cap \bigcap_{\varphi(\nu')<\xi} F^\sigma_{\nu'} .$$

Applying (27) with $F' = F$, $C = S'(\nu) \cap F'$, it follows that there are $F^\sigma_\nu \in \mathfrak{F}$ and $A(\nu) \subset S'(\nu) \cap F'$ so that $|A(\nu)| < \aleph_\alpha$, $F^\sigma_\nu \cap A(\nu) = \emptyset$ and

(35) $$F^\sigma_\nu \cap S'(\nu) \subset \bigcup \{\mathfrak{T}_1(B) : B \in U_{\mathfrak{m}_\sigma}(A(\nu))\} .$$

This defines $A(\nu)$ and F^σ_ν for all $\nu \in N_\sigma$. Since the F^σ_ν $(\nu \in N_\sigma)$ form a decreasing sequence in $\mathfrak{F}$, it follows by (22) (b) that there is $F_{\sigma+1} \in \mathfrak{F}$ such that

$$F_{\sigma+1} \subset \bigcap_{\nu \in N_\sigma} F^\sigma_\nu .$$

If $|A(\nu)| < \mathfrak{m}_\sigma$, then $F^\sigma_\nu \cap S'(\nu) = \emptyset$ by (35) and in this case we define $B(\nu, \nu_\sigma) = \emptyset$ $(\nu_\sigma < \omega_\beta)$. On the other hand, if $|A(\nu)| \geqq \mathfrak{m}_\sigma$, then there is a sequence $(B(\nu, \nu_\sigma))_{\nu_\sigma < \omega_\beta}$ which contains all the elements of $U_{\mathfrak{m}_\sigma}(A(\nu))$. This defines $B(\nu')$

for $\nu' \in N_{\sigma-1}$. Now define

$$S(\nu, \nu_\sigma) = S'(\nu) \cap F_{\sigma+1} \cap \mathfrak{T}_1(B(\nu, \nu_\sigma)) \qquad ((\nu, \nu_\sigma) \in N_{\sigma+1}) .$$

By (35) and the definitions of $F_{\sigma+1}$ and $S(\nu, \nu_\sigma)$, it follows that (30) holds with $\tau = \sigma$. It is clear from our definitions that (29), (31), (33) and (34) hold with $\tau = \sigma$, and it remains for us to verify (32). Let $x \in F_{\sigma+1}$. Let $\varkappa \leqq \sigma$ and suppose that we have already defined $\nu_\mu < \omega_\beta$ for $\xi < \varkappa$ so that $x \in S(\nu_0, \ldots, \nu_\mu)$ for $\mu < \varkappa$. Then $x \in S'(\nu_0, \ldots, \nu_\mu, \ldots)_{\mu<\varkappa}$ by the definition of this set. Therefore, since (30) holds for $\tau \leqq \sigma$, there is $\nu_\varkappa < \omega_\beta$ so that $x \in S(\nu_0, \ldots, \nu_\varkappa)$. This defines $\nu_\varkappa$ for $\varkappa \leqq \sigma$ so that $x \in S(\nu_0, \ldots, \nu_\sigma)$ and it follows that (32) holds for $\tau = \sigma$.

Considering that there is, by (22)(b), $F' \in \mathfrak{F}$ such that $F' \subset \bigcap_{\sigma<\varrho} F_{\sigma+1}$, it follows (just as in the proof of (32) above), that there is $\nu \in N_\sigma$ such that

$$\bigcap_{\sigma+1<\varrho} S(\nu \restriction \sigma + 1) \neq \emptyset .$$

For this ν put $B_\sigma = B(\nu \restriction \sigma + 1)$ for $\sigma < \varrho$. By (31) we have that $B_\sigma \neq \emptyset$ and so $|B_\sigma| = \mathfrak{m}_\sigma$ $(\sigma < \varrho)$. Also, if $\sigma < \sigma' < \varrho$, then it follows from (31) and the fact that $B_{\sigma'} \subset A(\nu \restriction \sigma' + 1) \subset S'(\nu \restriction \sigma' + 1) \subset S(\nu \restriction \sigma + 1)$, that B_σ, $B_{\sigma'}$ are disjoint and $[B_\sigma, B_{\sigma'}] \subset \mathfrak{T}_1$. If we put

$$C = \bigcup_{\substack{\sigma<\varrho \\ \sigma \text{ even}}} B_\sigma , \qquad D = \bigcup_{\substack{\sigma<\varrho \\ \sigma \text{ odd}}} B_\sigma ,$$

then $|C| = |D| = \aleph_\beta$, $C \cap D = \emptyset$ and $[C, D] \subset \mathfrak{T}_1$. This proves (13).

4. Examples of $\aleph_\alpha$-quasi filters

In this section we shall give examples of sets A and $\mathfrak{G} \subset \mathfrak{P}(A)$ having the property:

(36) *There is an $\aleph_{\beta+1}$-quasi filter $\mathfrak{F}$ on A such that $|\mathfrak{F}| \leqq \aleph_{\beta+1}$ and, for every $X \in \mathrm{Dense}(\mathfrak{F})$, then there is $Y \in \mathfrak{G}$ such that $Y \subset X$.*

It follows from (9), (10) and (13) that, if GCH holds and $\mathfrak{G}$ satisfies (36), then

$$A \to (\mathfrak{G}, \aleph_{cf(\beta)})^2 ,$$
$$A \to (\mathfrak{G}, [\mathfrak{m}, \mathfrak{G}])^2 \qquad (\mathfrak{m} < \aleph_\beta) ,$$
$$A \to (\mathfrak{G}, [\aleph_\beta, \aleph_\beta])^2 .$$

We shall first try to extend as far as possible the relations (7), (11), (14) (which are the respective corollaries of the above formulae for $\eta_{\beta+1}$-sets) to $\aleph_{\beta+1}$-saturated models.

The following definition is due to KEISLER [10] although we use a slightly different notation.

Definition 6. Let $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \varrho}$ be a relational system of type μ and let $L(\mu)$ be a first order logic with identity and $\mu(\lambda)$-ary predicate symbols P_λ ($\lambda < \varrho$). Let $F(\mu)$ be the set of formulas of $L(\mu)$. If $\Phi(x_0, \ldots, x_n) \in F(\mu)$ is a formula with $n + 1$ free variables and $a_1, \ldots, a_n \in A$, we put

$$E^{\mathfrak{A}}(\Phi, a_1, \ldots, a_n) = \{a_0 \in A : \Phi^{\mathfrak{A}}(a_0, \ldots, a_n) \text{ is true}\} .$$

Let $\mathfrak{F}(\mathfrak{A}) = \{E^{\mathfrak{A}}(\Phi, a_1, \ldots, a_n) : \Phi \in F(\mu), \Phi$ has $n + 1$ free variables and $a_1, \ldots, a_n \in A\}$. The relational system $\mathfrak{A}$ is said to be $\aleph_\alpha$-*saturated* if $\mathfrak{F}(\mathfrak{A})$ is $\aleph_\alpha$-saturated.

We shall outline the proofs of the following two examples of sets A and $\mathfrak{G}$ satisfying (36).

(37) *Assume* GCH. *Let* $\mathfrak{A} = \langle A, R_\lambda \rangle_{\lambda < \varrho}$ *be an* $\aleph_{\beta+1}$-*saturated relational system,* $|\mathfrak{A}| = |A| = \aleph_{\beta+1}$, $\varrho < \omega_{\beta+1}$. *Let* $\mathfrak{G}$ *be the set of all sets* $X \subset A$ *for which there is* A', $X \subset A' \subset A$, *satisfying the following three conditions*:

(i) $\mathfrak{A}' = \mathfrak{A} \restriction A'$ is $\aleph_{\beta+1}$ *saturated and* $|\mathfrak{A}'| = \aleph_{\beta+1}$;

(ii) $X \cap B \neq \emptyset$ *for every infinite set* $B \in \mathfrak{F}(\mathfrak{A}')$;

(iii) $\mathfrak{F}(\mathfrak{A}') \restriction X = \{X \cap B : B \in \mathfrak{F}(\mathfrak{A}')\}$ *is* $\aleph_{\beta+1}$-*saturated.*

Then A, $\mathfrak{G}$ *satisfy* (36).

(38) *Suppose the same hypothesis as in* (37) *holds. Suppose further that, for every* $\Phi \in F(\mu)$ *with* $n + 1$ *free variables and* $a_1, \ldots, a_n \in A$,

$$E^{\mathfrak{A}}(\Phi, a_1, \ldots, a_n) \not\subset \{a_1, \ldots, a_n\} \to |E^{\mathfrak{A}}(\Phi, a_1, \ldots, a_n)| \geqq \aleph_0 .$$

Then A *and* $\mathfrak{G}' = \{X \subset A : \mathfrak{A} \restriction X$ *is* $\aleph_{\beta+1}$-*saturated and has power* $\aleph_{\beta+1}\}$ *satisfy* (36).

We remark that the conclusion of (38) is the desirable analogue of the results for $\eta_{\beta+1}$, but as H. J. Keisler pointed out to us that there are $\aleph_{\beta+1}$-saturated relational systems for which the set $\mathfrak{G}$ defined in (37) satisfies (36) but the set $\mathfrak{G}'$ defined in (38) does not. The additional condition of (38) is true e.g. if $\mathfrak{A} = \langle A, < \rangle$ and $<$ is a dense ordering of A.

Outline proof of (37). It is known (e.g. [11]) that any two elementarily equivalent $\aleph_{\beta+1}$-saturated structures of power $\aleph_{\beta+1}$ are isomorphic. Let $\mathfrak{B} = \langle B, S_\lambda \rangle_{\lambda < \varrho}$ be isomorphic to $\mathfrak{A}$. By Theorem 2.1 of [10], we can assume that $\mathfrak{A} = \mathfrak{B}^I/D$, where $|I| = \aleph_\beta$ and D is an $\aleph_{\beta+1}$ good ultra filter on I (for the special properties of D see [10]). Put $\mathfrak{F}_0 = \{F \subset A | F = \prod_{i \in I} F_i/D$, where $F_i \in \mathfrak{F}(\mathfrak{B})$ $(i \in I)\}$

Then $\mathfrak{F}_0$ satisfies the following conditions (α), (β).

(α) For every $F \in \mathfrak{F}_0$, $\mathfrak{F}(\mathfrak{A} \restriction F) \subset \mathfrak{F}_0$.

(β) If $\mathfrak{F}' \subset \mathfrak{F}_0$, $|\mathfrak{F}'| \leqq \aleph_\beta$ and $\mathfrak{F}'$ has the finite intersection property, then there is $F \in \mathfrak{F}_0$, $F \neq \emptyset$ such that $F \subset \bigcap \mathfrak{F}'$. Also, if $|\bigcap \mathfrak{F}''| \geqq \aleph_0$ for every finite $\mathfrak{F}'' \subset \mathfrak{F}'$, then F can be chosen so that $|F| = \aleph_{\beta+1}$.

Note that (α) holds since, for every $\Phi \in F(\mu)$ (of n free variables) and every $f_1, \ldots, f_n \in \mathfrak{A} \upharpoonright F$, $\Phi^{\mathfrak{A} \upharpoonright F}(f_1, \ldots, f_n)$ holds iff $\{i \in I: \Phi^{\mathfrak{B} \upharpoonright F_i}(f_1(i), \ldots, f_n(i))\} \in D$ and, for every i, $\mathfrak{F}(\mathfrak{B} \upharpoonright F_i) \subset \mathfrak{F}(\mathfrak{B})$. On the other hand, ($\beta$) follows by a standard argument from Theorem 2.1 of [10].

Now put $\mathfrak{F} = \mathfrak{F}_0 \cap [A]^{\aleph_{\beta+1}}$. It follows from ($\beta$) that $\mathfrak{F} = \mathfrak{F}_0 \cap [A]^{\geqq \aleph_0}$, and that $\mathfrak{F}$ is an $\aleph_{\beta+1}$-quasi filter. Suppose $X' \in \mathrm{Dense}(\mathfrak{F})$. We will show that there is $X \subset X'$ such that $X \in \mathfrak{G}$. Since $X' \in \mathrm{Dense}(\mathfrak{F})$, there is $A' \in \mathfrak{F} \subset \mathfrak{F}_0$ such that

$$X' \cap B \neq \emptyset \qquad \text{for every } B \subset A', B \in \mathfrak{F} .$$

We verify that conditions (i), (ii) and (iii) of (37) hold for $X = X' \cap A'$ and A'.

(i) From (β) it follows that $\mathfrak{F}_0$ is $\aleph_{\beta+1}$-saturated and hence $\mathfrak{F}(\mathfrak{A}')$ ($\subset \mathfrak{F}_0$) is also $\aleph_{\beta+1}$-saturated. By the definition of $\mathfrak{F}$, we have that $|A'| = \aleph_{\beta+1}$.

(ii) Suppose $B \in \mathfrak{F}(\mathfrak{A}')$, $|B| \geqq \aleph_0$. Then $B \in \mathfrak{F}_0$ by (α) and, by (β), $|B| = \aleph_{\beta+1}$, i.e. $B \in \mathfrak{F}$. Therefore, $X \cap B \neq \emptyset$ by the definition of A'.

(iii) Let $\mathfrak{H} \subset \mathfrak{F}(\mathfrak{A}') \upharpoonright X$, $|\mathfrak{H}| \leqq \aleph_\beta$ and suppose that $\mathfrak{H}$ has the finite intersection property. Then there is $\mathfrak{F}' \subset \mathfrak{F}(\mathfrak{A}') \subset \mathfrak{F}_0$ such that $|\mathfrak{F}'| \leqq \aleph_\beta$ and $\mathfrak{H} = \{X \cap B: B \in \mathfrak{F}'\}$. If there is a finite subset $\mathfrak{H}' \subset \mathfrak{H}$ such that $|\bigcap \mathfrak{H}'| < \aleph_0$, then trivially $\bigcap \mathfrak{H} \neq \emptyset$ by the finite intersection property. So we can assume that $|\bigcap \mathfrak{H}'| \geqq \aleph_0$ for every finite set $\mathfrak{H}' \subset \mathfrak{H}$. Therefore, $|\bigcap \mathfrak{F}''| \geqq \aleph_0$ for every finite set $\mathfrak{F}'' \subset \mathfrak{F}'$ and so, by (β), there is $B \in \mathfrak{F}$ such that $B \subset \bigcap \mathfrak{F}'$. Since $B \subset A'$, we have

$$\emptyset \neq X \cap B \subset X \cap \bigcap \mathfrak{F}' = \bigcap \mathfrak{H}.$$

Outline proof of (38). Let $\mathfrak{B}$, $\mathfrak{A}$, $\mathfrak{F}$, $\mathfrak{F}_0$, X, A' be as defined in the proof of (37). The additional assumption of (38) can be formulated so that for every $\Phi \in F(\mu)$, $m < \omega$

$$\mathfrak{A} \models \forall x_1 \ldots \forall x_m \left(\exists x_0 \left(\Phi(x_0, x_1, \ldots, x_n) \wedge \bigwedge_{i=1}^{n} x_0 \neq x_i\right)\right.$$
$$\left.\Rightarrow \exists y \left(\Phi(y, x_1, \ldots, x_n) \wedge \bigwedge_{i=0}^{m} y \neq x_i\right)\right).$$

The same holds for $\mathfrak{B}$ and hence for every $\mathfrak{B} \upharpoonright F_i$ with $F_i \in \mathfrak{F}(\mathfrak{B})$ since $\mathfrak{F}(\mathfrak{B} \upharpoonright F_i) \subset \mathfrak{F}(\mathfrak{B})$. Since $\mathfrak{A}' = \mathfrak{A} \upharpoonright A' = \prod_{i \in I} \mathfrak{B} \upharpoonright F_i/D$, where $F_i \in \mathfrak{F}(\mathfrak{B})$ ($i \in I$), the same also holds for $\mathfrak{A}'$. Thus we have that

$$E^{\mathfrak{A}'}(\Phi, a_1, \ldots, a_n) \not\subset \{a_1, \ldots, a_n\} \Rightarrow |E^{\mathfrak{A}'}(\Phi, a_1, \ldots, a_n)| \geqq \aleph_0$$

for every $a_1, \ldots, a_n \in A'$ and $\Phi \in F(\mu)$. It follows from (37)(ii) that $\mathfrak{A}'$ is an elementary extension of $\mathfrak{A} \upharpoonright X$. Therefore, $\mathfrak{F}(\mathfrak{A}' \upharpoonright X) \subset \mathfrak{F}(\mathfrak{A}') \upharpoonright X$ and hence, by (iii) of (37), $\mathfrak{A} \upharpoonright X = \mathfrak{A}' \upharpoonright X$ is $\aleph_{\beta+1}$-saturated.

Finally, we mention one further simple instance of sets A, $\mathfrak{G}$ satisfying (36).

(39) *Assume* GCH. *Let* $\mathfrak{J}$ *be an* $\aleph_{\beta+1}$*-complete ideal in* A *generated by at most* $\aleph_{\beta+1}$ *elements and let* $\mathfrak{G} = \mathfrak{P}(A) - \mathfrak{J}$. *Then* (36) *holds.*

For let $\mathfrak{H} \subset \mathfrak{J}$ be any set such that $|\mathfrak{H}| \leq \aleph_{\beta+1}$ and such that each set $I \in \mathfrak{J}$ is contained in some $H \in \mathfrak{H}$. Then $\mathfrak{F} = \{A - H: H \in \mathfrak{H}\}$ is an $\aleph_{\beta+1}$-quasi filter on A, $|\mathfrak{F}| = \aleph_{\beta+1}$ and $X \in \mathrm{Dense}(\mathfrak{F})$ iff $X \notin \mathfrak{J}$.

As a corollary of (39) we regain the following known result of [1].

(40) *Assume* GCH. *Let R be the set of reals and let $\mathfrak{G}_1$ be the set of subsets of R having positive Lebesgue outer measure and let $\mathfrak{G}_2$ be the set of subsets of R of second category. Then* (36) *holds with $\beta = 0$, $A = R$ and $\mathfrak{G} = \mathfrak{G}_i$* $(i = 1, 2)$.

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